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Joint-Tree Model and the Maximum Genus of Graphs

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Abstract: The vertex v of a graph G is called a 1-*critical-vertex* for the maximum genus of the graph, or for simplicity called 1-*critical-vertex*, if $G-v$ is a connected graph and $\gamma_M(G-v)=\gamma_M(G)-1$. In this paper, through the *joint-tree* model, we obtained some types of 1-*critical-vertex*, and get the upper embeddability of the *Spiral* S_m^n.

Key Words: Joint-tree, maximum genus, graph embedding, Smarandache $\mathscr{P}$-drawing.

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§1. Introduction

In 1971, Nordhaus, Stewart and White [12] introduced the idea of the maximum genus of graphs. Since then many researchers have paid attention to this object and obtained many interesting results, such as the results in [2-8,13,15,17] etc. In this paper, by means of the joint-tree model, which is originated from the early works of Liu ([8]) and is formally established in [10] and [11], we offer a method which is different from others to find the maximum genus of some types of graphs.

Surfaces considered here are compact 2-dimensional manifolds without boundary. An orientable surface S can be regarded as a polygon with even number of directed edges such that both a and a^{-1} occurs once on S for each $a \in S$, where the power "-1"means that the direction of a^{-1} is opposite to that of a on the polygon. For convenience, a polygon is represented by a linear sequence of lowercase letters. An elementary result in algebraic topology states that

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each orientable surface is equivalent to one of the following standard forms of surfaces:

$$O_p = \begin{cases} a_0 a_0^{-1}, & p = 0, \\ \prod\limits_{i=1}^{p} a_i b_i a_i^{-1} b_i^{-1}, & p \geq 1 \ . \end{cases}$$

which are the sphere ($p = 0$), torus ($p = 1$), and the orientable surfaces of genus p ($p \geq 2$). The genus of a surface S is denoted by $g(S)$. Let A, B, C, D, and E be possibly empty linear sequence of letters. Suppose $A = a_1 a_2 \ldots a_r, r \geq 1$, then $A^{-1} = a_r^{-1} \ldots a_2^{-1} a_1^{-1}$ is called the *inverse* of A. If $\{a, b, a^{-1}, b^{-1}\}$ appear in a sequence with the form as $AaBbCa^{-1}Db^{-1}E$, then they are said to be an *interlaced set*; otherwise, a *parallel set*. Let $\widetilde{S}$ be the set of all surfaces. For a surface $S \in \widetilde{S}$, we obtain its genus $g(S)$ by using the following transforms to determine its equivalence to one of the standard forms.

Transform 1 $Aaa^{-1} \sim A$, where $A \in \widetilde{S}$ and $a \notin A$.

Transform 2 $AabBb^{-1}a^{-1} \sim AcBc^{-1}$.

Transform 3 $(Aa)(a^{-1}B) \sim (AB)$.

Transform 4 $AaBbCa^{-1}Db^{-1}E \sim ADCBEaba^{-1}b^{-1}$.

In the above transforms, the parentheses stand for cyclic order. For convenience, the parentheses are always omitted when unnecessary to distinguish cyclic or linear order. For more details concerning surfaces, the reader is referred to [10-11] and [14].

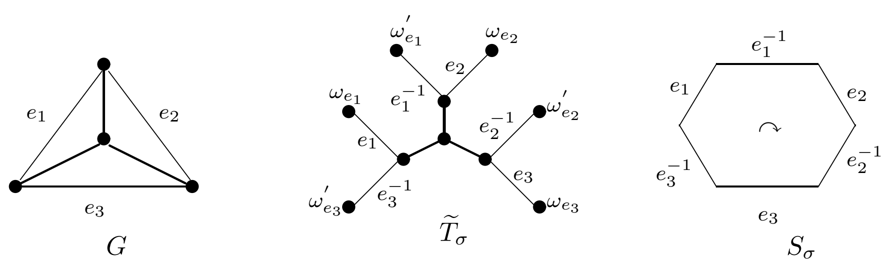

Fig. 1.

For a graphical property $\mathscr{P}$, a *Smarandache* $\mathscr{P}$*-drawing* of a graph G is such a good drawing of G on the plane with minimal intersections for its each subgraph $H \in \mathscr{P}$ and *optimal* if $\mathscr{P} = G$ with minimized crossings. Let T be a spanning tree of a graph $G = (V, E)$, then $E = E_T + E_T^*$, where E_T consists of all the tree edges, and $E_T^* = \{e_1, e_2, \ldots e_\beta\}$ consists of all the co-tree edges, where $\beta = \beta(G)$ is the cycle rank of G. Split each co-tree edge $e_i = (\mu_{e_i}, \nu_{e_i}) \in E_T^*$ into two semi-edges $(\mu_{e_i}, \omega_{e_i})$, $(\nu_{e_i}, \omega_{e_i}')$, denoted by e_i^{+1} (or simply by e_i if no confusion) and e_i^{-1} respectively. Let $\widetilde{T} = (V + V_1, E + E_1)$, where $V_1 = \{\omega_{e_i},\ \omega_{e_i}' \mid 1 \leqslant i \leqslant \beta\}$, $E_1 = \{(\mu_{e_i}, \omega_{e_i}),\ (\nu_{e_i}, \omega_{e_i}') \mid 1 \leqslant i \leqslant \beta\}$. Obviously, $\widetilde{T}$ is a tree. A *rotation at a vertex* v, which is denoted by σ_v, is a cyclic permutation of edges incident on v. A rotation system $\sigma = \sigma_G$ for a graph G is a set $\{\sigma_v | \forall v \in V(G)\}$. The tree $\widetilde{T}$ with a rotation system of G is called a *joint-tree* of G, and is denoted by $\widetilde{T}_\sigma$. Because it ia a tree, it can be embedded in the plane. By reading the lettered semi-edges of $\widetilde{T}_\sigma$ in a fixed direction (clockwise or anticlockwise), we can get an algebraic representation of the surface which is represented by a $2\beta-$polygon. Such a surface, which is denoted by S_σ, is called an associated surface of $\widetilde{T}_\sigma$. A joint-tree $\widetilde{T}_\sigma$ of G and its associated surface is illustrated by Fig.1, where the rotation at each vertex of G complies with the clockwise rotation. From [10], there is 1-1 correspondence between associated surfaces (or joint-trees) and embeddings of a graph.

To *merge* a vertex of degree two is that replace its two incident edges with a single edge joining the other two incident vertices. *Vertex-splitting* is such an operation as follows. Let v be a vertex of graph G. We replace v by two new vertices v_1 and v_2. Each edge of G joining v to another vertex u is replaced by an edge joining u and v_1, or by an edge joining u and v_2. A graph is called a *cactus* if all circuits are independent, *i.e.*, pairwise vertex-disjoint. The *maximum genus* $\gamma_M(G)$ of a connected graph G is the maximum integer k such that there exists an embedding of G into the orientable surface of genus k. Since any embedding must have at least one face, the Euler characteristic for one face leads to an upper bound on the maximum genus

$$\gamma_M(G) \leq \lfloor \frac{|E(G)| - |V(G)| + 1}{2} \rfloor.$$

A graph G is said to be *upper embeddable* if $\gamma_M(G) = \lfloor \frac{\beta(G)}{2} \rfloor$, where $\beta(G) = |E(G)| - |V(G)| + 1$ denotes the *Betti number* of G. Obviously, the maximum genus of a cactus is zero. The vertex v of a graph G is called a 1-*critical-vertex* for the maximum genus of the graph, or for simplicity called 1-*critical-vertex*, if $G-v$ is a connected graph and $\gamma_M(G-v) = \gamma_M(G)-1$. Graphs considered here are all connected, undirected, and with minimum degree at least three. In addition, the surfaces are all orientable. Notations and terminologies not defined here can be seen in [1] and [9-11].

Lemma 1.0 *If there is a joint-tree $\widetilde{T}_\sigma$ of G such that the genus of its associated surface equals $\lfloor \beta(G)/2 \rfloor$ then G is upper embeddable.*

Proof According to the definition of joint-tree, associated surface, and upper embeddable graph, Lemma 1.0 can be easily obtained. □

Lemma 1.1 *Let AB be a surface. If $x \notin A \cup B$, then $g(AxBx^{-1}) = g(AB)$ or $g(AxBx^{-1}) = g(AB) + 1$.*

Proof First discuss the topological standard form of the surface AB.

(I) According to the left to right direction, let $\{x_1, y_1, x_1^{-1}, y_1^{-1}\}$ be the first interlaced set appeared in A. Performing Transform 4 on $\{x_1, y_1, x_1^{-1}, y_1^{-1}\}$ we will get $A'Bx_1y_1x_1^{-1}y_1^{-1}$ ($\sim AB$). Then perform Transform 4 on the first interlaced set in A'. And so on. Eventually we will get $\widetilde{A}B \prod\limits_{i=1}^{r} x_iy_ix_i^{-1}y_i^{-1}$ ($\sim AB$), where there is no interlaced set in $\widetilde{A}$.

(II) For the surface $\widetilde{A}B \prod\limits_{i=1}^{r} x_iy_ix_i^{-1}y_i^{-1}$, from the left of B, successively perform Transform 4 on B similar to that on A in (I). Eventually we will get $\widetilde{A}\widetilde{B} \prod\limits_{i=1}^{r} x_iy_ix_i^{-1}y_i^{-1} \prod\limits_{j=1}^{s} a_jb_ja_j^{-1}b_j^{-1}$ ($\sim$

AB), where there is no interlaced set in $\widetilde{B}$.

(III) For the surface $\widetilde{A}\widetilde{B}\prod_{i=1}^{r} x_i y_i x_i^{-1} y_i^{-1} \prod_{j=1}^{s} a_j b_j a_j^{-1} b_j^{-1}$, from the left of $\widetilde{A}\widetilde{B}$, successively perform Transform 4 on $\widetilde{A}\widetilde{B}$ similar to that on A in (I). At last, we will get $\prod_{i=1}^{p} a_i b_i a_i^{-1} b_i^{-1}$, which is the topologically standard form of the surface AB.

As for the surface $AxBx^{-1}$, perform Transform 4 on A and B similar to that on A in (I) and B in (II) respectively. Eventually $\widetilde{A}x\widetilde{B}x^{-1}\prod_{i=1}^{r} x_i y_i x_i^{-1} y_i^{-1} \prod_{j=1}^{s} a_j b_j a_j^{-1} b_j^{-1}$ ($\sim AxBx^{-1}$) will be obtained. Then perform the same Transform 4 on $\widetilde{A}x\widetilde{B}x^{-1}$ as that on $\widetilde{A}\widetilde{B}$ in (III), and at last, one more Transform 4 than that in (III) may be needed because of x and x^{-1} in $\widetilde{A}x\widetilde{B}x^{-1}$. Eventually $\prod_{i=1}^{p} a_i b_i a_i^{-1} b_i^{-1}$ or $\prod_{i=1}^{p+1} a_i b_i a_i^{-1} b_i^{-1}$, which is the topologically standard form of the surface $AxBx^{-1}$, will be obtained.

From the above, Lemma 1.1 is obtained. □

Lemma 1.2 *Among all orientable surfaces represented by the linear sequence consisting of a_i and a_i^{-1} $(i = 1, \ldots, n)$, the surface $a_1 a_2 \ldots a_n a_1^{-1} a_2^{-1} \ldots a_n^{-1}$ is one whose genus is maximum.*

Proof According to Transform 4, Lemma 1.2 can be easily obtained. □

Lemma 1.3 *Let G be a graph with minimum degree at least three, and $\bar{G}$ be the graph obtained from G by a sequence of vertex-splitting, then $\gamma_M(\bar{G}) \le \gamma_M(G)$. Furthermore, if $\bar{G}$ is upper embeddable then G is upper embeddable as well.*

Proof Let v be a vertex of degree $n(\ge 4)$ in G, and $G^{'}$ be the graph obtained from G by splitting the vertex v into two vertices such that both their degrees are at least three. First of all, we prove that the maximum genus will not increase after one vertex-splitting operation, *i.e.*, $\gamma_M(G^{'}) \le \gamma_M(G)$.

Let e_1, e_2, $\ldots$ e_n be the n edges incident to v, and v be split into v_1 and v_2. Without loss of generality, let e_{i_1}, e_{i_2}, $\ldots$ e_{i_r} be incident to v_1, and $e_{i_{r+1}}$, $\ldots$ e_{i_n} be incident to v_2, where $2 \le i_r \le n-2$. Select such a spanning tree T of G that e_{i_1} is a tree edge, and e_{i_2}, $\ldots$ e_{i_n} are all co-tree edges. As for graph $G^{'}$, select T^* be a spanning tree such that both e_{i_1} and (v_1, v_2) are tree edges, and the other edges of T^* are the same as the edges in T. Obviously, e_{i_2}, $\ldots$ e_{i_n} are co-tree edges of T^*. Let $\mathcal{T}$=$\{\hat{T}_\sigma | \hat{T}_\sigma = \overline{(T-v)}_\sigma$, where $\overline{(T-v)}_\sigma$ is a joint-tree of $G - v\}$, $\mathcal{T}^*$=$\{\hat{T}^*_\sigma | \hat{T}^*_\sigma = \overline{(T^* - \{v_1, v_2\})}_\sigma$, where $\overline{(T^* - \{v_1, v_2\})}_\sigma$ is a joint-tree of $G^{'} - \{v_1, v_2\}\}$. It is obvious that $\mathcal{T} = \mathcal{T}^*$. Let $\mathcal{S}$ be the set of all the associated surfaces of the joint-trees of G, and $\mathcal{S}^*$ be the set of all the associated surfaces of the joint trees of $G^{'}$. Obviously, $\mathcal{S}^* \subseteq \mathcal{S}$. Furthermore, $|\mathcal{S}^*| = r! \times (n-r)! \times |\mathcal{T}^*| < |\mathcal{S}| = (n-1)! \times |\mathcal{T}|$. So $\mathcal{S}^* \subset \mathcal{S}$, and we have $\gamma_M(G^{'}) \le \gamma_M(G)$.

Reiterating this procedure, we can get that $\gamma_M(\bar{G}) \le \gamma_M(G)$. Furthermore, because $\beta(G) = \beta(\bar{G})$, it can be obtained that if $\bar{G}$ is upper embeddable then $\lfloor \frac{\beta(G)}{2} \rfloor = \lfloor \frac{\beta(\bar{G})}{2} \rfloor = \gamma_M(\bar{G}) \le \gamma_M(G) \le \lfloor \frac{\beta(G)}{2} \rfloor$. So, $\gamma_M(G) = \lfloor \frac{\beta(G)}{2} \rfloor$, and G is upper embeddable. □

§2. Results Related to 1-Critical-Vertex

The *neckband* $\mathcal{N}_{2n}$ is such a graph that $\mathcal{N}_{2n} = C_{2n} + R$, where C_{2n} is a 2n-cycle, and $R = \{a_i | a_i = (v_{2i-1}, v_{2i+2}).\ (i = 1, 2, \ldots, n,\ 2i + 2 \equiv r(mod\ 2n),\ 1 \leq r < 2n)\}$. The *möbius ladder* $\mathcal{M}_{2n}$ is such a cubic circulant graph with 2n vertices, formed from a 2n-cycle by adding edges (called "rungs") connecting opposite pairs of vertices in the cycle. For example, Fig. 2.1 and Fig. 2.5 is a graph of $\mathcal{N}_8$ and $\mathcal{M}_{2n}$ respectively. A vertex like the solid vertex in Fig. 2.2, Fig. 2.3, Fig. 2.4, Fig. 2.5, and Fig. 2.6 is called an *α-vertex*, *β-vertex*, *γ-vertex*, *δ-vertex*, and *η-vertex* respectively, where Fig. 2.6 is a neckband.

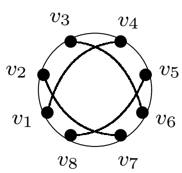

Fig. 2.1.

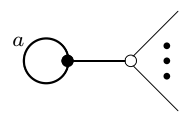

Fig. 2.2

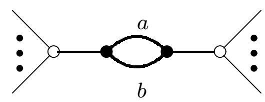

Fig. 2.3

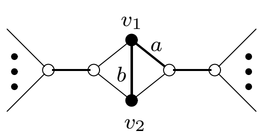

Fig. 2.4

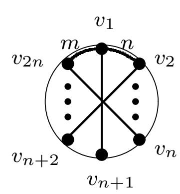

Fig. 2.5

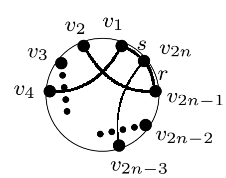

Fig. 2.6

Theorem 2.1 *If v is an α-vertex of a graph G, then $\gamma_M(G - v) = \gamma_M(G)$. If v is a β-vertex, or a γ-vertex, or a δ-vertex, or an η-vertex of a graph G, and $G - v$ is a connected graph, then $\gamma_M(G - v) = \gamma_M(G) - 1$, i.e., β-vertex, γ-vertex, δ-vertex and η-vertex are 1-critical-vertex.*

Proof If v is an *α-vertex* of the graph G, then it is easy to get that $\gamma_M(G - v) = \gamma_M(G)$. In the following, we will discuss the other cases.

Case 1 v is an *β-vertex* of G.

According to Fig. 2.3, select such a spanning tree T of G such that both a and b are co-tree edges. It is obvious that the associated surface for each joint-tree of G must be one of the following four forms:

(i) $AabBa^{-1}b^{-1} \sim ABaba^{-1}b^{-1}$;
(ii) $AabBb^{-1}a^{-1} \sim AcBc^{-1}$;
(iii) $AbaBa^{-1}b^{-1} \sim AcBc^{-1}$;
(iv) $AbaBb^{-1}a^{-1} \sim ABbab^{-1}a^{-1}$.

On the other hand, for each joint-tree $\widetilde{T}^*_\sigma$, which is a joint-tree of $G - v$, its associated surface must be the form as AB, where A and B are the same as that in the above four forms.

According to (i)-(iv), Lemma 1.1, and $g(ABaba^{-1}b^{-1})$=$g(AB)+1$, we can get that $\gamma_M(G-v)$ $=\gamma_M(G)-1$.

Case 2 v is an γ-*vertex* of G.

As illustrated by Fig.2.4, both v_1 and v_2 are γ-*vertex*. Without loss of generality, we only prove that $\gamma_M(G-v_1)=\gamma_M(G)-1$. Select such a spanning tree T of G such that both a and b are co-tree edges. The associated surface for each joint-tree of G must be one of the following 16 forms:

$$\begin{array}{llll}
Aabb^{-1}a^{-1}B, & Aabb^{-1}Ba^{-1}, & Aaba^{-1}Bb^{-1}, & AabBa^{-1}b^{-1}, \\
Abab^{-1}a^{-1}B, & Abab^{-1}Ba^{-1}, & Abaa^{-1}Bb^{-1}, & AbaBa^{-1}b^{-1}, \\
Ab^{-1}a^{-1}Bab, & Ab^{-1}Ba^{-1}ab, & Aa^{-1}Bb^{-1}ab, & ABa^{-1}b^{-1}ab, \\
Ab^{-1}a^{-1}Bba, & Ab^{-1}Ba^{-1}ba, & Aa^{-1}Bb^{-1}ba, & ABa^{-1}b^{-1}ba.
\end{array}$$

Furthermore, each of these 16 types of surfaces is topologically equivalent to one of such surfaces as AB, $ABaba^{-1}b^{-1}$, and $AcBc^{-1}$. On the other hand, for each joint-tree $\widetilde{T}^*_\sigma$, which is a joint-tree of $G-v_1$, its associated surface must be the form of AB, where A and B are the same as that in the above 16 forms. According to Lemma 1.1 and $g(ABaba^{-1}b^{-1})$=$g(AB)+1$, we can get that $\gamma_M(G-v)=\gamma_M(G)-1$.

Case 3 v is an δ-*vertex* of G.

In Fig.2.5, let $a_i=(v_i,v_{n+i}), i=1,2,\ldots,n$. Without loss of generality, we only prove that $\gamma_M(G-v_1)=\gamma_M(G)-1$. Select such a joint-tree $\widetilde{T}_\sigma$ of Fig. 2.5, which is illustrated by Fig.3, where the edges of the spanning tree are represented by solid line. It is obvious that the associated surface of $\widetilde{T}_\sigma$ is $mnm^{-1}n^{-1}a_2a_3\ldots a_na_2^{-1}a_3^{-1}\ldots a_n^{-1}$. On the other hand, $a_2a_3\ldots a_na_2^{-1}a_3^{-1}\ldots a_n^{-1}$ is the associated surface of one of the joint-trees of $G-v_1$. From Lemma 1.2 and $g(mnm^{-1}n^{-1}a_2a_3\ldots a_na_2^{-1}a_3^{-1}\ldots a_n^{-1})$=$g(a_2a_3\ldots a_na_2^{-1}a_3^{-1}\ldots a_n^{-1})+1$, we can get that $\gamma_M(G-v)=\gamma_M(G)-1$.

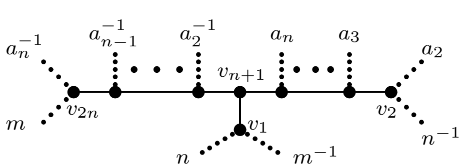

Fig. 3.

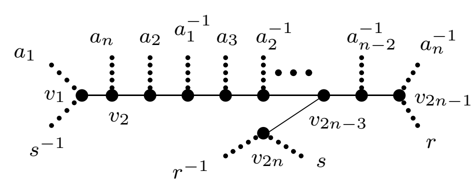

Fig. 4.

Case 4 v is an η-*vertex* of G.

As illustrated by Fig.2.6, every vertex in Fig. 2.6 is a η-*vertex*. Without loss of generality, we only prove that $\gamma_M(G-v_{2n})=\gamma_M(G)-1$.

A joint-tree $\widetilde{T}_\sigma$ of Fig.2.6 is depicted by Fig.4. It can be read from Fig.4 that the associated surface of $\widetilde{T}_\sigma$ is $S=a_1a_n(\prod_{i=1}^{n-3}a_{i+1}a_i^{-1})a_{n-2}^{-1}a_n^{-1}rsr^{-1}s^{-1}$. Performing a sequence of Transform

4 on S, we have

$$\begin{array}{rcl} S & = & a_1a_n(\prod_{i=1}^{n-3} a_{i+1}a_i^{-1})a_{n-2}^{-1}a_n^{-1}rsr^{-1}s^{-1} \\ \text{(Transform 4)} & \sim & (\prod_{i=2}^{n-3} a_{i+1}a_i^{-1})a_{n-2}^{-1}a_2rsr^{-1}s^{-1}a_1a_na_1^{-1}a_n^{-1} \\ \text{(Transform 4)} & \sim & (\prod_{i=4}^{n-3} a_{i+1}a_i^{-1})a_{n-2}^{-1}a_4rsr^{-1}s^{-1}a_1a_na_1^{-1}a_n^{-1}a_3a_2a_3^{-1}a_2^{-1} \\ & \cdots & \cdots \\ \text{(Transform 4)} & \sim & \begin{cases} rsr^{-1}s^{-1}a_1a_na_1^{-1}a_n^{-1}(\prod_{i=2}^{n-4} a_{i+1}a_ia_{i+1}^{-1}a_i^{-1}) & n \equiv 0(mod\ 2); \\ rsr^{-1}s^{-1}a_1a_na_1^{-1}a_n^{-1}(\prod_{i=2}^{n-3} a_{i+1}a_ia_{i+1}^{-1}a_i^{-1}) & n \equiv 1(mod\ 2). \end{cases} \end{array} \quad (1)$$

It is known from (1) that

$$g(S) = \gamma_M(G) \quad (2)$$

On the other hand, $S' = a_1a_n(\prod_{i=1}^{n-3} a_{i+1}a_i^{-1})a_{n-2}^{-1}a_n^{-1}$ is the associated surface of $\widetilde{T}_\sigma^*$, where $\widetilde{T}_\sigma^*$ is a joint-tree of $G - v_{2n}$. Performing a sequence of Transform 4 on S', we have

$$\begin{array}{rcl} S' & = & a_1a_n(\prod_{i=1}^{n-3} a_{i+1}a_i^{-1})a_{n-2}^{-1}a_n^{-1} \\ & \sim & \begin{cases} a_1a_na_1^{-1}a_n^{-1}(\prod_{i=2}^{n-4} a_{i+1}a_ia_{i+1}^{-1}a_i^{-1}) & n \equiv 0(mod\ 2); \\ a_1a_na_1^{-1}a_n^{-1}(\prod_{i=2}^{n-3} a_{i+1}a_ia_{i+1}^{-1}a_i^{-1}) & n \equiv 1(mod\ 2). \end{cases} \end{array} \quad (3)$$

It can be inferred from (3) that

$$g(S') = \gamma_M(G - v_{2n}). \quad (4)$$

From (1) and (3) we have

$$g(S) = g(S') + 1. \quad (5)$$

From (2), (4), and (5) we have $\gamma_M(G - v_{2n}) = \gamma_M(G) - 1$.

According to the above, we can get Theorem 2.1. □

Let G be a connected graph with minimum degree at least 3. The following algorithm can be used to get the maximum genus of G.

Algorithm I:

Step 1 Input $i = 0$, $G_0 = G$.

Step 2 If there is a 1-*critical-vertex* v in G_i, then delete v from G_i and go to Step 3. Else, go to Step 4.

Step 3 Deleting all the vertices of degree one and merging all the vertices of degree two in $G_i - v$, we get a new graph G_{i+1}. Let $i = i+1$, then go back to Step 2.

Step 4 Output $\gamma_M(G) = \gamma_M(G_i) + i$.

Remark Using Algorithm I, the computing of the maximum genus of G can be reduced to the computing of the maximum genus of G_i, which may be much easier than that of G.

§3. Upper Embeddability of Graphs

An *ear* of a graph G, which is the same as the definition offered in [16], is a path that is maximal with respect to internal vertices having degree 2 in G and is contained in a cycle in G. An *ear decomposition* of G is a decomposition $p_0, \ldots, p_k$ such that p_0 is a cycle and p_i for $i \geqslant 1$ is an ear of $p_0 \cup \cdots \cup p_i$. A *spiral* $\mathcal{S}_m^n$ is the graph which has an ear decomposition $p_0, \ldots, p_n$ such that p_0 is the m-cycle $(v_1 v_2 \ldots v_m)$, p_i for $1 \leqslant i \leqslant m-1$ is the 3-path $v_{m+2i-2}v_{m+2i-1}v_{m+2i}v_i$ which joining v_{m+2i-2} and v_i, and p_i for $i > m-1$ is the 3-path $v_{m+2i-2}v_{m+2i-1}v_{m+2i}v_{2i-m+1}$ which joining v_{m+2i-2} and v_{2i-m+1}. If some edges in $\mathcal{S}_m^n$ are replaced by the graph depicted by Fig. 6, then the graph is called an *extended-spiral*, and is denoted by $\tilde{\mathcal{S}}_m^n$. Obviously, both the vertex v_1 and v_2 in Fig. 6 are γ-vertex. For convenience, a graph of $\mathcal{S}_5^6$ is illustrated by Fig.5, and Fig.7 is the graph which is obtained from $\mathcal{S}_5^6$ by replacing the edge (v_{13}, v_{14}) with the graph depicted by Fig.6.

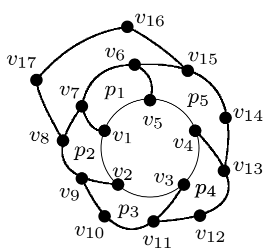

Fig. 5.

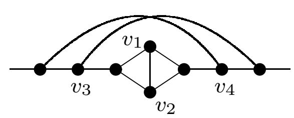

Fig. 6.

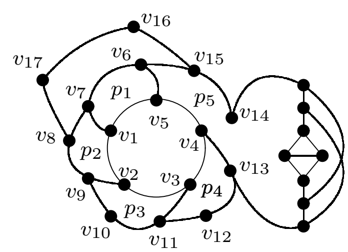

Fig. 7.

Theorem 3.1 *The graph $\mathcal{S}_5^n$ is upper embeddable. Furthermore, $\gamma_M(\mathcal{S}_5^n - v_{2n+3}) = \gamma_M(\mathcal{S}_5^n) - 1$, i.e., v_{2n+3} is a 1-critical-vertex of $\mathcal{S}_5^n$.*

Proof According to the definition of $\mathcal{S}_5^n$, when $n \le 4$, it is not a hard work to get the upper embeddability of $\mathcal{S}_5^n$. So the following 5 cases will be considered.

Case 1 $n = 5j$, where j is an integer no less than 1.

Without loss of generality, a spanning tree T of $\mathcal{S}_5^n$ can be chosen as $T = T_1 \cup T_2$, where T_1 is the path $v_2v_1v_5v_4v_3\{\prod_{i=1}^{j-1} v_{10i+1}v_{10i}v_{10i-1}v_{10i-2}v_{10i-3}v_{10i-4}v_{10i+5}v_{10i+4}v_{10i+3}v_{10i+2}\}v_{2n+1}$- v_{2n} $v_{2n-1}v_{2n-2}v_{2n-3}v_{2n-4}v_{2n+5}v_{2n+4}v_{2n+3}$, $T_2 = (v_{2n+1}, v_{2n+2})$. Obviously, the $n+1$ co-tree edges of $\mathcal{S}_5^n$ with respect to T are $e_1 = (v_2, v_3)$, $e_2 = (v_2, v_9)$, $e_3 = (v_1, v_7)$, $\prod_{i=1}^{j-1}\{e_{5i-1} = (v_{10i-5}, v_{10i-4}), e_{5i} = (v_{10i-6}, v_{10i+3}), e_{5i+1} = (v_{10i+1}, v_{10i+2}), e_{5i+2} = (v_{10i}, v_{10i+9}), e_{5i+3} =$

$(v_{10i-2}, v_{10i+7})\}$, $e_{n-1} = (v_{2n-5}, v_{2n-4})$, $e_n = (v_{2n-6}, v_{2n+3})$, $e_{n+1} = (v_{2n+2}, v_{2n+3})$. Select such a joint-tree $\widetilde{T}_\sigma$ of $\mathcal{S}_5^n$ which is depicted by Fig.8. After a sequence of Transform 4, the associated surface S of $\widetilde{T}_\sigma$ has the form as

$$\begin{aligned} S &= e_1e_2e_1^{-1}e_3e_4e_5\{\prod_{i=1}^{j-2} e_{5i+1}e_{5i+2}e_{5i-3}^{-1}e_{5i+3}e_{5i-2}^{-1}e_{5i-1}^{-1}e_{5i+4}e_{5i+5}e_{5i}^{-1}e_{5i+1}^{-1}\} \\ &\quad e_{n-4}e_{n-3}e_{n-8}^{-1}e_{n-2}e_{n-7}^{-1}e_{n-6}^{-1}e_{n-1}e_{n-5}^{-1}e_{n-4}^{-1}e_{n-3}^{-1}e_{n-2}^{-1}e_{n-1}^{-1}e_{n+1}e_ne_{n+1}^{-1}e_n^{-1} \\ &\sim \prod_{i=1}^{\lfloor\frac{n+1}{2}\rfloor} e_{i1}e_{i2}e_{i1}^{-1}e_{i2}^{-1}, \end{aligned}$$

where e_{ij}, $e_{ij}^{-1} \in \{e_1, \ldots, e_{n+1}, e_1^{-1}, \ldots, e_{n+1}^{-1}\}$; $i = 1, \ldots, \lfloor\frac{n+1}{2}\rfloor$; $j = 1, 2$. Obviously, $g(S) = \lfloor\frac{n+1}{2}\rfloor$. So, when $n = 5j$, $\mathcal{S}_5^n$ is upper embeddable.

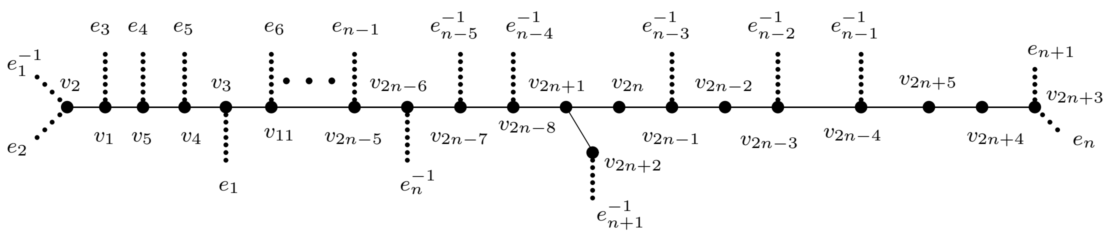

Fig. 8.

Case 2 $n = 5j + 1$, where j is an integer no less than 1.

Without loss of generality, select $T = T_1 \cup T_2$ to be a spanning tree of $\mathcal{S}_5^n$, where T_1 is the path $v_3v_2v_1\{\prod_{i=1}^{j} v_{10i-3}v_{10i-4}v_{10i-5}v_{10i-6}v_{10i+3}v_{10i+2}v_{10i+1}v_{10i}v_{10i-1}v_{10i-2}\}v_{2n+5}v_{2n+4}v_{2n+3}$, $T_2 = (v_{2n+1}, v_{2n+2})$. It is obviously that the $n+1$ co-tree edges of $\mathcal{S}_5^n$ with respect to T are $e_1 = (v_1, v_5)$, $e_2 = (v_3, v_4)$, $e_3 = (v_3, v_{11})$, $e_4 = (v_2, v_9)$, $\prod_{i=1}^{j-1}\{e_{5i} = (v_{10i-3}, v_{10i-2}), e_{5i+1} = (v_{10i-4}, v_{10i+5}), e_{5i+2} = (v_{10i+3}, v_{10i+4}), e_{5i+3} = (v_{10i+2}, v_{10i+11}), e_{5i+4} = (v_{10i}, v_{10i+9})\}$, $e_{n-1} = (v_{2n-5}, v_{2n-4})$, $e_n = (v_{2n-6}, v_{2n+3})$, $e_{n+1} = (v_{2n+2}, v_{2n+3})$. Similar to Case 1, select a joint tree $\widetilde{T}_\sigma$ of $\mathcal{S}_5^n$. After a sequence of Transform 4, the associated surface S of $\widetilde{T}_\sigma$ has the form as

$$\begin{aligned} S &= e_1e_2e_3e_4e_5e_6e_1^{-1}e_2^{-1}e_7e_8e_3^{-1}e_9\{\prod_{i=1}^{j-2} e_{5i-1}^{-1}e_{5i}^{-1}e_{5i+5}e_{5i+6}e_{5i+1}^{-1}e_{5i+2}^{-1}e_{5i+7} \\ &\quad e_{5i+8}e_{5i+3}^{-1}e_{5i+9}\}e_{n-7}^{-1}e_{n-6}^{-1}e_{n-1}e_{n-5}^{-1}e_{n-4}^{-1}e_{n-3}^{-1}e_{n-2}^{-1}e_{n-1}^{-1}e_{n+1}e_ne_{n+1}^{-1}e_n^{-1} \\ &\sim \prod_{i=1}^{\lfloor\frac{n+1}{2}\rfloor} e_{i1}e_{i2}e_{i1}^{-1}e_{i2}^{-1}, \end{aligned}$$

where e_{ij}, $e_{ij}^{-1} \in \{e_1, \ldots, e_{n+1}, e_1^{-1}, \ldots, e_{n+1}^{-1}\}$; $i = 1, \ldots, \lfloor\frac{n+1}{2}\rfloor$; $j = 1, 2$. Obviously, $g(S) = \lfloor\frac{n+1}{2}\rfloor$. So, when $n = 5j + 1$, $\mathcal{S}_5^n$ is upper embeddable.

Case 3 $n = 5j + 2$, where j is an integer no less than 1.

Without loss of generality, select a spanning tree of $\mathcal{S}_5^n$ to be $T = T_1 \cup T_2$, where T_1 is the path $v_1v_5v_4v_3v_2\{\prod_{i=1}^{j} v_{10i-1}v_{10i-2}v_{10i-3}v_{10i-4}v_{10i+5}v_{10i+4}v_{10i+3}v_{10i+2}v_{10i+1}v_{10i}\}v_{2n+5}v_{2n+4}v_{2n+3}$, $T_2 = (v_{2n+1}, v_{2n+2})$. It is obviously that the $n+1$ co-tree edges of $\mathcal{S}_5^n$ with respect to T are $e_1 = (v_1, v_2)$, $e_2 = (v_1, v_7)$, $e_3 = (v_5, v_6)$, $e_4 = (v_4, v_{13})$, $e_5 = (v_3, v_{11})$, $\prod_{i=1}^{j-1}\{e_{5i+1} = (v_{10i-1}, v_{10i}), e_{5i+2} = (v_{10i-2}, v_{10i+7}), e_{5i+3} = (v_{10i+5}, v_{10i+6}), e_{5i+4} = (v_{10i+4}, v_{10i+13}), e_{5i+5} = (v_{10i+2}, v_{10i+11})\}$, $e_{n-1} = (v_{2n-5}, v_{2n-4})$, $e_n = (v_{2n-6}, v_{2n+3})$, $e_{n+1} = (v_{2n+2}, v_{2n+3})$. Similar to Case 1, select a joint-tree $\widetilde{T}_\sigma$ of $\mathcal{S}_5^n$. After a sequence of Transform 4, the associated surface S of $\widetilde{T}_\sigma$ has the form as

$$\begin{aligned} S &= e_1e_2e_3e_4e_5e_1^{-1}e_6e_7e_2^{-1}e_3^{-1}e_8e_9e_4^{-1}e_{10}\{\prod_{i=1}^{j-2} e_{5i}^{-1}e_{5i+1}^{-1}e_{5i+6}e_{5i+7}e_{5i+2}^{-1}e_{5i+3}^{-1}e_{5i+8} \\ &\quad e_{5i+9}e_{5i+4}^{-1}e_{5i+10}\}e_{n-7}^{-1}e_{n-6}^{-1}e_{n-1}e_{n-5}^{-1}e_{n-4}^{-1}e_{n-3}^{-1}e_{n-2}^{-1}e_{n-1}^{-1}e_{n+1}e_ne_{n+1}^{-1}e_n^{-1} \\ &\sim \prod_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} e_{i1}e_{i2}e_{i1}^{-1}e_{i2}^{-1}, \end{aligned}$$

where e_{ij}, $e_{ij}^{-1} \in \{e_1, \ldots, e_{n+1}, e_1^{-1}, \ldots, e_{n+1}^{-1}\}$; $i = 1, \ldots, \lfloor \frac{n+1}{2} \rfloor$; $j = 1, 2$. Obviously, $g(S) = \lfloor \frac{n+1}{2} \rfloor$. So, when $n = 5j + 2$, $\mathcal{S}_5^n$ is upper embeddable.

Case 4 $n = 5j + 3$, where j is an integer no less than 1.

Without loss of generality, a spanning tree T of $\mathcal{S}_5^n$ can be chosen as $T = T_1 \cup T_2$, where T_1 is the path $v_2v_1v_7v_6v_5v_4v_3\{\prod_{i=1}^{j} v_{10i+1}v_{10i}v_{10i-1}v_{10i-2}v_{10i+7}v_{10i+6}v_{10i+5}v_{10i+4}v_{10i+3}$-$v_{10i+2}\}v_{2n+5}v_{2n+4}v_{2n+3}$, $T_2 = (v_{2n+1}, v_{2n+2})$. It is obviously that the $n+1$ co-tree edges of $\mathcal{S}_5^n$ with respect to T are $e_1 = (v_1, v_5)$, $e_2 = (v_2, v_3)$, $e_3 = (v_2, v_9)$, $\prod_{i=1}^{j}\{e_{5i-1} = (v_{10i-3}, v_{10i-2}), e_{5i} = (v_{10i-4}, v_{10i+5}), e_{5i+1} = (v_{10i-6}, v_{10i+3}), e_{5i+2} = (v_{10i+1}, v_{10i+2}), e_{5i+3} = (v_{10i}, v_{10i+9})\}$, $e_{n+1} = (v_{2n+2}, v_{2n+3})$. Similar to Case 1, select a joint-tree $\widetilde{T}_\sigma$ of $\mathcal{S}_5^n$. After a sequence of Transform 4, the associated surface S of $\widetilde{T}_\sigma$ has the form as

$$\begin{aligned} S &= e_1e_2e_3e_4e_5e_1^{-1}e_6e_2^{-1}\{\prod_{i=1}^{j-1} e_{5i+2}e_{5i+3}e_{5i-2}^{-1}e_{5i-1}^{-1}e_{5i+4}e_{5i+5}e_{5i}^{-1}e_{5i+6} \\ &\quad e_{5i+1}^{-1}e_{5i+2}^{-1}\}e_{n-1}e_{n-5}^{-1}e_{n-4}^{-1}e_{n-3}^{-1}e_{n-2}^{-1}e_{n-1}^{-1}e_{n+1}e_ne_{n+1}^{-1}e_n^{-1} \\ &\sim \prod_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} e_{i1}e_{i2}e_{i1}^{-1}e_{i2}^{-1}, \end{aligned}$$

where e_{ij}, $e_{ij}^{-1} \in \{e_1, \ldots, e_{n+1}, e_1^{-1}, \ldots, e_{n+1}^{-1}\}$; $i = 1, \ldots, \lfloor \frac{n+1}{2} \rfloor$; $j = 1, 2$. Obviously, $g(S) = \lfloor \frac{n+1}{2} \rfloor$. So, when $n = 5j + 3$, $\mathcal{S}_5^n$ is upper embeddable.

Case 5 $n = 5j + 4$, where j is an integer no less than 1.

Without loss of generality, a spanning tree T of $\mathcal{S}_5^n$ can be chosen as $T = T_1 \cup T_2 \cup T_3$, where T_1 is the path $v_1v_2\{\prod_{i=1}^{j} v_{10i-1}v_{10i-2}v_{10i-3}v_{10i-4}v_{10i-5}v_{10i-6}v_{10i+3}v_{10i+2}v_{10i+1}v_{10i}\}v_{2n+1}$-$v_{2n}v_{2n-1}v_{2n-2}v_{2n-3}v_{2n-4}v_{2n+5}v_{2n+4}v_{2n+3}$, $T_2 = (v_2, v_3)$, $T_3 = (v_{2n+1}, v_{2n+2})$. It is obviously that the $n+1$ co-tree edges of $\mathcal{S}_5^n$ with respect to T are $e_1 = (v_1, v_5)$, $e_2 = (v_1, v_7)$, $e_3 = (v_3, v_4)$, $e_4 = (v_3, v_{11})$, $\prod_{i=1}^{j}\{e_{5i} = (v_{10i-1}, v_{10i}), e_{5i+1} = (v_{10i-2}, v_{10i+7}), e_{5i+2} = (v_{10i-4}, v_{10i+5}), e_{5i+3} = (v_{10i+3}, v_{10i+4}), e_{5i+4} = (v_{10i+2}, v_{10i+11})\}$, $e_{n+1} = (v_{2n+2}, v_{2n+3})$. Similar to Case 1, select a joint-tree $\widetilde{T}_\sigma$ of $\mathcal{S}_5^n$. After a sequence of Transform 4, the associated surface S of $\widetilde{T}_\sigma$ has the form as

$$\begin{aligned} S &= e_2e_1e_3e_4e_5e_6e_2^{-1}e_7e_1^{-1}e_3^{-1}\{\prod_{i=1}^{j-1} e_{5i+3}e_{5i+4}e_{5i-1}^{-1}e_{5i}^{-1}e_{5i+5}e_{5i+6}e_{5i+1}^{-1} \\ &\quad e_{5i+7}e_{5i+2}^{-1}e_{5i+3}^{-1}\}e_{n-1}e_{n-5}^{-1}e_{n-4}^{-1}e_{n-3}^{-1}e_{n-2}^{-1}e_{n-1}^{-1}e_{n+1}e_ne_{n+1}^{-1}e_n^{-1} \\ &\sim \prod_{i=1}^{\lfloor\frac{n+1}{2}\rfloor} e_{i1}e_{i2}e_{i1}^{-1}e_{i2}^{-1}, \end{aligned}$$

where $e_{ij},\ e_{ij}^{-1} \in \{e_1, \ldots, e_{n+1}, e_1^{-1}, \ldots, e_{n+1}^{-1}\}$; $i = 1, \ldots, \lfloor\dfrac{n+1}{2}\rfloor$; $j = 1, 2$. Obviously, $g(S) = \lfloor\dfrac{n+1}{2}\rfloor$. So, when $n = 5j+4$, $\mathcal{S}_5^n$ is upper embeddable.

From the Case 1-5, the upper embeddability of $\mathcal{S}_5^n$ can be obtained.

Similar to the Case 1-5, for each $n \geq 5$, there exists a joint-tree $\widetilde{T}_\sigma^*$ of $\mathcal{S}_5^n - v_{2n+3}$ such that its associated surface is $S' = S - \{e_{n+1}e_ne_{n+1}^{-1}e_n^{-1}\}$. It is obvious that S' is the surface into which the embedding of $\mathcal{S}_5^n - v_{2n+3}$ is the maximum genus embedding. Furthermore, $g(S') = g(S) - 1$, *i.e.*, $\gamma_M(\mathcal{S}_5^n - v_{2n+3}) = \gamma_M(\mathcal{S}_5^n) - 1$. So, v_{2n+3} is a 1-*critical-vertex* of $\mathcal{S}_5^n$. □

Similar to the proof of Theorem 3.1, we can get the following conclusions.

Theorem 3.2 *The graph $\mathcal{S}_m^n$ is upper embeddable. Furthermore, $\gamma_M(\mathcal{S}_m^n - v_{m+2n-2}) = \gamma_M(\mathcal{S}_m^n) - 1$, i.e., v_{m+2n-2} is a 1-critical-vertex of $\mathcal{S}_m^n$.*

Corollary 3.1 *Let G be a graph with minimum degree at least three. If G, through a sequence of vertex-splitting operations, can be turned into a spiral $\mathcal{S}_m^n$, then G is upper embeddable.*

Proof According to Lemma 1.3, Theorem 3.2, and the upper embeddability of graphs, Corollary 3.1 can be obtained. □

In the following, we will offer an algorithm to obtain the maximum genus of the *extended-spiral* $\tilde{\mathcal{S}}_m^n$.

Algorithm II:

Step 1 Input $i = 0$ and $j = 0$. Let G_0 be the *extended-spiral* $\tilde{\mathcal{S}}_m^n$.

Step 2 If there is a γ-vertex v in G_i, then delete v from G_i, and go to Step 3. Else, go to Step 4.

Step 3 Deleting all the vertices of degree one and merging some vertices of degree two in $G_i - v$, we get a new graph G_{i+1}. Let $i = i+1$. If G_i is a *spiral* $\mathcal{S}_m^n$, then go to Step 4. Else, go back to Step 2.

Step 4 Let G_{i+j} be the *spiral* $\mathcal{S}_m^n$. Deleting v_{m+2n-2} from $\mathcal{S}_m^n$, we will get a new graph G_{i+j+1}, (obviously, G_{i+j+1} is either a *spiral* $\mathcal{S}_m^{n-2}$ or a *cactus*).

Step 5 If G_{i+j+1} is a *cactus*, then go to Step 6. Else, Let $n = n-2$, $j = j+1$ and go back to Step 4.

Step 6 Output $\gamma_M(\tilde{\mathcal{S}}_m^n) = i + j + 1$.

Remark 1. In the graph G depicted by Fig.6, after deleting a γ-vertex v_1 (or v_2) from G, the vertex v_3 (or v_4) is still a γ-vertex of the remaining graph.

2. From Algorithm II we can get that the *extended-spiral* $\tilde{\mathcal{S}}_m^n$ is upper embeddable.

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