Scientia Magna
Vol. 3 (2007), No. 1, 54-80

An introduction to Smarandache multi-spaces and mathematical combinatorics[1]

Linfan Mao
Chinese Academy of Mathematics and System Science
Beijing 100080, P.R.China
E-mail: maolinfan@163.com

Received Jan. 28, 2007

Abstract These Smarandache spaces are right theories for objectives by logic. However, the mathematical combinatorics is a combinatorial theory for branches in classical mathematics motivated by a combinatorial speculation. Both of them are unifying theories for sciences and contribute more and more to mathematics in the 21st century. In this paper, I introduce these two subjects and mainly concentrate on myself research works on mathematical combinatorics finished in past three years, such as those of map geometries, pseudo-manifolds of dimensional n, topological or differential structures on smoothly combinatorial manifolds. All of those materials have established the pseudo-manifold geometry and combinatorially Finsler geometry or Riemannian geometry. Other works for applications of Smarandache multi-spaces to algebra and theoretical physics are also partially included in this paper.

Keywords Smarandache multi-space, mathematical combinatorics, Smarandache n-manifold, map geometry, topological and differential structures, geometrical inclusions.

§1. Introduction

Today, we have known two heartening mathematical theories for sciences. One of them is the Smarandache multi-space theory, came into being by purely logic ([22] − [23]). Another is the mathematical combinatorics motivated by a combinatorial speculation for branches in classical mathematics([7], [16]). The former is more like a philosophical notion. However, the later can be enforced in practice, which opened a new way for mathematics in the 21st century, namely generalizing classical mathematics by its combinatorialization.

Then what is a Smarandache multi-space? Let us begin from a famous proverb. See Fig.1.1. In this proverb, six blind men were asked to determine what an elephant looked like by feeling different parts of the elephant's body.

---

[1] Reported at The Third International Conference on Number Theory and Smarandache's Problems of China, Mar. 23-25, 2007, Xi'an, P.R.China

Fig.1.1

The man touched the elephant's leg, tail, trunk, ear, belly or tusk claims it's like a pillar, a rope, a tree branch, a hand fan, a wall or a solid pipe, respectively. They entered into an endless argument. Each of them insisted his view right. All of you are right!A wise man explains to them: Why are you telling it differently is because each one of you touched the different part of the elephant. So, actually the elephant has all those features what you all said.

Certainly, Smarandache multi-spaces are related with the natural space. For this space, a view of the sky by eyes of a man stand on the earth is shown in Fig.1.2. The bioelectric structure of human's eyes decides that he or she can not see too far, or too tiny thing without the help of precision instruments. The picture shown in Fig.1.3 was made by the Hubble telescope in 1995.

Fig.1.2

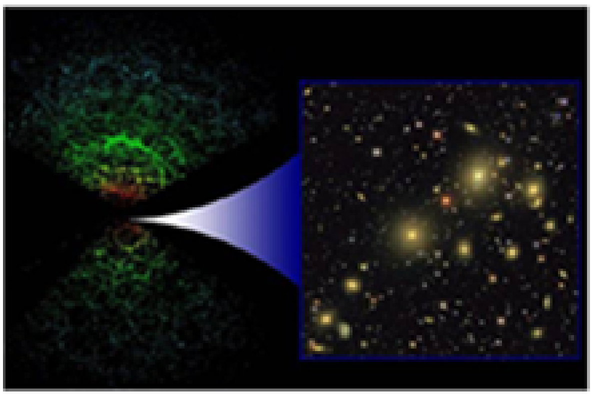

Fig.1.3

Physicists are usually to write (t, x_1, x_2, x_3) in $\mathbf{R}^4$ to represent an *event*. For two events $A_1 = (t_1, x_1, x_2, x_3)$ and $A_2 = (t_2, y_1, y_2, y_3)$, their *spacetime interval* $\triangle s$ is defined by

$$\triangle^2 s = -c^2 \triangle t^2 + \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2},$$

where c is the speed of light in the vacuum. The Einstein's general relativity states that all laws of physics take the same form in any reference system and his equivalence principle says that there are no difference for physical effects of the inertial force and the gravitation in a field small enough.

Combining his two principles, Einstein got his gravitational equation

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \lambda g_{\mu\nu} = -8\pi G T_{\mu\nu},$$

where

$R_{\mu\nu} = R_{\nu\mu} = R^{\alpha}_{\mu\alpha\nu}$ and $R = g^{\nu\mu}R_{\nu\mu}$, $R^{\alpha}_{\mu i\nu} = \frac{\partial\Gamma^i_{\mu i}}{\partial x^\nu} - \frac{\partial\Gamma^i_{\mu\nu}}{\partial x^i} + \Gamma^{\alpha}_{\mu i}\Gamma^i_{\alpha\nu} - \Gamma^{\alpha}_{\mu\nu}\Gamma^i_{\alpha i}$, $\Gamma^g_{mn} = \frac{1}{2}g^{pq}(\frac{\partial g_{mp}}{\partial u^n} + \frac{\partial g_{np}}{\partial u^m} - \frac{\partial g_{mn}}{\partial u^p})$.

Applying the Einstein's equation of gravitational field and the cosmological principle, namely there are no difference at different points and different orientations at a point of a cosmos on the metric $10^4 l.y.$ with the Robertson-Walker metric

$$ds^2 = -c^2dt^2 + a^2(t)[\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)].$$

Friedmann got a standard model of the universe which classifies universes into three types: static, contracting and expanding. This model also brought about the birth of the Big Bang model in thirties of the 20th century. The following diagram describes the developing process of our cosmos in different periods after the Big Bang.

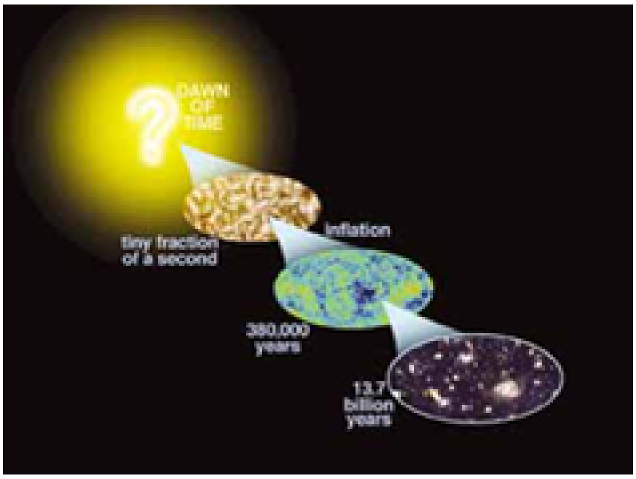

Fig.1.4

Today, more and more evidences indicate that our universe is in accelerating expansion. In 1934, R.Tolman first showed that blackbody radiation in an expanding universe cools but retains its thermal distribution and remains a blackbody. G.Gamow, R.Alpher and R.Herman predicted that a Big Bang universe will have a blackbody cosmic microwave background with temperature about 5K in 1948. Afterward, A.Penzias and R.Wilson discovered the 3K cosmic microwave background (CMB) radiation in 1965, which made the two physicists finally won the Noble Prize of physics in 1978. G.F.Smoot and J.C.Mather also won the Noble Prize of physics for their discovery of the blackbody form and anisotry of the cosmic microwave background radiation in 2006. In Fig.1.5, the CMB timeline and a drawing by a artificial satellite WMAP in 2003 are shown.

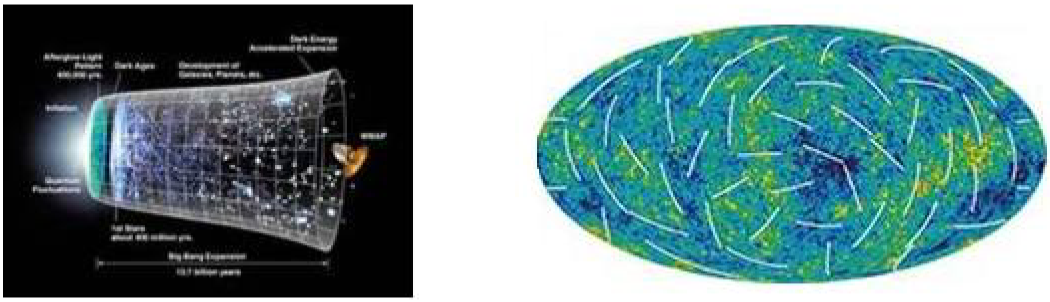

Fig.1.5

We have known that all matters are made of atoms and sub-atomic particles, held together by four fundamental forces, i.e., gravity, electromagnetism, strong nuclear force and weak force. They are partially explained by Quantum Theory (electromagnetism, strong nuclear force and weak force) and Relativity Theory(gravity). The Einstein' s unifying theory of fields wishs to describe the four fundamental forces, i.e., combine Quantum Theory and Relativity Theory. His target was nearly realized in 80s in last century, namely the establishing of string/M-theory.

There are five already known string theories, i.e., $E_8 \times E_8$ heterotic string, $SO(32)$ heterotic string, $SO(32)$ Type I string, Type IIA and Type IIB, each of them is an extreme theory of M-theory such as those shown in Fig.1.6.

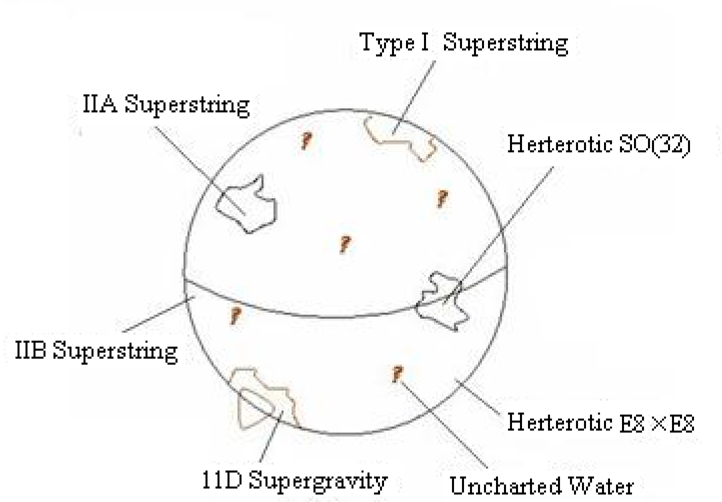

Fig.1.6

Then what is the right theory for the universe? A right theory for the universe Σ should be

$$\begin{aligned}\Sigma &= \{E_8 \times E_8 \text{ heterotic string}\}\bigcup\{SO(32) \text{ heterotic string}\}\\ &\bigcup\{SO(32) \text{ type I string}\}\bigcup\{\text{type IIA string}\}\\ &\bigcup\{\text{type IIB string}\}\bigcup A\bigcup\cdots\bigcup B\cdots\bigcup C,\end{aligned}$$

where $A, \cdots, B, \cdots, C$ denote some unknown theories for the universe Σ.

Generally, what is a right theory for an objective Δ? We all know that the foundation of science is the measures and metrics. Different characteristic A_i by different metric describes the different side Δ_i of Δ. Therefore, a right theory for Δ should be

$$\Delta = \bigcup_{i\geq 1} \Delta_i = \bigcup_{i\geq 1} A_i.$$

Now Smarandache multi-spaces are formally defined in the next, which convinces us that Smarandache multi-spaces are nothing but mathematics for right theories of objectives.

Definition 1.1.([9],[22]) A Smarandache multi-space is a union of n different spaces equipped with some different structures for an integer $n \geq 2$.

For example, let n be an integer, $Z_1 = (\{0, 1, 2, \cdots, n-1\}, +)$ an additive group (mod n) and $P = (0, 1, 2, \cdots, n-1)$ a permutation. For any integer $i, 0 \leq i \leq n-1$, define

$$Z_{i+1} = P^i(Z_1),$$

such that $P^i(k) +_i P^i(l) = P^i(m)$ in Z_{i+1} if $k + l = m$ in Z_1, where $+_i$ denotes the binary operation $+_i : (P^i(k), P^i(l)) \to P^i(m)$. Then we know that $\bigcup_{i=1}^{n} Z_i$ is a Smarandache multi-space.

The mathematical combinatorics is a combinatorial theory for classical mathematics established by the following conjecture on mathematical sciences.

Conjecture 1.1.([7], [16]) Every mathematical science can be reconstructed from or made by combinatorization.

This conjecture means that

(i) One can selects finite combinatorial rulers to reconstruct or make generalization for classical mathematics and

(ii) One can combine different branches into a new theory and this process ended until it has been done for all mathematical sciences.

Applications of the mathematical combinatorics to geometry, algebra and physics can be found in these references [9] − [17]. For terminologies and notations not defined in this paper, we follow [1], [4] for differential geometry and [21], [24] for topology.

§2. Smaradache Geometries

2.1. **Geometrical multi-space**

A multi-metric space is defined in the following.

Definition 2.1 A multi-metric space is a union $\bigcup_{1}^{m} M_i$ such that each M_i is a space with a metric ρ_i for any integer i, $1 \leq i \leq m$.

2.2. **Smarandache geometries**

The axiom system of the Euclid geometry consists following five axioms.

(A1) There is a straight line between any two points.

(A2) A finite straight line can produce a infinite straight line continuously.

(A3) Any point and a distance can describe a circle.

(A4) All right angles are equal to one another.

(A5) If a straight line falling on two straight lines make the interior angles on the same side less than two right angles, then the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

The axiom (A5) can be also replaced by:

(A5') Given a line and a point exterior this line, there is one line parallel to this line.

The Lobachevshy-Bolyai-Gauss geometry, also called hyperbolic geometry is a geometry with axioms $(A1)-(A4)$ and the following axiom $(L5)$:

(L5) There are infinitely many line parallels to a given line passing through an exterior point.

The Riemann geometry, also called elliptic geometry is a geometry with axioms $(A1)-(A4)$ and the following axiom $(R5)$:

(R5) There is no parallel to a given line passing through an exterior point.

These two geometries are mixed non-Euclid geometry. F.Smarandache asked the following question in 1969 for new mixed non-euclid geometries.

Question 2.1. Are there other geometries by denying axioms in Euclid geometry not like the hyperbolic or Riemann geometry?

He also specified his question to the following concrete question.

Question 2.2. Are there paradoxist geometry, non-geometry, counter-projective geometry and anti-geometry defined by definitions follows?

2.2.1. **Paradoxist geometry**

In this geometry, its axioms are $(A1)-(A4)$ and with one of the following as the axiom $(P5)$.

(i) There are at least a straight line and a point exterior to it in this space for which any line that passes through the point intersect the initial line.

(ii) There are at least a straight line and a point exterior to it in this space for which only one line passes through the point and does not intersect the initial line.

(iii) There are at least a straight line and a point exterior to it in this space for which only a finite number of lines $l_1, l_2, \cdots, l_k, k \geq 2$ pass through the point and do not intersect the initial line.

(iv) There are at least a straight line and a point exterior to it in this space for which an infinite number of lines pass through the point (but not all of them) and do not intersect the initial line.

(v) There are at least a straight line and a point exterior to it in this space for which any line that passes through the point and does not intersect the initial line.

2.2.2. **Non-Geometry**

The non-geometry is a geometry by denial some axioms of $(A1)-(A5)$ such as follows.

$(A1^-)$ It is not always possible to draw a line from an arbitrary point to another arbitrary point.

$(A2^-)$ It is not always possible to extend by continuity a finite line to an infinite line.

$(A3^-)$ It is not always possible to draw a circle from an arbitrary point and of an arbitrary interval.

$(A4^-)$ Not all the right angles are congruent.

$(A5^-)$ If a line, cutting two other lines, forms the interior angles of the same side of it strictly less than two right angle, then not always the two lines extended towards infinite cut each other in the side where the angles are strictly less than two right angle.

2.2.3. **Counter-Projective geometry**

Denoted by P the point set, L the line set and R a relation included in $P \times L$. A counter-projective geometry is a geometry with counter-axioms following.

$(C1)$ There exist: either at least two lines, or no line, that contains two given distinct points.

$(C2)$ Let p_1, p_2, p_3 be three non-collinear points, and q_1, q_2 two distinct points. Suppose that $\{p_1.q_1, p_3\}$ and $\{p_2, q_2, p_3\}$ are collinear triples. Then the line containing p_1, p_2 and the line containing q_1, q_2 do not intersect.

$(C3)$ Every line contains at most two distinct points.

2.2.4. **Anti-Geometry**

A geometry by denial some axioms of the Hilbert's 21 axioms of Euclidean geometry.

Definition 2.2.([6]) An axiom is said Smarandachely denied if the axiom behaves in at least two different ways within the same space, i.e., validated and invalided, or only invalided but in multiple distinct ways.

A Smarandache geometry is a geometry which has at least one Smarandachely denied axiom(1969).

For example, let us consider an Euclidean plane $\mathbf{R}^2$ and three non-collinear points A, B and C. Define s-points as all usual Euclidean points on $\mathbf{R}^2$ and s-lines as any Euclidean line that passes through one and only one of points A, B and C. This geometry then is a Smarandache geometry because two axioms are Smarandachely denied comparing with an Euclid geometry.

(i) The axiom (A5) that through a point exterior to a given line there is only one parallel passing through it is now replaced by two statements: one parallel and no parallel. Let L be an s-line passing through C and not parallel to AB in the Euclidean sense. Notice that through any s-point collinear with A or B there is one s-line parallel to L and through any other s-point there are no s-lines parallel to L such as those shown in Fig.2.1(a).

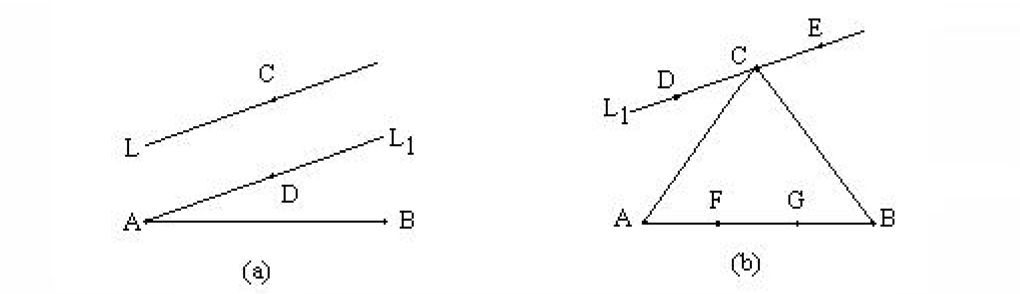

Fig.2.1

(ii) The axiom that through any two distinct points there exists one line passing through them is now replaced by; one s-line and no s-line. Notice that through any two distinct s-points D, E collinear with one of A, B and C, there is one s-line passing through them and through any two distinct s-points F, G lying on AB or non-collinear with one of A, B and C, there is no s-line passing through them such as those shown in Fig.2.1(b).

Iseri constructed s-manifolds for dimensional 2 Smarandache manifolds in [5] as follows.

An s-manifold is any collection of these equilateral triangular disks T_i, $1 \leq i \leq n$ satisfying conditions following:

(i) Each edge e is the identification of at most two edges e_i, e_j in two distinct triangular disks $T_i, T_j, 1 \leq i \leq n$ and $i \neq j$;

(ii) Each vertex v is the identification of one vertex in each of five, six or seven distinct triangular disks, called elliptic, euclidean or hyperbolic point.

These vertices are classified by the number of the disks around them. A vertex around five, six or seven triangular disks is called respective an elliptic vertex, an Euclid vertex or a hyperbolic vertex, which can be realized in $\mathbf{R}^3$ such as shown in Fig.2.2 for an elliptic point and Fig.2.3 for a hyperbolic point.

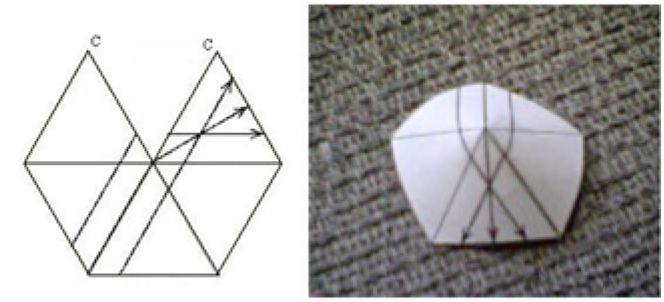

Fig.2.2

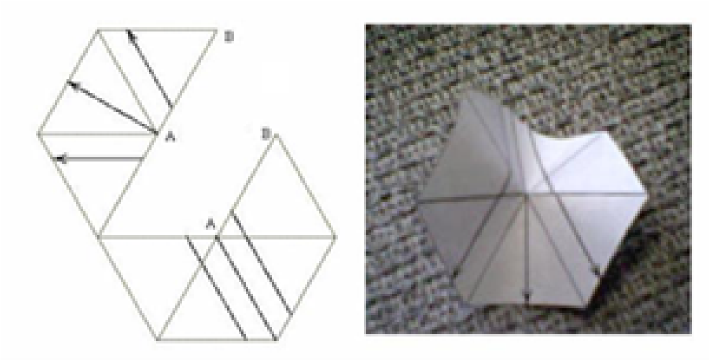

Fig.2.3

Iseri proved in [5] that there are Smarandache geometries, particularly, paradoxist geometries, non-geometries, counter-projective geometries and anti-geometries in s-manifolds.

Now let $\Delta_i, 1 \leq i \leq 7$ denote those of closed s-manifolds with vertex valency $5, 6, 7$, 5 or 6, 5 or 7, 6 or 7, 5 or 6 or 7, respectively. Then a classification for closed s-manifolds was obtained in [7].

Theorem 2.1.([7]) $|\Delta_i| = +\infty$ for $i = 2, 3, 4, 6, 7$ and $|\Delta_1| = 2, |\Delta_5| \geq 2$.

2.3. **Smarandache manifolds**

For any integer n, $n \geq 1$, an n-manifold is a Hausdorff space M^n, i.e., a space that satisfies the T_2 separation axiom, such that for any $p \in M^n$, there is an open neighborhood U_p, $p \in U_p$ a subset of M^n and a homeomorphism $\varphi_p : U_p \to \mathbf{R}^n$ or $\mathbf{C^n}$, respectively.

A Smarandache manifold is an n-dimensional manifold that support a Smarandache geometry.

Question 2.3. Can we construct Smarandache n-manifolds for any integer $n \geq 2$?

§3. Constructing Smarandache 2-manifolds

3.1. **Maps geometries**

Closed s-manifolds in Iseri's model is essentially Smarandache 2-manifolds, special triangulations of spheres with vertex valency 5, 6 or 7. A generalization of his idea induced a general construction for Smarandache 2-manifolds, namely map geometries on 2-manifolds.

Let us introduce some terminologies in graph theory first. A graph G is an ordered 3-tuple $(V, E; I)$, where V, E are finite sets, $V \neq \emptyset$ and $I : E \to V \times V$. Call V the vertex set and E the edge set of G, denoted by $V(G)$ and $E(G)$, respectively. A graph can be represented by a diagram on the plane, in which vertices are elements in V and two vertices u, v is connected by an edge e if and only if there is a $\varsigma \in I$ enabling $\varsigma(e) = (u, v)$.

The classification theorem for 2-dimensional manifolds in topology says that each 2-manifold is homomorphic to the sphere P_0, or to a 2-manifold P_p by adding p handles on P_0, or to a 2-manifold N_q by adding q crosscaps on P_0. By definition, the former is said an orientable 2-manifold of genus p and the later a non-orientable 2-manifold of genus q. This classification for 2-dimensional manifolds can be also described by polygon representations of 2-manifolds with even sides stated following again.

Any compact 2-manifold is homeomorphic to one of the following standard 2-manifolds:

(P_0) the sphere: aa^{-1};

(P_n) the connected sum of $n, n \geq 1$ tori:

$$a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1};$$

(Q_n) the connected sum of $n, n \geq 1$ projective planes:

$$a_1 a_1 a_2 a_2 \cdots a_n a_n.$$

A combinatorial map M is a connected topological graph cellularly embedded in a 2-manifold M^2. For example, the graph K^4 on the Klein bottle with one face length 4 and another 8 is shown in Fig.3.1.

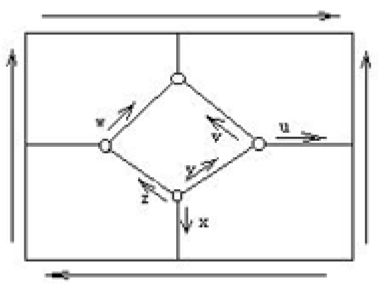

Fig.3.1

Definition 3.7. For a combinatorial map M with each vertex valency≥ 3, endow each vertex $u, u \in V(M)$ a real number $\mu(u), 0 < \mu(u) < \frac{4\pi}{\rho_M(u)}$. Call (M, μ) a map geometry without boundary, $\mu : V(M) \to R$ an angle function on M.

As an example, Fig.3.2 presents a map geometry without boundary on a map K^4 on the plane.

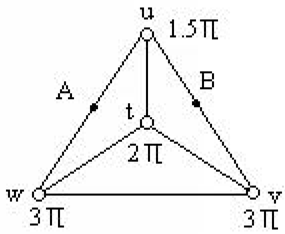

Fig.3.2

In this map geometry, lines behaviors are shown in Fig.3.3.

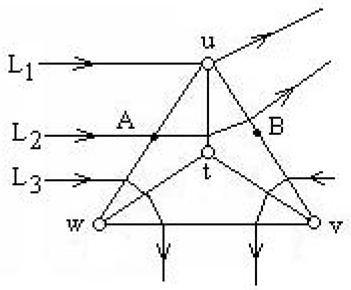

Fig.3.3

Definition 3.8. For a map geometry (M, μ) without boundary and faces $f_1, f_2, \cdots, f_l \in F(M), 1 \leq l \leq \phi(M) - 1$, if $S(M) \setminus \{f_1, f_2, \cdots, f_l\}$ is connected, then call $(M, \mu)^{-l} = (S(M) \setminus \{f_1, f_2, \cdots, f_l\}, \mu)$ a map geometry with boundary $f_1, f_2, \cdots, f_l$, where $S(M)$ denotes the locally orientable 2-manifold underlying M.

An example for map geometries with boundary is presented in Fig.3.4

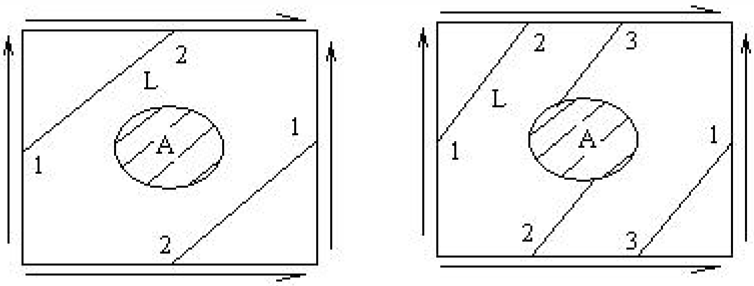

Fig.3.4

Similar to these results of Iseri, we obtained a result for Smarandache 2-manifolds in [9].

Theorem 3.1.([9]) There are Smarandache 2-manifolds in map geometries with or without boundary, particularly,

(1) For a map M on a 2-manifold with order≥ 3, vertex valency≥ 3 and a face $f \in F(M)$, there is an angle factor µ such that (M, μ) and $(M, \mu)^{-1}$ is a paradoxist geometry by denial the axiom (A5) with these axioms (A5), (L5) and (R5).

(2) There are non-geometries in map geometries with or without boundary.

(3) Unless axioms I-3, II-3, III-2, V-1 and V-2 in the Hilbert's axiom system for an Euclid geometry, an anti-geometry can be gotten from map geometries with or without boundary by denial other axioms in this axiom system.

(4) Unless the axiom (C3), a counter-projective geometry can be gotten from map geometries with or without boundary by denial axioms (C1) and (C2).

§4. Constructing Smarandache n-manifolds

The constructions applied in map geometries can be generalized to differential n-manifolds for Smarandache n-manifolds, which also enables us to affirm that Smarandache geometries include nearly all existent differential geometries, such as Finsler geometry and Riemannian geometry, etc..

4.1. **Differentially Smarandache n-manifolds**

A differential n-manifold $(M^n, \mathcal{A})$ is an n-manifold $M^n, M^n = \bigcup\limits_{i \in I} U_i$, endowed with a C^r differential structure $\mathcal{A} = \{(U_\alpha, \varphi_\alpha) | \alpha \in I\}$ on M^n for an integer r with following conditions hold.

(1) $\{U_\alpha; \alpha \in I\}$ is an open covering of M^n;

(2) For $\forall \alpha, \beta \in I$, atlases $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are *equivalent*, i.e., $U_\alpha \bigcap U_\beta = \emptyset$ or $U_\alpha \bigcap U_\beta \neq \emptyset$ but the *overlap maps*

$$\varphi_\alpha \varphi_\beta^{-1} : \varphi_\beta(U_{\alpha \bigcap U_\beta}) \to \varphi_\beta(U_\beta) \text{ and } \varphi_\beta \varphi_\alpha^{-1} : \varphi_\beta(U_{\alpha \bigcap U_\beta}) \to \varphi_\alpha(U_\alpha)$$

are C^r;

(3) $\mathcal{A}$ is maximal, i.e., if (U, φ) is an atlas of M^n equivalent with one atlas in $\mathcal{A}$, then $(U, \varphi) \in \mathcal{A}$.

An n-manifold is smooth if it is endowed with a C^∞ differential structure.

Construction 4.1 Let M^n be an n-manifold with an atlas $\mathcal{A} = \{(U_p, \varphi_p) | p \in M^n\}$. For $\forall p \in M^n$ with a local coordinates $(x_1, x_2, \cdots, x_n)$, define a spatially directional mapping $\omega : p \to \mathbf{R}^n$ action on φ_p by

$$\omega : p \to \varphi_p^\omega(p) = \omega(\varphi_p(p)) = (\omega_1, \omega_2, \cdots, \omega_n),$$

i.e., if a line L passes through $\varphi(p)$ with direction angles $\theta_1, \theta_2, \cdots, \theta_n$ with axes $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n$ in $\mathbf{R}^n$, then its direction becomes

$$\theta_1 - \frac{\vartheta_1}{2} + \sigma_1, \theta_2 - \frac{\vartheta_2}{2} + \sigma_2, \cdots, \theta_n - \frac{\vartheta_n}{2} + \sigma_n,$$

after passing through $\varphi_p(p)$, where for any integer $1 \leq i \leq n$, $\omega_i \equiv \vartheta_i (mod 4\pi)$, $\vartheta_i \geq 0$ and

$$\sigma_i = \begin{cases} \pi, & if \ \ 0 \leq \omega_i < 2\pi, \\ 0, & if \ \ 2\pi < \omega_i < 4\pi. \end{cases}$$

A manifold M^n endowed with such a spatially directional mapping $\omega : M^n \to \mathbf{R}^n$ is called an n-dimensional pseudo-manifold, denoted by $(M^n, \mathcal{A}^\omega)$.

Definition 4.1. A spatially directional mapping $\omega : M^n \to \mathbf{R}^n$ is euclidean if for any point $p \in M^n$ with a local coordinates $(x_1, x_2, \cdots, x_n)$, $\omega(p) = (2\pi k_1, 2\pi k_2, \cdots, 2\pi k_n)$ with $k_i \equiv 1(mod2)$ for $1 \leq i \leq n$, otherwise, non-euclidean.

Definition 4.2. Let $\omega : M^n \to \mathbf{R}^n$ be a spatially directional mapping and $p \in (M^n, \mathcal{A}^\omega)$, $\omega(p)(\mod 4\pi) = (\omega_1, \omega_2, \cdots, \omega_n)$. Call a point p elliptic, euclidean or hyperbolic in direction $\mathbf{e}_i$, $1 \leq i \leq n$ if $0 \leq \omega_i < 2\pi$, $\omega_i = 2\pi$ or $2\pi < \omega_i < 4\pi$.

Then we got serval results for Smarandache n-manifolds following.

Theorem 4.1.([14]) For a point $p \in M^n$ with local chart (U_p, φ_p), $\varphi_p^\omega = \varphi_p$ if and only if $\omega(p) = (2\pi k_1, 2\pi k_2, \cdots, 2\pi k_n)$ with $k_i \equiv 1(\mod 2)$ for $1 \leq i \leq n$.

Corollary 4.1. Let $(M^n, \mathcal{A}^\omega)$ be a pseudo-manifold. Then $\varphi_p^\omega = \varphi_p$ if and only if every point in M^n is euclidean.

Theorem 4.2.([14]) Let $(M^n, \mathcal{A}^\omega)$ be an n-dimensional pseudo-manifold and $p \in M^n$. If there are euclidean and non-euclidean points simultaneously or two elliptic or hyperbolic points in a same direction in (U_p, φ_p), then $(M^n, \mathcal{A}^\omega)$ is a Smarandache n-manifold.

4.2. **Tangent and cotangent vector spaces**

The tangent vector space at a point of a smoothly Smarandache n-manifold is introduced in the following.

Definition 4.3. Let $(M^n, \mathcal{A}^\omega)$ be a smoothly differential Smarandache n-manifold and $p \in M^n$. A tangent vector v at p is a mapping $v : X_p \to \mathbf{R}$ with these following conditions hold.

(1) $\forall g, h \in X_p, \forall \lambda \in \mathbf{R},\ v(h + \lambda h) = v(g) + \lambda v(h)$;

(2) $\forall g, h \in X_p, v(gh) = v(g)h(p) + g(p)v(h)$.

Denote all tangent vectors at a point $p \in (M^n, \mathcal{A}^\omega)$ by T_pM^n and define addition "$+$" and scalar multiplication "$\cdot$" for $\forall u, v \in T_pM^n, \lambda \in \mathbf{R}$ and $f \in X_p$ by

$$(u + v)(f) = u(f) + v(f), \quad (\lambda u)(f) = \lambda \cdot u(f).$$

Then it can be shown immediately that T_pM^n is a vector space under these two operations "$+$" and "$\cdot$" with basis determined in the next theorem.

Theorem 4.3.([14]) For any point $p \in (M^n, \mathcal{A}^\omega)$ with a local chart (U_p, φ_p), $\varphi_p(p) = (x_1^{\cdot} x_2^0, \cdots, x_n^0)$, if there are just s euclidean directions along $\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \cdots, \mathbf{e}_{i_s}$ for a point , then the dimension of T_pM^n is

$$\dim T_pM^n \ = \ 2n - s$$

with a basis

$$\{\frac{\partial}{\partial x^{i_j}}|_p \mid 1 \leq j \leq s\} \bigcup \{\frac{\partial^-}{\partial x^l}|_p, \frac{\partial^+}{\partial x^l}|_p \mid 1 \leq l \leq n \text{ and } l \neq i_j, 1 \leq j \leq s\}.$$

The cotangent vector space at a point of $(M^n, \mathcal{A}^\omega)$ is defined in the next.

Definition 4.4. For $\forall p \in (M^n, \mathcal{A}^\omega)$, the dual space $T_p^* M^n$ is called a co-tangent vector space at p.

Definition 4.5. For $f \in \Im_p, d \in T_p^* M^n$ and $v \in T_p M^n$, the action of d on f, called a differential operator $d : \Im_p \to \mathbf{R}$, is defined by

$$df \ = \ v(f).$$

Then we immediately got the basis of cotangent vector space at a point.

Theorem 4.4.([14]) For any point $p \in (M^n, \mathcal{A}^\omega)$ with a local chart (U_p, φ_p), $\varphi_p(p) = (x_1^\cdot x_2^0, \cdots, x_n^0)$, if there are just s euclidean directions along $\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \cdots, \mathbf{e}_{i_s}$ for a point , then the dimension of $T_p^* M^n$ is

$$\dim T_p^* M^n \ = \ 2n - s$$

with a basis

$$\{dx^{i_j}|_p \mid 1 \leq j \leq s\} \bigcup \{d^- x_p^l, d^+ x^l|_p \mid 1 \leq l \leq n \ and \ l \neq i_j, 1 \leq j \leq s\},$$

where

$$dx^i|_p(\frac{\partial}{\partial x^j}|_p) = \delta_j^i \ and \ d^{\epsilon_i} x^i|_p(\frac{\partial^{\epsilon_i}}{\partial x^j}|_p) = \delta_j^i,$$

for $\epsilon_i \in \{+, -\}, 1 \leq i \leq n$.

4.3. **Pseudo-manifold geometries**

Here we introduce Minkowski norms on these pseudo-manifolds $(M^n, \mathcal{A}^\omega)$.

Definition 4.6. A Minkowski norm on a vector space V is a function $F : V \to \mathbf{R}$ such that

(1) F is smooth on $V \backslash \{0\}$ and $F(v) \geq 0$ for $\forall v \in V$;

(2) F is 1-homogenous, i.e., $F(\lambda v) = \lambda F(v)$ for $\forall \lambda > 0$;

(3) For all $y \in V \backslash \{0\}$, the symmetric bilinear form $g_y : V \times V \to \mathbf{R}$ with

$$g_y(u, v) = \sum_{i,j} \frac{\partial^2 F(y)}{\partial y^i \partial y^j}$$

is positive definite for $u, v \in V$.

Denote by $TM^n = \bigcup\limits_{p \in (M^n, \mathcal{A}^\omega)} T_p M^n$.

Definition 4.7. A pseudo-manifold geometry is a pseudo-manifold $(M^n, \mathcal{A}^\omega)$ endowed with a Minkowski norm F on TM^n.

Then we found the following result.

Theorem 4.5.([14]) There are pseudo-manifold geometries.

4.4. **Principal fiber bundles and connections**

Although the dimension of each tangent vector space maybe different, we can also introduce principal fiber bundles and connections on pseudo-manifolds as follows.

Definition 4.8. A principal fiber bundle (PFB) consists of a pseudo-manifold $(P, \mathcal{A}_1^\omega)$, a projection $\pi : (P, \mathcal{A}_1^\omega) \to (M, \mathcal{A}_0^{\pi(\omega)})$, a base pseudo-manifold $(M, \mathcal{A}_0^{\pi(\omega)})$ and a Lie group G, denoted by (P, M, ω^π, G) such that (1), (2) and (3) following hold.

(1) There is a right freely action of G on $(P, \mathcal{A}_1^\omega)$, i.e., for $\forall g \in G$, there is a diffeomorphism $R_g : (P, \mathcal{A}_1^\omega) \to (P, \mathcal{A}_1^\omega)$ with $R_g(p^\omega) = p^\omega g$ for $\forall p \in (P, \mathcal{A}_1^\omega)$ such that $p^\omega(g_1 g_2) = (p^\omega g_1) g_2$ for

$\forall p \in (P, \mathcal{A}_1^\omega)$, $\forall g_1, g_2 \in G$ and $p^\omega e = p^\omega$ for some $p \in (P^n, \mathcal{A}_1^\omega)$, $e \in G$ if and only if e is the identity element of G.

(2) The map $\pi : (P, \mathcal{A}_1^\omega) \rightarrow (M, \mathcal{A}_0^{\pi(\omega)})$ is onto with $\pi^{-1}(\pi(p)) = \{pg | g \in G\}$, $\pi\omega_1 = \omega_0\pi$, and regular on spatial directions of p, i.e., if the spatial directions of p are $(\omega_1, \omega_2, \cdots, \omega_n)$, then ω_i and $\pi(\omega_i)$ are both elliptic, or euclidean, or hyperbolic and $|\pi^{-1}(\pi(\omega_i))|$ is a constant number independent of p for any integer $i, 1 \leq i \leq n$.

(3) For $\forall x \in (M, \mathcal{A}_0^{\pi(\omega)})$ there is an open set U with $x \in U$ and a diffeomorphism $T_u^{\pi(\omega)}$: $(\pi)^{-1}(U^{\pi(\omega)}) \rightarrow U^{\pi(\omega)} \times G$ of the form $T_u(p) = (\pi(p^\omega), s_u(p^\omega))$, where $s_u : \pi^{-1}(U^{\pi(\omega)}) \rightarrow G$ has the property $s_u(p^\omega g) = s_u(p^\omega)g$ for $\forall g \in G, p \in \pi^{-1}(U)$.

Definition 4.9. Let (P, M, ω^π, G) be a PFB with $\dim G = r$. A subspace family $H = \{H_p | p \in (P, \mathcal{A}_1^\omega), \dim H_p = \dim T_{\pi(p)} M\}$ of TP is called a connection if conditions (1) and (2) following hold.

(1) For $\forall p \in (P, \mathcal{A}_1^\omega)$, there is a decomposition

$$T_p P = H_p \bigoplus V_p$$

and the restriction $\pi_*|_{H_p} : H_p \rightarrow T_{\pi(p)} M$ is a linear isomorphism.

(2) H is invariant under the right action of G, i.e., for $p \in (P, \mathcal{A}_1^\omega)$, $\forall g \in G$,

$$(R_g)_{*p}(H_p) = H_{pg}.$$

Then we obtained an interesting dimensional formula for V_p.

Theorem 4.6.([14]) Let (P, M, ω^π, G) be a PFB with a connection H. $\forall p \in (P, \mathcal{A}_1^\omega)$, if the number of euclidean directions of p is $\lambda_P(p)$, then

$$\dim V_p = \frac{(\dim P - \dim M)(2\dim P - \lambda_P(p))}{\dim P}.$$

4.5. **Geometrical inclusions in Smarandache geometries**

We obtained geometrical theorems and inclusions in Smarandache geometries following.

Theorem 4.7.([14]) A pseudo-manifold geometry (M^n, φ^ω) with a Minkowski norm on TM^n is a Finsler geometry if and only if all points of (M^n, φ^ω) are euclidean.

Corollary 4.2. There are inclusions among Smarandache geometries, Finsler geometry, Riemann geometry and Weyl geometry:

$$\{Smarandache\ geometries\} \supset \{pseudo-manifold\ geometries\}$$
$$\supset \{Finsler\ geometry\} \supset \{Riemann\ geometry\} \supset \{Weyl\ geometry\}.$$

Theorem 4.8.([14]) A pseudo-manifold geometry (M_c^n, φ^ω) with a Minkowski norm on TM^n is a Kähler geometry if and only if F is a Hermite inner product on M_c^n with all points of (M^n, φ^ω) being euclidean.

Corollary 4.3. There are inclusions among Smarandache geometries, pseudo-manifold geometry and Kähler geometry:

$$\{Smarandache\ geometries\} \supset \{pseudo-manifold\ geometries\} \supset \{K\ddot{a}hler\ geometry\}.$$

§5. Geometry on Combinatorial manifolds

The combinatorial speculation for geometry on manifolds enables us to consider these geometrical objects consisted by manifolds with different dimensions, i.e., combinatorial manifolds. Certainly, each combinatorial manifold is a Smarandache manifold itself. Similar to the construction of Riemannian geometry, by introducing metrics on combinatorial manifolds we can construct topological or differential structures on them and obtained an entirely new geometrical theory, which also convinces us those inclusions of geometries in Smarandache geometries established in Section 4 again.

For an integer $s \geq 1$, let $n_1, n_2, \cdots, n_s$ be an integer sequence with $0 < n_1 < n_2 < \cdots < n_s$. Choose s open unit balls $B_1^{n_1}, B_2^{n_2}, \cdots, B_s^{n_s}$, where $\bigcap\limits_{i=1}^{s} B_i^{n_i} \neq \emptyset$ in $\mathbf{R}^{n_1+2+\cdots n_s}$. Then a *unit open combinatorial ball of degree* s is a union

$$\widetilde{B}(n_1, n_2, \cdots, n_s) = \bigcup_{i=1}^{s} B_i^{n_i}.$$

Definition 5.1. For a given integer sequence $n_1, n_2, \cdots, n_m, m \geq 1$ with $0 < n_1 < n_2 < \cdots < n_s$, a combinatorial manifold $\widetilde{M}$ is a Hausdorff space such that for any point $p \in \widetilde{M}$, there is a local chart (U_p, φ_p) of p, i.e., an open neighborhood U_p of p in $\widetilde{M}$ and a homoeomorphism $\varphi_p : U_p \to \widetilde{B}(n_1(p), n_2(p), \cdots, n_{s(p)}(p))$ with $\{n_1(p), n_2(p), \cdots, n_{s(p)}(p)\} \subseteq \{n_1, n_2, \cdots, n_m\}$ and $\bigcup\limits_{p \in \widetilde{M}} \{n_1(p), n_2(p), \cdots, n_{s(p)}(p)\} = \{n_1, n_2, \cdots, n_m\}$, denoted by $\widetilde{M}(n_1, n_2, \cdots, n_m)$ or $\widetilde{M}$ on the context and

$$\widetilde{\mathcal{A}} = \{(U_p, \varphi_p) | p \in \widetilde{M}(n_1, n_2, \cdots, n_m))\},$$

an atlas on $\widetilde{M}(n_1, n_2, \cdots, n_m)$. The maximum value of $s(p)$ and the dimension $\widehat{s}(p)$ of $\bigcap\limits_{i=1}^{s(p)} B_i^{n_i}$ are called the dimension and the intersectional dimensional of $\widetilde{M}(n_1, n_2, \cdots, n_m)$ at the point p, respectively.

A combinatorial manifold $\widetilde{M}$ is called finite if it is just combined by finite manifolds.

A finite combinatorial manifold is given in Fig.5.1.

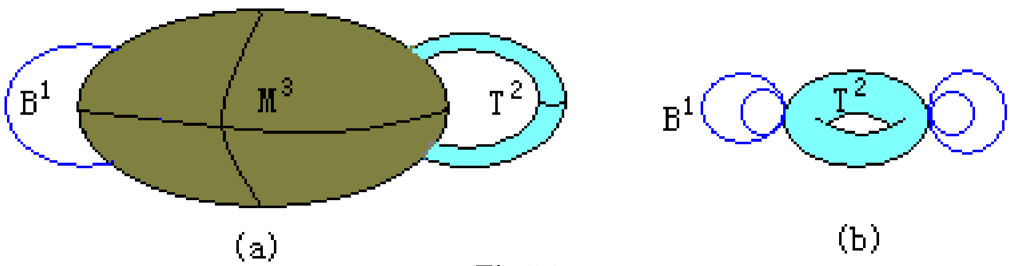

Fig.5.1

5.1. **Topological structures**

5.1.1. **Connectedness**

Definition 5.1. For two points p, q in a finitely combinatorial manifold $\widetilde{M}(n_1, n_2, \cdots, n_m)$, if there is a sequence $B_1, B_2, \cdots, B_s$ of d-dimensional open balls with two conditions following hold.

(1) $B_i \subset \widetilde{M}(n_1, n_2, \cdots, n_m)$ for any integer $i, 1 \leq i \leq s$ and $p \in B_1$, $q \in B_s$;

(2) The dimensional number $\dim(B_i \bigcap B_{i+1}) \geq d$ for $\forall i, 1 \leq i \leq s-1$.

Then points p, q are called d-dimensional connected in $\widetilde{M}(n_1, n_2, \cdots, n_m)$ and the sequence $B_1, B_2, \cdots, B_e$ a d-dimensional path connecting p and q, denoted by $P^d(p, q)$.

If each pair p, q of points in the finitely combinatorial manifold $\widetilde{M}(n_1, n_2, \cdots, n_m)$ is d-dimensional connected, then $\widetilde{M}(n_1, n_2, \cdots, n_m)$ is called d-pathwise connected and say its connectivity$\geq d$.

Let $\widetilde{M}(n_1, n_2, \cdots, n_m)$ be a finitely combinatorial manifold and $d, d \geq 1$ an integer. We construct a labelled graph $G^d[\widetilde{M}(n_1, n_2, \cdots, n_m)]$ by

$$V(G^d[\widetilde{M}(n_1, n_2, \cdots, n_m)]) = V_1 \bigcup V_2,$$

where

$V_1 = \{n_i - \text{manifolds } M^{n_i} \text{ in } \widetilde{M}(n_1, n_2, \cdots, n_m) | 1 \leq i \leq m\}$,

and

$V_2 = \{$isolated intersection points $O_{M^{n_i}, M^{n_j}}$ ofM^{n_i}, M^{n_j} in $\widetilde{M}(n_1, n_2, \cdots, n_m)$ for $1 \leq i, j \leq m\}$.

Label n_i for each n_i-manifold in V_1 and 0 for each vertex in V_2 and

$$E(G^d[\widetilde{M}(n_1, n_2, \cdots, n_m)]) = E_1 \bigcup E_2,$$

where

$E_1 = \{(M^{n_i}, M^{n_j}) | \dim(M^{n_i} \bigcap M^{n_j}) \geq d, 1 \leq i, j \leq m\}$,

and

$E_2 = \{(O_{M^{n_i}, M^{n_j}}, M^{n_i}), (O_{M^{n_i}, M^{n_j}}, M^{n_j}) | M^{n_i}$ tangent M^{n_j} at the point $O_{M^{n_i}, M^{n_j}}$ for $1 \leq i, j \leq m\}$.

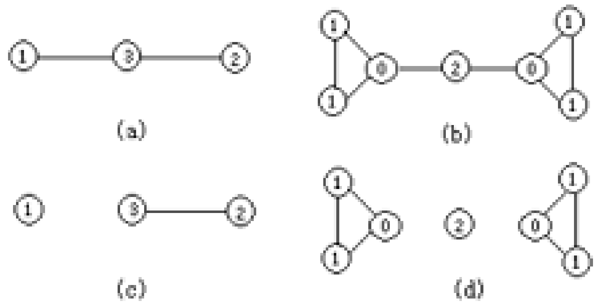

Fig.5.2

For example, these correspondent labelled graphs gotten from finitely combinatorial manifolds in Fig.5.1 are shown in Fig.5.2, in where $d = 1$ for (a) and (b), $d = 2$ for (c) and (d).

For a given integer sequence $1 \leq n_1 < n_2 < \cdots < n_m, m \geq 1$, denote by $\mathcal{H}^d(n_1, n_2, \cdots, n_m)$ all these finitely combinatorial manifolds $\widetilde{M}(n_1, n_2, \cdots, n_m)$ with connectivity$\geq d$, where $d \leq n_1$ and $\mathcal{G}(n_1, n_2, \cdots, n_m)$ all these connected graphs $G[n_1, n_2, \cdots, n_m]$ with vertex labels 0, n_1, n_2, $\cdots$, n_m and conditions following hold.

(1) The induced subgraph by vertices labelled with 1 in G is a union of complete graphs;

(2) All vertices labelled with 0 can only be adjacent to vertices labelled with 1.

Then we knew a relation between sets $\mathcal{H}^d(n_1, n_2, \cdots, n_m)$ and $\mathcal{G}(n_1, n_2, \cdots, n_m)$.

Theorem 5.1.([17]) Let $1 \le n_1 < n_2 < \cdots < n_m, m \ge 1$ be a given integer sequence. Then every finitely combinatorial manifold $\widetilde{M} \in \mathcal{H}^d(n_1, n_2, \cdots, n_m)$ defines a labelled connected graph $G[n_1, n_2, \cdots, n_m] \in \mathcal{G}(n_1, n_2, \cdots, n_m)$. Conversely, every labelled connected graph $G[n_1, n_2, \cdots, n_m] \in \mathcal{G}(n_1, n_2, \cdots, n_m)$ defines a finitely combinatorial manifold $\widetilde{M} \in \mathcal{H}^d(n_1, n_2, \cdots, n_m)$ for any integer $1 \le d \le n_1$.

5.1.2. **Homotopy**

Denoted by $f \simeq g$ two homotopic mappings f and g. Following the same pattern of homotopic spaces, we define homotopically combinatorial manifolds in the next.

Definition 5.2. Two finitely combinatorial manifolds $\widetilde{M}(k_1, k_2, \cdots, k_l)$ and $\widetilde{M}(n_1, n_2, \cdots, n_m)$ are said to be homotopic if there exist continuous maps

$f : \widetilde{M}(k_1, k_2, \cdots, k_l) \to \widetilde{M}(n_1, n_2, \cdots, n_m)$,

$g : \widetilde{M}(n_1, n_2, \cdots, n_m) \to \widetilde{M}(k_1, k_2, \cdots, k_l)$,

such that $gf \simeq$ identity

$: \widetilde{M}(k_1, k_2, \cdots, k_l) \to \widetilde{M}(k_1, k_2, \cdots, k_l)$

and

$fg \simeq$ identity $: \widetilde{M}(n_1, n_2, \cdots, n_m) \to \widetilde{M}(n_1, n_2, \cdots, n_m)$.

Then we obtained the following result.

Theorem 5.2.([17]) Let $\widetilde{M}(n_1, n_2, \cdots, n_m)$ and $\widetilde{M}(k_1, k_2, \cdots, k_l)$ be finitely combinatorial manifolds with an equivalence $\varpi : G[\widetilde{M}(n_1, n_2, \cdots, n_m)] \to G[\widetilde{M}(k_1, k_2, \cdots, k_l)]$. If for $\forall M_1, M_2 \in V(G[\widetilde{M}(n_1, n_2, \cdots, n_m)])$, M_i is homotopic to $\varpi(M_i)$ with homotopic mappings

$f_{M_i} : M_i \to \varpi(M_i)$, $g_{M_i} : \varpi(M_i) \to M_i$

such that

$f_{M_i}|_{M_i \bigcap M_j} = f_{M_j}|_{M_i \bigcap M_j}, \quad g_{M_i}|_{M_i \bigcap M_j} = g_{M_j}|_{M_i \bigcap M_j}$

providing $(M_i, M_j) \in E(G[\widetilde{M}(n_1, n_2, \cdots, n_m)])$ for $1 \le i, j \le m$, then $\widetilde{M}(n_1, n_2, \cdots, n_m)$ is homotopic to $\widetilde{M}(k_1, k_2, \cdots, k_l)$.

5.1.3. **Fundamental d-groups**

Definition 5.3. Let $\widetilde{M}(n_1, n_2, \cdots, n_m)$ be a finitely combinatorial manifold. For an integer $d, 1 \le d \le n_1$ and $\forall x \in \widetilde{M}(n_1, n_2, \cdots, n_m)$, a fundamental d-group at the point x, denoted by $\pi^d(\widetilde{M}(n_1, n_2, \cdots, n_m), x)$ is defined to be a group generated by all homotopic classes of closed d-pathes based at x.

If $d = 1$ and $\widetilde{M}(n_1, n_2, \cdots, n_m)$ is just a manifold M, we get that

$$\pi^d(\widetilde{M}(n_1, n_2, \cdots, n_m), x) = \pi(M, x).$$

Whence, fundamental d-groups are a generalization of fundamental groups in topology. We obtained the following characteristics for fundamental d-groups of finitely combinatorial manifolds.

Theorem 5.3.([17]) Let $\widetilde{M}(n_1, n_2, \cdots, n_m)$ be a d-connected finitely combinatorial manifold with $1 \le d \le n_1$. Then

(1) For $\forall x \in \widetilde{M}(n_1, n_2, \cdots, n_m)$,

$$\pi^d(\widetilde{M}(n_1, n_2, \cdots, n_m), x) \cong (\bigoplus_{M \in V(G^d)} \pi^d(M)) \bigoplus \pi(G^d),$$

where $G^d = G^d[\widetilde{M}(n_1, n_2, \cdots, n_m)]$, $\pi^d(M), \pi(G^d)$ denote the fundamental d-groups of a manifold M and the graph G^d, respectively and

(2) For $\forall x, y \in \widetilde{M}(n_1, n_2, \cdots, n_m)$,

$$\pi^d(\widetilde{M}(n_1, n_2, \cdots, n_m), x) \cong \pi^d(\widetilde{M}(n_1, n_2, \cdots, n_m), y).$$

A d-connected finitely combinatorial manifold $\widetilde{M}(n_1, n_2, \cdots, n_m)$ is said to be simply d-connected if $\pi^d(\widetilde{M}(n_1, n_2, \cdots, n_m), x)$ is trivial. As a consequence, we get the following result by Theorem 2.7.

Corollary 5.1. A d-connected finitely combinatorial manifold $\widetilde{M}(n_1, n_2, \cdots, n_m)$ is simply d-connected if and only if

(1) For $\forall M \in V(G^d[\widetilde{M}(n_1, n_2, \cdots, n_m)])$, M is simply d-connected

and

(2) $G^d[\widetilde{M}(n_1, n_2, \cdots, n_m)]$ is a tree.

5.1.4. **Euler-Poincare characteristic**

The integer

$$\chi(\mathfrak{M}) = \sum_{i=0}^{\infty}(-1)^i \alpha_i,$$

with α_i the number of i-dimensional cells in a CW-complex $\mathfrak{M}$ is called the *Euler-Poincare characteristic* of the complex $\mathfrak{M}$. Now define a clique sequence $\{Cl(i)\}_{i\geq 1}$ in the graph $G[\widetilde{M}]$ by the following programming.

STEP 1. Let $Cl(G[\widetilde{M}]) = l_0$. Construct

$$\begin{aligned} Cl(l_0) &= \{K_1^{l_0}, K_2^{l_0}, \cdots, K_p^{i_0} | K_i^{l_0} \succ G[\widetilde{M}] \text{ and } K_i^{l_0} \cap K_j^{l_0} = \emptyset, \\ &\quad \text{or a vertex} \in \mathrm{V(G[\widetilde{M}])} \text{ for } i \neq j, 1 \leq i, j \leq p\}. \end{aligned}$$

STEP 2. Let $G_1 = \bigcup\limits_{K^l \in Cl(l)} K^l$ and $Cl(G[\widetilde{M}] \setminus G_1) = l_1$. Construct

$$\begin{aligned} Cl(l_1) &= \{K_1^{l_1}, K_2^{l_1}, \cdots, K_q^{i_1} | K_i^{l_1} \succ G[\widetilde{M}] \text{ and } K_i^{l_1} \cap K_j^{l_1} = \emptyset \\ &\quad \text{or a vertex} \in \mathrm{V(G[\widetilde{M}])} \text{ for } i \neq j, 1 \leq i, j \leq q\}. \end{aligned}$$

STEP 3. Assume we have constructed $Cl(l_{k-1})$ for an integer $k \geq 1$. Let

$$G_k = \bigcup_{K^{l_{k-1}} \in Cl(l)} K^{l_{k-1}}$$

and

$$Cl(G[\widetilde{M}] \setminus (G_1 \cup \cdots \cup G_k)) = l_k.$$

We construct

$$\begin{aligned} Cl(l_k) &= \{K_1^{l_k}, K_2^{l_k}, \cdots, K_r^{l_k} | K_i^{l_k} \succ G[\widetilde{M}] \text{ and } K_i^{l_k} \cap K_j^{l_k} = \emptyset, \\ &\quad \text{or a vertex} \in \mathrm{V(G[\widetilde{M}])} \text{ for } i \neq j, 1 \leq i, j \leq r\}. \end{aligned}$$

STEP 4. Continue STEP 3 until we find an integer t such that there are no edges in $G[\widetilde{M}] \setminus \bigcup\limits_{i=1}^{t} G_i$.

By this clique sequence $\{Cl(i)\}_{i\geq 1}$, we calculated the Euler-Poincare characteristic of finitely combinatorial manifolds.

Theorem 5.4.([17]) Let $\widetilde{M}$ be a finitely combinatorial manifold. Then

$$\chi(\widetilde{M}) = \sum_{K^k\in Cl(k),k\geq 2} \sum_{M_{i_j}\in V(K^k),1\leq j\leq s\leq k} (-1)^{s+1}\chi(M_{i_1}\bigcap\cdots\bigcap M_{i_s}).$$

5.2. **Differential structures**

5.2.1. **Differentially combinatorial manifolds**

These differentially combinatorial manifolds are defined in next definition.

Definition 5.4. For a given integer sequence $1\leq n_1 < n_2 < \cdots < n_m$, a combinatorially C^h differential manifold $(\widetilde{M}(n_1,n_2,\cdots,n_m);\widetilde{\mathcal{A}})$ is a finitely combinatorial manifold $\widetilde{M}(n_1,n_2,\cdots,n_m)$, $\widetilde{M}(n_1,n_2,\cdots,n_m) = \bigcup\limits_{i\in I} U_i$, endowed with a atlas $\widetilde{\mathcal{A}} = \{(U_\alpha;\varphi_\alpha)|\alpha\in I\}$ on $\widetilde{M}(n_1,n_2,\cdots,n_m)$ for an integer $h, h\geq 1$ with conditions following hold.

(1) $\{U_\alpha;\alpha\in I\}$ is an open covering of $\widetilde{M}(n_1,n_2,\cdots,n_m)$;

(2) For $\forall\alpha,\beta\in I$, local charts $(U_\alpha;\varphi_\alpha)$ and $(U_\beta;\varphi_\beta)$ are *equivalent*, i.e., $U_\alpha\bigcap U_\beta=\emptyset$ or $U_\alpha\bigcap U_\beta\neq\emptyset$ but the *overlap maps*

$$\varphi_\alpha\varphi_\beta^{-1}:\varphi_\beta(U_\alpha\bigcap U_\beta)\to\varphi_\beta(U_\beta) \text{ and } \varphi_\beta\varphi_\alpha^{-1}:\varphi_\beta(U_\alpha\bigcap U_\beta)\to\varphi_\alpha(U_\alpha)$$

are C^h mappings;

(3) $\widetilde{\mathcal{A}}$ is maximal, i.e., if $(U;\varphi)$ is a local chart of $\widetilde{M}(n_1,n_2,\cdots,n_m)$ equivalent with one of local charts in $\widetilde{\mathcal{A}}$, then $(U;\varphi)\in\widetilde{\mathcal{A}}$.

Denote by $(\widetilde{M}(n_1,n_2,\cdots,n_m);\widetilde{\mathcal{A}})$ a combinatorially differential manifold. A finitely combinatorial manifold $\widetilde{M}(n_1,n_2,\cdots,n_m)$ is said to be smooth if it is endowed with a C^∞ differential structure.

5.2.2. **Tangent and cotangent vector spaces**

Definition 5.5. Let $(\widetilde{M}(n_1,n_2,\cdots,n_m),\widetilde{\mathcal{A}})$ be a smoothly combinatorial manifold and $p\in\widetilde{M}(n_1,n_2,\cdots,n_m)$. A tangent vector v at p is a mapping $v: X_p\to\mathbf{R}$ with conditions following hold.

(1) $\forall g,h\in X_p,\forall\lambda\in\mathbf{R},\ v(h+\lambda h)=v(g)+\lambda v(h)$;

(2) $\forall g,h\in X_p, v(gh)=v(g)h(p)+g(p)v(h)$.

Denoted all tangent vectors at $p\in\widetilde{M}(n_1,n_2,\cdots,n_m)$ by $T_p\widetilde{M}(n_1,n_2,\cdots,n_m)$ and define addition "$+$" and scalar multiplication "$\cdot$" for $\forall u,v\in T_p\widetilde{M}(n_1,n_2,\cdots,n_m)$, $\lambda\in\mathbf{R}$ and $f\in X_p$ by

$$(u+v)(f)=u(f)+v(f),\quad (\lambda u)(f)=\lambda\cdot u(f).$$

Then it can be shown immediately that $T_p\widetilde{M}(n_1,n_2,\cdots,n_m)$ is a vector space under these two operations "$+$" and "$\cdot$" with a basis determined in next theorem.

Theorem 5.5.([17]) For any point $p\in\widetilde{M}(n_1,n_2,\cdots,n_m)$ with a local chart $(U_p;[\varphi_p])$, the dimension of $T_p\widetilde{M}(n_1,n_2,\cdots,n_m)$ is

$$\dim T_p\widetilde{M}(n_1,n_2,\cdots,n_m)=\widehat{s}(p)+\sum_{i=1}^{s(p)}(n_i-\widehat{s}(p)),$$

with a basis matrix

$$[\frac{\partial}{\partial\overline{x}}]_{s(p)\times n_{s(p)}} =$$

$$\left[\begin{array}{cccccccc} \frac{1}{s(p)}\frac{\partial}{\partial x^{11}} & \cdots & \frac{1}{s(p)}\frac{\partial}{\partial x^{1\widehat{s}(p)}} & \frac{\partial}{\partial x^{1(\widehat{s}(p)+1)}} & \cdots & \frac{\partial}{\partial x^{1n_1}} & \cdots & 0 \\ \frac{1}{s(p)}\frac{\partial}{\partial x^{21}} & \cdots & \frac{1}{s(p)}\frac{\partial}{\partial x^{2\widehat{s}(p)}} & \frac{\partial}{\partial x^{2(\widehat{s}(p)+1)}} & \cdots & \frac{\partial}{\partial x^{2n_2}} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ \frac{1}{s(p)}\frac{\partial}{\partial x^{s(p)1}} & \cdots & \frac{1}{s(p)}\frac{\partial}{\partial x^{s(p)\widehat{s}(p)}} & \frac{\partial}{\partial x^{s(p)(\widehat{s}(p)+1)}} & \cdots & \cdots & \frac{\partial}{\partial x^{s(p)(n_{s(p)}-1)}} & \frac{\partial}{\partial x^{s(p)n_{s(p)}}} \end{array}\right],$$

where $x^{il} = x^{jl}$ for $1 \leq i, j \leq s(p), 1 \leq l \leq \widehat{s}(p)$, namely there is a smoothly functional matrix $[v_{ij}]_{s(p)\times n_{s(p)}}$ such that for any tangent vector $\overline{v}$ at a point p of $\widetilde{M}(n_1, n_2, \cdots, n_m)$,

$$\overline{v} = [v_{ij}]_{s(p)\times n_{s(p)}} \odot [\frac{\partial}{\partial\overline{x}}]_{s(p)\times n_{s(p)}},$$

where $[a_{ij}]_{k\times l} \odot [b_{ts}]_{k\times l} = \sum\limits_{i=1}^{k}\sum\limits_{j=1}^{l} a_{ij}b_{ij}$.

Definition 5.6. For $\forall p \in (\widetilde{M}(n_1, n_2, \cdots, n_m); \widetilde{\mathcal{A}})$, the dual space $T_p^*\widetilde{M}(n_1, n_2, \cdots, n_m)$ is called a co-tangent vector space at p.

Definition 5.7. For $f \in X_p, d \in T_p^*\widetilde{M}(n_1, n_2, \cdots, n_m)$ and $\overline{v} \in T_p\widetilde{M}(n_1, n_2, \cdots, n_m)$, the action of d on f, called a differential operator $d : X_p \to \mathbf{R}$, is defined by

$$df = \overline{v}(f).$$

Then we then obtained the result on the basis of cotangent vector space at a point following.

Theorem 5.6.([17]) For $\forall p \in (\widetilde{M}(n_1, n_2, \cdots, n_m); \widetilde{\mathcal{A}})$ with a local chart $(U_p; [\varphi_p])$, the dimension of $T_p^*\widetilde{M}(n_1, n_2, \cdots, n_m)$ is

$$\dim T_p^*\widetilde{M}(n_1, n_2, \cdots, n_m) = \widehat{s}(p) + \sum_{i=1}^{s(p)}(n_i - \widehat{s}(p)),$$

with a basis matrix

$$[d\overline{x}]_{s(p)\times n_{s(p)}} =$$

$$\left[\begin{array}{cccccccc} \frac{dx^{11}}{s(p)} & \cdots & \frac{dx^{1\widehat{s}(p)}}{s(p)} & dx^{1(\widehat{s}(p)+1)} & \cdots & dx^{1n_1} & \cdots & 0 \\ \frac{dx^{21}}{s(p)} & \cdots & \frac{dx^{2\widehat{s}(p)}}{s(p)} & dx^{2(\widehat{s}(p)+1)} & \cdots & dx^{2n_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ \frac{dx^{s(p)1}}{s(p)} & \cdots & \frac{dx^{s(p)\widehat{s}(p)}}{s(p)} & dx^{s(p)(\widehat{s}(p)+1)} & \cdots & \cdots & dx^{s(p)n_{s(p)}-1} & dx^{s(p)n_{s(p)}} \end{array}\right],$$

where $x^{il} = x^{jl}$ for $1 \leq i, j \leq s(p), 1 \leq l \leq \widehat{s}(p)$, namely for any co-tangent vector d at a point p of $\widetilde{M}(n_1, n_2, \cdots, n_m)$, there is a smoothly functional matrix $[u_{ij}]_{s(p)\times s(p)}$ such that,

$$d = [u_{ij}]_{s(p)\times n_{s(p)}} \odot [d\overline{x}]_{s(p)\times n_{s(p)}}.$$

5.2.3. **Tensor fields**

Definition 5.8. Let $\widetilde{M}(n_1, n_2, \cdots, n_m)$ be a smoothly combinatorial manifold and $p \in \widetilde{M}(n_1, n_2, \cdots, n_m)$. A tensor of type (r, s) at the point p on $\widetilde{M}(n_1, n_2, \cdots, n_m)$ is an $(r+s)$-multilinear function τ,

$$\tau : \underbrace{T_p^*\widetilde{M} \times \cdots \times T_p^*\widetilde{M}}_{r} \times \underbrace{T_p\widetilde{M} \times \cdots \times T_p\widetilde{M}}_{s} \to \mathbf{R},$$

where $T_p\widetilde{M} = T_p\widetilde{M}(n_1, n_2, \cdots, n_m)$ and $T_p^*\widetilde{M} = T_p^*\widetilde{M}(n_1, n_2, \cdots, n_m)$.

Then we found the next result.

Theorem 5.7([17]) Let $\widetilde{M}(n_1, n_2, \cdots, n_m)$ be a smoothly combinatorial manifold and $p \in \widetilde{M}(n_1, n_2, \cdots, n_m)$. Then

$$T_s^r(p, \widetilde{M}) = \underbrace{T_p\widetilde{M} \otimes \cdots \otimes T_p\widetilde{M}}_{r} \otimes \underbrace{T_p^*\widetilde{M} \otimes \cdots \otimes T_p^*\widetilde{M}}_{s},$$

where $T_p\widetilde{M} = T_p\widetilde{M}(n_1, n_2, \cdots, n_m)$ and $T_p^*\widetilde{M} = T_p^*\widetilde{M}(n_1, n_2, \cdots, n_m)$, particularly,

$$\dim T_s^r(p, \widetilde{M}) = (\widehat{s}(p) + \sum_{i=1}^{s(p)}(n_i - \widehat{s}(p)))^{r+s}.$$

5.2.4. **Exterior differentiations**

For the exterior differentiations on combinatorial manifolds, we find results following.

Theorem 5.8.([17]) Let $\widetilde{M}$ be a smoothly combinatorial manifold. Then there is a unique exterior differentiation $\widetilde{d} : \Lambda(\widetilde{M}) \to \Lambda(\widetilde{M})$ such that for any integer $k \geq 1$, $\widetilde{d}(\Lambda^k) \subset \Lambda^{k+1}(\widetilde{M})$ with conditions following hold.

(1) $\widetilde{d}$ is linear, i.e., for $\forall \varphi, \psi \in \Lambda(\widetilde{M})$, $\lambda \in \mathbf{R}$,

$$\widetilde{d}(\varphi + \lambda\psi) = \widetilde{d}\varphi \wedge \psi + \lambda\widetilde{d}\psi,$$

and for $\varphi \in \Lambda^k(\widetilde{M}), \psi \in \Lambda(\widetilde{M})$,

$$\widetilde{d}(\varphi \wedge \psi) = \widetilde{d}\varphi + (-1)^k\varphi \wedge \widetilde{d}\psi.$$

(2) For $f \in \Lambda^0(\widetilde{M})$, $\widetilde{d}f$ is the differentiation of f.

(3) $\widetilde{d}^2 = \widetilde{d} \cdot \widetilde{d} = 0$.

(4) $\widetilde{d}$ is a local operator, i.e., if $U \subset V \subset \widetilde{M}$ are open sets and $\alpha \in \Lambda^k(V)$, then $\widetilde{d}(\alpha|_U) = (\widetilde{d}\alpha)|_U$.

Theorem 5.9.([17]) Let $\omega \in \Lambda^1(\widetilde{M})$. Then for $\forall X, Y \in X(\widetilde{M})$,

$$\widetilde{d}\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]).$$

5.2.5. **Connections on combinatorial manifolds**

Definition 5.9. Let $\widetilde{M}$ be a smoothly combinatorial manifold. A connection on tensors of $\widetilde{M}$ is a mapping $\widetilde{D} : X(\widetilde{M}) \times T_s^r\widetilde{M} \to T_s^r\widetilde{M}$ with $\widetilde{D}_X\tau = \widetilde{D}(X, \tau)$ such that for $\forall X, Y \in X(\widetilde{M})$, $\tau, \pi \in T_s^r(\widetilde{M})$, $\lambda \in \mathbf{R}$ and $f \in C^\infty(\widetilde{M})$,

(1) $\widetilde{D}_{X+fY}\tau = \widetilde{D}_X\tau + f\widetilde{D}_Y\tau$; and $\widetilde{D}_X(\tau+\lambda\pi) = \widetilde{D}_X\tau + \lambda\widetilde{D}_X\pi$;

(2) $\widetilde{D}_X(\tau\otimes\pi) = \widetilde{D}_X\tau\otimes\pi + \sigma\otimes\widetilde{D}_X\pi$;

(3) For any contraction C on $T^r_s(\widetilde{M})$,

$$\widetilde{D}_X(C(\tau)) = C(\widetilde{D}_X\tau).$$

Then we got results following.

Theorem 5.10.([17]) Let $\widetilde{M}$ be a smoothly combinatorial manifold. Then there exists a connection $\widetilde{D}$ locally on $\widetilde{M}$ with a form

$$(\widetilde{D}_X\tau)|_U = X^{\sigma\varsigma}\tau^{(\mu_1\nu_1)(\mu_2\nu_2)\cdots(\mu_r\nu_r)}_{(\kappa_1\lambda_1)(\kappa_2\lambda_2)\cdots(\kappa_s\lambda_s),(\mu\nu)}\frac{\partial}{\partial x^{\mu_1\nu_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{\mu_r\nu_r}}\otimes dx^{\kappa_1\lambda_1}\otimes\cdots\otimes dx^{\kappa_s\lambda_s},$$

for $\forall Y\in X(\widetilde{M})$ and $\tau\in T^r_s(\widetilde{M})$, where

$$\begin{aligned}\tau^{(\mu_1\nu_1)(\mu_2\nu_2)\cdots(\mu_r\nu_r)}_{(\kappa_1\lambda_1)(\kappa_2\lambda_2)\cdots(\kappa_s\lambda_s),(\mu\nu)} &= \frac{\partial\tau^{(\mu_1\nu_1)(\mu_2\nu_2)\cdots(\mu_r\nu_r)}_{(\kappa_1\lambda_1)(\kappa_2\lambda_2)\cdots(\kappa_s\lambda_s)}}{\partial x^{\mu\nu}}\\ &+ \sum_{a=1}^{r}\tau^{(\mu_1\nu_1)\cdots(\mu_{a-1}\nu_{a-1})(\sigma\varsigma)(\mu_{a+1}\nu_{a+1})\cdots(\mu_r\nu_r)}_{(\kappa_1\lambda_1)(\kappa_2\lambda_2)\cdots(\kappa_s\lambda_s)}\Gamma^{\mu_a\nu_a}_{(\sigma\varsigma)(\mu\nu)}\\ &- \sum_{b=1}^{s}\tau^{(\mu_1\nu_1)(\mu_2\nu_2)\cdots(\mu_r\nu_r)}_{(\kappa_1\lambda_1)\cdots(\kappa_{b-1}\lambda_{b-1})(\mu\nu)(\sigma_{b+1}\varsigma_{b+1})\cdots(\kappa_s\lambda_s)}\Gamma^{\sigma\varsigma}_{(\sigma_b\varsigma_b)(\mu\nu)},\end{aligned}$$

and $\Gamma^{\kappa\lambda}_{(\sigma\varsigma)(\mu\nu)}$ is a function determined by

$$\widetilde{D}_{\frac{\partial}{\partial x^{\mu\nu}}}\frac{\partial}{\partial x^{\sigma\varsigma}} = \Gamma^{\kappa\lambda}_{(\sigma\varsigma)(\mu\nu)}\frac{\partial}{\partial x^{\sigma\varsigma}},$$

on $(U_p;[\varphi_p]) = (U_p;x^{\mu\nu})$ of a point $p\in\widetilde{M}$, also called the coefficient on a connection.

Theorem 5.11.([17]) Let $\widetilde{M}$ be a smoothly combinatorial manifold with a connection $\widetilde{D}$. Then for $\forall X,Y\in X(\widetilde{M})$,

$$\widetilde{T}(X,Y) = \widetilde{D}_XY - \widetilde{D}_YX - [X,Y]$$

is a tensor of type $(1,2)$ on $\widetilde{M}$.

If $T(\frac{\partial}{\partial x^{\mu\nu}},\frac{\partial}{\partial x^{\sigma\varsigma}})\equiv 0$, we call T torsion-free. This enables us getting the next useful result.

Theorem 5.12.([17]) A connection $\widetilde{D}$ on tensors of a smoothly combinatorial manifold $\widetilde{M}$ is torsion-free if and only if $\Gamma^{\kappa\lambda}_{(\mu\nu)(\sigma\varsigma)} = \Gamma^{\kappa\lambda}_{(\sigma\varsigma)(\mu\nu)}$.

5.2.6. **Combinatorially Finsler geometry**

Definition 5.10. A combinatorially Finsler geometry is a smoothly combinatorial manifold $\widetilde{M}$ endowed with a Minkowski norm $\widetilde{F}$ on $T\widetilde{M}$, denoted by $(\widetilde{M};\widetilde{F})$.

Then we got the following result.

Theorem 5.13.([17]) There are combinatorially Finsler geometries.

Theorem 5.14.([17]) A combinatorially Finsler geometry $(\widetilde{M}(n_1,n_2,\cdots,n_m);\widetilde{F})$ is a Smarandache geometry if $m\geq 2$.

Because combinatorially Finsler geometries are subsets of Smarandache geometries, we obtained the next consequence.

Corollary 5.2. There are inclusions among Smarandache geometries, Finsler geometry, Riemannian geometry and Weyl geometry:

$$\{Smarandache\ geometries\} \supset \{combinatorially\ Finsler\ geometries\}$$
$$\supset \{Finsler\ geometry\}\ and\ \{combinatorially\ Riemannian\ geometries\}$$
$$\supset \{Riemannian\ geometry\} \supset \{Weyl\ geometry\}.$$

5.2.7. **Integration on combinatorial manifolds**

For a smoothly combinatorial manifold $\widetilde{M}(n_1, \cdots, n_m)$, there must be an atlas $C = \{(\widetilde{U}_\alpha, [\varphi_\alpha])|\alpha \in \widetilde{I}\}$ on $\widetilde{M}(n_1, \cdots, n_m)$ consisting of positively oriented charts such that for $\forall\alpha \in \widetilde{I}$, $\widehat{s}(p)+\sum\limits_{i=1}^{s(p)}(n_i-\widehat{s}(p))$ is an constant $\widetilde{n}_{\widetilde{U}_\alpha}$ for $\forall p \in \widetilde{U}_\alpha$. Denote such atlas on $\widetilde{M}(n_1, \cdots, n_m)$ by $C_{\widetilde{M}}$ and an integer family $\mathcal{H}_{\widetilde{M}}(n, m) = \{n_{\widetilde{U}_\alpha}|\alpha \in \widetilde{I}\}$.

Definition 5.11. Let $\widetilde{M}$ be a smoothly combinatorial manifold with orientation O and $(\widetilde{U}; [\varphi])$ a positively oriented chart with a constant $\widetilde{n} \in \mathcal{H}_{\widetilde{M}}(n, m)$. Suppose $\omega \in \Lambda^{\widetilde{n}_{\widetilde{U}}}(\widetilde{M}), \widetilde{U} \subset \widetilde{M}$ has compact support $\widetilde{C} \subset \widetilde{U}$. Then define

$$\int_{\widetilde{C}} \omega = \int \varphi_*(\omega|_{\widetilde{U}}).$$

Now if $C_{\widetilde{M}}$ is an atlas of positively oriented charts with an integer set $H_{\widetilde{M}}$, let $\widetilde{P} = \{(\widetilde{U}_\alpha, \varphi_\alpha, g_\alpha)|\alpha \in \widetilde{I}\}$ be a partition of unity subordinate to $C_{\widetilde{M}}$. For $\forall\omega \in \Lambda^n(\widetilde{M})$, $\widetilde{n} \in \mathcal{H}_{\widetilde{M}}(n, m)$, an integral of ω on $\widetilde{P}$ is defined by

$$\int_{\widetilde{P}} \omega = \sum_{\alpha\in\widetilde{I}} \int g_\alpha \omega.$$

Definition 5.12. Let $\widetilde{M}$ be a smoothly combinatorial manifold. A subset D of $\widetilde{M}$ is with boundary if its points can be classified into two classes following.

Class 1 **(interior point Int**D**)** For $\forall p \in \mathrm{Int}D$, there is a neighborhood V_p of p enable $V_p \subset D$.

Case 2 **(boundary** ∂D**)** For $\forall p \in \partial D$, there is integers μ, ν for a local chart $(U_p; [\varphi_p])$ of p such that $x^{\mu\nu}(p) = 0$ but

$$U_p \bigcap D = \{q|q \in U_p, x^{\kappa\lambda} \geq 0 \text{ for } \forall\{\kappa, \lambda\} \neq \{\mu, \nu\}\}.$$

We then generalized the famous *Stokes' theorem* on manifolds in next theorem.

Theorem 5.15.([18]) Let $\widetilde{M}$ be a smoothly combinatorial manifold with an integer set $\mathcal{H}_{\widetilde{M}}(n, m)$ and $\widetilde{D}$ a boundary subset of $\widetilde{M}$. For $\widetilde{n} \in \mathcal{H}_{\widetilde{M}}$ if $\omega \in \Lambda^{\widetilde{n}}(\widetilde{M})$ has compact support, then

$$\int_{\widetilde{D}} d\omega = \int_{\partial\widetilde{D}} \omega,$$

with the convention $\int_{\partial\widetilde{D}} \omega = 0$ while $\partial\widetilde{D} = \emptyset$.

Corollaries following are immediately obtained by Theorem 5.15.

Corollary 5.3. Let $\widetilde{M}$ be a homogenously combinatorial manifold with an integer set $\mathcal{H}_{\widetilde{M}}(n,m)$ and $\widetilde{D}$ a boundary subset of $\widetilde{M}$. For $\widetilde{n} \in H_{\widetilde{M}}(n,m)$ if $\omega \in \Lambda^{\widetilde{n}}(\widetilde{M})$ has a compact support, then

$$\int_{\widetilde{D}} d\omega = \int_{\partial\widetilde{D}} \omega,$$

particularly, if $\widetilde{M}$ is nothing but a manifold, the Stokes theorem holds.

Corollary 5.4. Let $\widetilde{M}$ be a smoothly combinatorial manifold with an integer set $\mathcal{H}_{\widetilde{M}}(n,m)$. For $\widetilde{n} \in \mathcal{H}_{\widetilde{M}}(n,m)$, if $\omega \in \Lambda^{\widetilde{n}}(\widetilde{M})$ has a compact support, then

$$\int_{\widetilde{M}} \omega = 0.$$

§6. Applications to other fields

6.1. **Applications to algebra**

The mathematical combinatorics can be also used to generalize algebraic systems, groups, rings, vector spaces, ... etc. in algebra as follows ([10] − [12]).

Definition 6.1 For any integers $n, n \geq 1$ and $i, 1 \leq i \leq n$, let A_i be a set with an operation set $O(A_i)$ such that $(A_i, O(A_i))$ is a complete algebraic system. Then the union

$$\bigcup_{i=1}^{n} (A_i, O(A_i))$$

is called an n multi-algebra system.

Definition 6.2 Let $\widetilde{G} = \bigcup\limits_{i=1}^{n} G_i$ be a complete multi-algebra system with a binary operation set $O(\widetilde{G}) = \{\times_i, 1 \leq i \leq n\}$. If for any integer $i, 1 \leq i \leq n$, $(G_i; \times_i)$ is a group and for $\forall x, y, z \in \widetilde{G}$ and any two binary operations "$\times$" and "$\circ$", $\times \neq \circ$, there is one operation, for example the operation $\times$ satisfying the distribution law to the operation "$\circ$" provided their operation results exist , i.e.,

$$x \times (y \circ z) = (x \times y) \circ (x \times z),$$

$$(y \circ z) \times x = (y \times x) \circ (z \times x),$$

then $\widetilde{G}$ is called a multi-group.

Definition 6.3. Let $\widetilde{R} = \bigcup\limits_{i=1}^{m} R_i$ be a complete multi-algebra system with a double binary operation set $O(\widetilde{R}) = \{(+_i, \times_i), 1 \leq i \leq m\}$. If for any integers $i, j,\ i \neq j, 1 \leq i, j \leq m$, $(R_i; +_i, \times_i)$ is a ring and for $\forall x, y, z \in \widetilde{R}$,

$$(x +_i y) +_j z = x +_i (y +_j z), \quad (x \times_i y) \times_j z = x \times_i (y \times_j z),$$

and

$$x \times_i (y +_j z) = x \times_i y +_j x \times_i z, \quad (y +_j z) \times_i x = y \times_i x +_j z \times_i x,$$

provided all these operation results exist, then $\widetilde{R}$ is called a multi-ring. If for any integer $1 \leq i \leq m$, $(R; +_i, \times_i)$ is a filed, then $\widetilde{R}$ is called a multi-filed.

Definition 6.4. Let $\widetilde{V} = \bigcup\limits_{i=1}^{k} V_i$ be a complete multi-algebra system with a binary operation set $O(\widetilde{V}) = \{(\dot{+}_i, \cdot_i) \mid 1 \leq i \leq m\}$ and $\widetilde{F} = \bigcup\limits_{i=1}^{k} F_i$ a multi-filed with a double binary operation set $O(\widetilde{F}) = \{(+_i, \times_i) \mid 1 \leq i \leq k\}$. If for any integers $i, j,\ 1 \leq i, j \leq k$ and $\forall \mathbf{a}, \mathbf{b}, \mathbf{c} \in \widetilde{V}$, $k_1, k_2 \in \widetilde{F}$,

(i) $(V_i; \dot{+}_i, \cdot_i)$ is a vector space on F_i with vector additive $\dot{+}_i$ and scalar multiplication $\cdot_i$;

(ii) $(\mathbf{a}\dot{+}_i\mathbf{b})\dot{+}_j\mathbf{c} = \mathbf{a}\dot{+}_i(\mathbf{b}\dot{+}_j\mathbf{c})$;

(iii) $(k_1 +_i k_2) \cdot_j \mathbf{a} = k_1 +_i (k_2 \cdot_j \mathbf{a})$;

provided all those operation results exist, then $\widetilde{V}$ is called a multi-vector space on the multi-filed $\widetilde{F}$ with a binary operation set $O(\widetilde{V})$, denoted by $(\widetilde{V}; \widetilde{F})$.

Elementary structural results for these multi-groups, multi-rings, multi-vector spaces,... can be found in references [9] − [13].

6.2. **Applications to theoretical physics**

Some physicists had applied Smarandache multi-spaces to solve many world problem by conservation laws, such as works in [2]. In fact, although the Bag Bang model is an application of the Einstein's gravitational equation to the universe, it throughout persists in the uniqueness of universes since one can not see other things happening in the spatial beyond the visual sense of mankind. This situation have been modified by physicists in theoretical physics such as those of gauge theory and string/M-theory adhered to a microspace at each point ([3]).

According the geometrical theory established in the last section, we can also introduce curvature tensors $R_{(\alpha\beta)(\mu\nu)}$ on smoothly combinatorial manifolds in the following way.

Definition 6.1. Let $\widetilde{M}$ be a smoothly combinatorial manifold with a connection $\widetilde{D}$. For $\forall X, Y, Z \in X(\widetilde{M})$, define a combinatorially curvature operator $\widetilde{\mathcal{R}}(X, Y) : X(\widetilde{M}) \to X(\widetilde{M})$ by

$$\widetilde{\mathcal{R}}(X, Y)Z = \widetilde{D}_X\widetilde{D}_Y Z - \widetilde{D}_Y\widetilde{D}_X Z - \widetilde{D}_{[X,Y]}Z,$$

and a combinatorially curvature tensor

$\widetilde{\mathcal{R}} : X(\widetilde{M}) \times X(\widetilde{M}) \times X(\widetilde{M}) \to X(\widetilde{M})$ by $\widetilde{\mathcal{R}}(Z, X, Y) = \widetilde{\mathcal{R}}(X, Y)Z$.

Then at each point $p \in \widetilde{M}$, there is a type $(1, 3)$ tensor $\widetilde{\mathcal{R}}_p : T_p\widetilde{M} \times T_p\widetilde{M} \times T_p\widetilde{M} \to T_p\widetilde{M}$ determined by $\widetilde{\mathcal{R}}(w, u, v) = \widetilde{\mathcal{R}}(u, v)w$ for $\forall u, v, w \in T_p\widetilde{M}$. Now let $(U_p; [\varphi_p])$ be a local chart at the point p, applying Theorems 5.5 and 5.6, we can find that

$$\widetilde{\mathcal{R}}(\frac{\partial}{\partial x^{\mu\nu}}, \frac{\partial}{\partial x^{\kappa\lambda}})\frac{\partial}{\partial x^{\sigma\varsigma}} = \widetilde{\mathcal{R}}^{\eta\theta}_{(\sigma\varsigma)(\mu\nu)(\kappa\lambda)}\frac{\partial}{\partial x^{\eta\theta}},$$

where

$$\widetilde{\mathcal{R}}^{\eta\theta}_{(\sigma\varsigma)(\mu\nu)(\kappa\lambda)} = \frac{\partial\Gamma^{\eta\theta}_{(\sigma\varsigma)(\kappa\lambda)}}{\partial x^{\mu\nu}} - \frac{\partial\Gamma^{\eta\theta}_{(\sigma\varsigma)(\mu\nu)}}{\partial x^{\kappa\lambda}} + \Gamma^{\vartheta\iota}_{(\sigma\varsigma)(\kappa\lambda)}\Gamma^{\eta\theta}_{(\vartheta\iota)(\mu\nu)} - \Gamma^{\vartheta\iota}_{(\sigma\varsigma)(\mu\nu)}\Gamma^{\eta\theta}_{(\vartheta\iota)(\kappa\lambda)},$$

and $\Gamma^{\sigma\varsigma}_{(\mu\nu)(\kappa\lambda)} \in C^{\infty}(U_p)$ determined by

$$\widetilde{D}_{\frac{\partial}{\partial x^{\mu\nu}}} \frac{\partial}{\partial x^{\kappa\lambda}} = \Gamma^{\sigma\varsigma}_{(\kappa\lambda)(\mu\nu)} \frac{\partial}{\partial x^{\sigma\varsigma}}.$$

Now we define $\widetilde{\mathcal{R}}_{(\mu\nu)(\kappa\lambda)} = \widetilde{\mathcal{R}}_{(\kappa\lambda)(\nu\mu)} = \widetilde{\mathcal{R}}^{\sigma\varsigma}_{(\mu\nu)(\sigma\varsigma)(\kappa\lambda)}$ and $R = g^{(\kappa\lambda)(\mu\nu)}\widetilde{\mathcal{R}}_{(\kappa\lambda)(\nu\mu)}$.

Then similar to the establishing of Einstein's gravitational equation, we know that

$$\widetilde{\mathcal{R}}_{(\mu\nu)(\kappa\lambda)} - \frac{1}{2}Rg_{(\mu\nu)(\kappa\lambda)} = -8\pi G T_{(\mu\nu)(\kappa\lambda)},$$

if we take smoothly combinatorial manifolds to describe the spacetime. Thereby there are Smarandache multi-space solutions in the Einstein's gravitational equation, particularly, solutions of combinatorially Euclidean spaces. For example, let

$$d\Omega^2(r,\theta,\varphi) = \frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2).$$

Then we can choose a multi-time system $\{t_1, t_2, \cdots, t_n\}$ to get a cosmic model of $n, n \geq 2$ combinatorially $\mathbf{R}^4$ spaces with line elements

$$ds_1^2 = -c^2dt_1^2 + a^2(t_1)d\Omega^2(r,\theta,\varphi),$$

$$ds_2^2 = -c^2dt_2^2 + a^2(t_2)d\Omega^2(r,\theta,\varphi),$$

$$\cdots\cdots,$$

$$ds_n^2 = -c^2dt_n^2 + a^2(t_n)d\Omega^2(r,\theta,\varphi).$$

As a by-product for the universe $\mathbf{R}^3$, there are maybe $n-1$ beings in the universe with different time system two by two for an integer $n \geq 2$ not alike that of humanity. So it is very encouraging for scientists looking for those beings in theory or experiments.

References

[1] R. Abraham and Marsden, Foundation of Mechanics(2nd edition), Addison-Wesley, Reading, Mass, 1978.

[2] G. Bassini and S.Capozziello, Multi-spaces and many worlds from conservation laws, Progress in Physics, **4**(2006), 65-72.

[3] M. Carmeli, Classical Fields–General Relativity and Gauge Theory, World Scientific, 2001.

[4] W. H. Chern and X. X. Li, Introduction to Riemannian Geometry, Peking University Press, 2002.

[5] H. Iseri, Smarandache manifolds, American Research Press, Rehoboth, NM, 2002.

[6] L. Kuciuk and M. Antholy, An Introduction to Smarandache Geometries, JP Journal of Geometry and Topology, **5**(2005), No.1, 77-81.

[7] L. F. Mao, Automorphism Groups of Maps, Surfaces and Smarandache Geometries, American Research Press, 2005.

[8] L. F. Mao, On Automorphisms groups of Maps, Surfaces and Smarandache geometries, Sientia Magna, **1**(2005), No. 2, 55-73.

[9] L. F. Mao, Smarandache multi-space theory, Hexis, Phoenix, AZ，2006.

[10] L. F. Mao, On multi-metric spaces, Scientia Magna, **2**(2006), No.1, 87-94.

[11] L. F. Mao, On algebraic multi-group spaces, Scientia Magna, **2**(2006), No.1, 64-70.

[12] L. F. Mao, On algebraic multi-ring spaces, Scientia Magna, **2**(2006), No. 2, 48-54.

[13] L. F. Mao, On algebraic multi-vector spaces, Scientia Magna, **2**(2006), No. 2, 1-6.

[14] L. F. Mao, Pseudo-Manifold Geometries with Applications, e-print, arXiv: math. GM/0610307.

[15] L. F. Mao, A new view of combinatorial maps by Smarandache's notion, Selected Papers on Mathematical Combinatorics(I), 49-74, World Academic Union, 2006.

[16] L. F. Mao, Combinatorial speculation and the combinatorial conjecture for mathematics, Selected Papers on Mathematical Combinatorics(I), 1-22, World Academic Union, 2006.

[17] L. F. Mao, Geometrical theory on combinatorial manifolds, *JP J. Geometry and Topology*, **7**(2007), No.1, 65-114.

[18] L. F. Mao, A generalization of Stokes theorem on combinatorial manifolds, e-print, arXiv: math. GM/0703400.

[19] W. S. Massey, Algebraic topology, an introduction, Springer-Verlag, New York, etc., 1977.

[20] V. V. Nikulin and I.R.Shafarevlch, Geometries and Groups, Springer-Verlag Berlin Heidelberg, 1987.

[21] Joseph J. Rotman, An introduction to algebraic topology, Springer-Verlag New York Inc., 1988.

[22] F. Smarandache, Mixed noneuclidean geometries, eprint arXiv: math.GM/0010119, 10/2000.

[23] F. Smarandache, A unifying field in logic: neutrosophic field, Multi-value Logic, **8**(2002), No.3, 385-438.

[24] J. Stillwell, Classical topology and combinatorial group theory, Springer-Verlag New York Inc., 1980.