On Iterations That Alternate the Pseudo-Smarandache and Classic Functions of Number Theory

Charles Ashbacher
Charles Ashbacher Technologies
Box 294
Hiawatha, IA 52233 USA

The Pseudo-Smarandache function was recently defined in a book by Kashihara[1].

Definition: For n > 0 and an integer, the value of the Pseudo-Smarandache function Z(n) is the smallest number m such that n evenly divides

$$\sum_{k=1}^{m} k$$

Note: It is well-known that this is equivalent to n evenly dividing $\frac{m(m+1)}{2}$.

The classic functions of number theory are also well-known and have the following definitions.

Let n be any integer greater than 1 with prime factorization

$$n = p_1^{\alpha 1} p_2^{\alpha 2} \dots p_k^{\alpha k}$$

Definition: For n > 0 and an integer, the number of divisors function is denoted by d(n). It is well-known that $d(n) = (\alpha 1 + 1)(\alpha 2 + 1) \dots (\alpha k + 1)$.

Definition: For n > 0 and an integer, the sum of the positive divisors of n is denoted by $\sigma(n)$. It is well-known that

$$\sigma(n) = (1 + p_1 + p_1^2 + \dots + p_1^{\alpha 1}) \dots (1 + p_k + p_k^2 + \dots + p_k^{\alpha k})$$

Definition: For n > 0 and an integer, the Euler phi function $\phi(n)$, is the number of integers k, $1 \leq k < n$ that are relatively prime to n. It is well-known that

$$\phi(n) = p_1^{\alpha 1 - 1}(p_1 - 1) \dots p_k^{\alpha k - 1}(p_k - 1)$$

Choosing a number n and repeatedly iterating a function is something that has been done many times before. What we will do here now is alternate the iterations between two different functions.

Example:
Construct the sequence of function iterations by starting with the number of divisors function and then alternate it with the Pseudo-Smarandache function.

For example, if n = 3, then the iterations would be

$$d(3) = 2 \quad Z(2) = 3. \; d(3) = 2 \quad Z(2) = 3 \quad \ldots$$

Or notationally

$$Z(\ldots d(Z(d(Z(d(n))))) \ldots)$$

For reference purposes, we will refer to this is as the Z-d sequence.

Note that for n = 3, the sequence of numbers is a two-cycle. This is no accident and it is easy to prove that this is a general result. We do this in a roundabout way.

Theorem 1: Let p be a prime, then the Z-d sequence will always be the two cycle

2 3 2 3 2 3 . . .

Proof: Since d(p) = 2 for p any prime and Z(2) = 3, which is also a prime, the result is immediate.

This behavior is even more general.

Theorem 2: If $n = p_1p_2$, then the Z-d sequence will always be the two cycle

4 7 2 3 2 3 . . .

Proof: Since d(n) = 4 and Z(4) = 7 and d(7) = 2, which starts the repeated 2 3 2 3 . . . cycle.

Theorem 3: If $n = p^2$, the Z-d sequence will always be the two cycle

3 2 2 3 2 3 2 3 2 . . .

Proof: Since d(n) = 3, Z(3) = 2, d(2) = 2, Z(2) = 3 and d(3) = 2. the result follows.

This behavior is a general one that is easy to prove.

Theorem: If n is any integer greater than 1, then the Z-d sequence will always reduce to the 2 cycle

2 3 2 3 2 3 2 3 . . .

Proof: Explicitly testing all numbers n ≤ 50, the result holds. To complete the proof, we rely on two simple lemmas.

Lemma 1: If n > 50, then $\frac{n}{d(n)} > 3$.

Proof: By a double induction on the number of prime factors and their exponents.

Basis: If p > 50 is prime, then d(n) = 2.

Inductive 1: Assume that n = $p^k > 50$ and that $\frac{n}{d(n)} > 3$. Then the ratio of
$\frac{p^{k+1}}{d(p^{k+1})} = \frac{p*p^k}{k+1+1} > \frac{p^k}{k+1} > 3$.

Inductive 2: Assume that for n = $p_1^{\alpha 1} \ldots p_j^{\alpha j}$, $\frac{n}{d(n)} > 3$. Add an additional prime factor to n to some power. Since the additional prime factor is not necessarily larger than the others, we will call it q and append it to the end noting that the primes are not necessarily in ascending order.

$$\frac{n*q^m}{d(n*q^m)} = \frac{n*q^m}{d(n)*d(q^m)}$$

since d is multiplicative. Furthermore, it is well-known that d(n) $\leq$ n for all n > 1. Therefore, we have

$$\frac{n*q^m}{d(n*q^m)} = \frac{n*q^m}{d(n)*d(q^m)} \geq \frac{n}{d(n)} > 3.$$

and the proof is complete.

Lemma 2: If n > 1 is an integer, then the largest value that the ratio $\frac{Z(n)}{n}$ can have is 2.

Proof: It is well-known that if n = 2^k, then Z(n) = 2^{k+1} -1. For all other values of n, Z(n) is at most n.

Therefore, we have an alternating sequence of numbers where one at most doubles the previous value and the other always reduces it by at least a factor of three. Since the one that always reduces by a factor of three is done first this guarantees that the iterations will eventually reduce the value to a number less than 50.

Since the iteration of the Z-d sequence always goes to the same 2-cycle, the result will be the same if the order of the iterations is reversed to the d-Z sequence.

Another iterated sequence that can be constructed involves the Euler phi function and the Pseudo-Smarandache function.

Definition: For n $>$ 1, the Z-phi sequence is the alternating iteration of the Euler phi function followed by the Pseudo-Smarandache function.

$$Z(\ldots(\phi(Z(\phi(n))))\ldots)$$

The sequence is the rather boring

1 1 1 1 1 . . .

for n=2.

This result is not universal, as for all numbers 3 $\leq$ n $\leq$ 14, the iterations move to the same

2 3 2 3 2 3

2-cycle. However, this is not a universal result, as when n = 15, the iteration creates the 2-cycle

8 15 8 15 8 15 8 15 . . .

which is also the cycle for 16 and 17.

Examining the behavior of the Z-phi iteration for all numbers n $\leq$ 254, all create either the

2 3 2 3 2 3 . . .

or

8 15 8 15 8 15 8 . . .

2-cycles. However, when n =255, the iteration creates the new 2-cycle

128 255 128 255 128 . . .

which is also the cycle for 256.

Creating and running a computer program to check for 2-cycles that are different from the previous three, no new 2-cycle is encountered until for n=65535

32768 65535 32768 65535 32768 . . .

which is also the cycle for n=65536.

The pattern so far is clear and is summarized in the following chart

Pattern	First Instance
2 -3	$3 = 2^2 - 1$
8 - 15	$15 = 2^4 - 1$
128 - 255	$255 = 2^8 - 1$
32768 - 65535	$65535 = 2^{16} - 1$

which raises several questions.

1) Does the Z-phi sequence always reduce to a 2-cycle of the form $2^{2^k-1} - 2^{2^k} - 1$ for $k \geq 1$?

2) Does any additional patterns always appear first for $n = 2^{2^k} - 1$?

A computer search was conducted to test these questions.

Definition: For $n > 1$, the Z-sigma sequence is the alternating iteration of the sigma, sum of divisors function followed by the Pseudo-Smarandache function.

$$Z(\ldots(\sigma(Z(\sigma(n))))\ldots)$$

For n = 2, the Z-sigma sequence creates the 2-cycle

3 2 3 2 . . .

and for $3 \leq n \leq 15$, the Z-sigma sequence creates the 2-cycle

24 15 24 15 24 15 . . .

However, for n = 16, we get our first cycle that is not a 2-cycle, but is in fact a 12-cycle.

63 104 64 127 126 312 143 168 48 124 31 32 63 104 64 127 126 312 143 168 48 ...

The numbers $17 \leq n \leq 19$ all generate the 2-cycle 24 15 24 15 . . . , but n = 20 generates the 2-cycle

42 20 42 20 42 20 . . .

and n = 21 generates the 12-cycle

63 104 64 127 126 312 143 168 48 124 31 32 63 104 64 127 126 312 143 168 48 ...

The numbers 20 $\leq$ n $\leq$ 24 all generate the 2-cycle 24 15, but n = 25 generates the 12-cycle

63 104 64 127 126 312 143 168 48 124 31 32 63 104 64 127 126 312 143 168 48 ...

and n = 26 the 2-cycle 42 20 42 20 . . .

It is necessary to go up to n = 381 before we get the new cycle 1023 1536 1023 1536 . . . and a search up to n = 552,000 revealed no additional generated cycles. This leads to some obvious additional questions.

1) Is there another cycle generated by the Z-sigma sequence?

2) Is there an infinite number of numbers n that generate the 2-cycle 42 20?

3) Are there any other numbers n that generate the two cycle 2 3?

4) Is there a pattern to the first appearance of a new cycle?

In conclusion, the iterated sequences created by alternating a classic function of number theory with the Pseudo-Smarandache functions yields some interesting results that are only touched upon here. The author strongly encourages others to further explore these problems and is interested in hearing of any additional work in this area.

Reference

1. K. Kashihara, **Comments and Topics on Smarandache Notions and Problems**, Erhus University Press, Vail, AZ., 1996.

* This paper was presented at the April, 1999 meeting of the Iowa section of the Mathematical Association of America.