ON THE k-POWER FREE NUMBER SEQUENCE

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ABSTRACT. The main purpose of this paper is to study the distribution properties of k-power free numbers, and give an interesting asymptotic formula.

1. INTRODUCTION AND RESULTS

A natural number n is called a k-power free number if it can not be divided by any p^k, where p is a prime number. One can obtain all k-power free number by the following method: From the set of natural numbers (except 0 and 1)

-take off all multiples of 2^k(i.e. 2^k, 2^{k+1}, 2^{k+2} ...).

-take off all multiples of 3^k.

-take off all multiples of 5^k.

...and so on (take off all multiples of all k-power primes).

Now the k-power free number sequence is $2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14, 15, 17, \cdots$. In reference [1], Professor F. Smarandache asked us to study the properties of the k-power free number sequence. About this problem, it seems that none had studied it before. In this paper, we use the analytic method to study the distribution properties of this sequence, and obtain an interesting asymptotic formula. For convenience, we define $\omega(n)$ as following: $\omega(n) = r$, if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$. Then we have the following:

Theorem. *Let $\mathcal{A}$ denotes the set of all k-power free numbers. Then we have the asymptotic formula ,*

$$\sum_{\substack{n \leq x \\ n \in \mathcal{A}}} \omega^2(n) = \frac{x(\ln \ln x)^2}{\zeta(k)} + O\left(x \ln \ln x\right),$$

where $\zeta(k)$ is the Riemann zeta-function.

Key words and phrases. k-power free numbers; Mean Value; Asymptotic formula.

2. Several lemmas

Lemma 1. *For any real number $x \geq 2$, we have the asymptotic formula*

$$\sum_{n\leq x} \omega(n) = x\ln\ln x + Ax + O\left(\frac{x}{\ln x}\right),$$
$$\sum_{n\leq x} \omega^2(n) = x(\ln\ln x)^2 + O\left(x\ln\ln x\right).$$

where $A = \gamma + \sum\limits_{p}\left(\ln\left(1-\frac{1}{p}\right)+\frac{1}{p}\right)$.

Proof. (See reference [2]).

Lemma 2. *Let $\mu(n)$ is Möbius function, then for any real number $x \geq 2$, we have the following identity*

$$\sum_{n=1}^{\infty}\frac{\mu(n)\omega(n)}{n^s} = -\frac{1}{\zeta(s)}\sum_{p}\frac{1}{p^s-1}.$$

Proof. From the definition of $\omega(n)$ and $\mu(n)$, we have

$$\sum_{n=1}^{\infty}\frac{\mu(n)\omega(n)}{n^s} = \sum_{n=2}^{\infty}\frac{\mu(n)\sum_{p|n}1}{n^s} = \sum_{p}\sum_{\substack{n=1\\(n,p)=1}}^{\infty}\frac{\mu(np)}{n^sp^s} = -\sum_{p}\frac{1}{p^s}\sum_{\substack{n=1\\(n,p)=1}}^{\infty}\frac{\mu(n)}{n^s}$$
$$= -\sum_{p}\frac{1}{p^s}\left(\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}\right)\left(1-\frac{1}{p^s}\right)^{-1} = -\frac{1}{\zeta(s)}\sum_{p}\frac{1}{p^s-1}.$$

This proves Lemma 2.

Lemma 3. *Let $k \geq 2$ is a fixed integer, then for any real number $x \geq 2$, we have the asymptotic formula*

$$\sum_{d^km\leq x}\omega^2(m)\mu(d) = \frac{x(\ln\ln x)^2}{\zeta(k)} + O\left(x\ln\ln x\right).$$

Proof. From Lemma 1, we have

$$\sum_{d^km\leq x}\omega^2(m)\mu(d) = \sum_{d\leq x^{\frac{1}{k}}}\mu(d)\sum_{m\leq x/d^k}\omega^2(m)$$
$$= \sum_{d\leq x^{\frac{1}{k}}}\mu(d)\left(\frac{x}{d^k}(\ln\ln\frac{x}{d^k})^2 + O\left(\frac{x}{d^k}\ln\ln\frac{x}{d^k}\right)\right)$$
$$= x\sum_{d\leq x^{\frac{1}{k}}}\frac{\mu(d)}{d^k}\left(\ln\ln x + \ln\ln\left(1-\frac{k\ln d}{\ln x}\right)\right)^2 + O\left(x\ln\ln x\right)$$
$$= x\left(\ln\ln x\right)^2\sum_{d=1}^{\infty}\frac{\mu(d)}{d^k} + O\left(x\ln\ln x\sum_{d\leq x^{\frac{1}{k}}}\frac{\ln d}{d^k\ln x}\right) + O(x\ln\ln x)$$
$$= \frac{x(\ln\ln x)^2}{\zeta(k)} + O\left(x\ln\ln x\right).$$

This proves Lemma 3.

Lemma 4. *For any real number $x \geq 2$, we have the estimate*
$$\sum_{d^k m \leq x} \omega^2(d)\mu(d) = O(x).$$

Proof. From Lemma 1, we have
$$\sum_{d^k m \leq x} \omega^2(d)\mu(d) = \sum_{d \leq x^{\frac{1}{k}}} \omega^2(d)\mu(d) \sum_{m \leq x/d^k} 1 = \sum_{d \leq x^{\frac{1}{k}}} \omega^2(d)\mu(d) \left[\frac{x}{d^k}\right]$$
$$= x \sum_{d \leq x^{\frac{1}{k}}} \frac{\omega^2(d)\mu(d)}{d^k} + O\left(\sum_{d \leq x^{\frac{1}{k}}} \omega^2(d)\mu(d)\right) = O(x).$$
This proves Lemma 4.

Lemma 5. *For any real number $x \geq 2$, we have the estimate*
$$\sum_{d^k m \leq x} \omega^2((d,m))\mu(d) = O(x).$$

Proof. Assume that (u,v) is the greatest common divisor of u and v, then we have
$$\sum_{d^k m \leq x} \omega^2((d,m))\mu(d) = \sum_{d \leq x^{\frac{1}{k}}} \mu(d) \sum_{u|d} \sum_{\substack{m \leq x/d^k \\ u|m}} \omega^2(u) = \sum_{d \leq x^{\frac{1}{k}}} \mu(d) \sum_{u|d} \omega^2(u) \left[\frac{x}{ud^k}\right]$$
$$= x \sum_{d=1}^{\infty} \frac{\mu(d) \sum\limits_{u|d} \frac{\omega^2(u)}{u}}{d^k} + O\left(\sum_{d \leq x^{\frac{1}{k}}} \mu(d) \sum_{u|d} \omega^2(u)\right) = O(x).$$
This proves Lemma 5.

Lemma 6. *For any real number $x \geq 2$, we have the asymptotic formula*
$$\sum_{d^k m \leq x} \omega(d)\omega(m)\mu(d) = Cx \ln\ln x + O(x),$$
where $C = -\frac{1}{\zeta(k)} \sum\limits_{p} \frac{1}{p^k - 1}$.

Proof. From Lemma 1 and Lemma 2 we have
$$\sum_{d^k m \leq x} \omega(d)\omega(m)\mu(d) = \sum_{d \leq x^{\frac{1}{k}}} \omega(d)\mu(d) \sum_{m \leq x/d^k} \omega(m)$$
$$= \sum_{d \leq x^{\frac{1}{k}}} \omega(d)\mu(d) \left(\frac{x \ln\ln \frac{x}{d^k}}{d^k} + \frac{Ax}{d^k} + O\left(\frac{x}{d^k \ln \frac{x}{d^k}}\right)\right)$$
$$= x \sum_{d \leq x^{\frac{1}{k}}} \frac{\omega(d)\mu(d)}{d^k} \left(\ln\ln x + \ln\ln\left(1 - \frac{k \ln d}{\ln x}\right)\right) + Ax \sum_{d \leq x^{\frac{1}{k}}} \frac{\omega(d)\mu(d)}{d^k} + O\left(\frac{x}{\ln x}\right)$$
$$= (x \ln\ln x + Ax) \sum_{d=1}^{\infty} \frac{\omega(d)\mu(d)}{d^k} + O\left(x \sum_{d \leq x^{\frac{1}{k}}} \frac{\ln d}{d^k \ln x}\right) + O\left(\frac{x}{\ln x}\right)$$
$$= Cx \ln\ln x + O(x).$$
This proves Lemma 6.

3. Proof of the Theorem

In this section, we shall complete the proof of the Theorem. For convenience we define the characteristic function of k-power free numbers as follows:

$$u(n) = \begin{cases} 1, & \text{if } n \text{ is a } k\text{-power free number;} \\ 0, & \text{otherwise.} \end{cases}$$

From Lemma 3, Lemma 4, Lemma 5 and Lemma 6, we have

$$\begin{aligned}
&\sum_{\substack{n\le x\\ n\in A}} \omega^2(n) = \sum_{n\le x} \omega^2(n) \sum_{d^k|n} \mu(d) = \sum_{d^k m\le x} \omega^2(d^k m)\mu(d) \\
&= \sum_{d^k m\le x} \left(\omega(d)+\omega(m)-\omega((d,m))\right)^2 \mu(d) \\
&= \sum_{d^k m\le x} \omega^2(m)\mu(d) + \sum_{d^k m\le x} \omega^2(d)\mu(d) + \sum_{d^k m\le x} \omega^2((d,m))\mu(d) \\
&\quad + 2\left(\sum_{d^k m\le x} \omega(d)\omega(m)\mu(d)\right) + O\left(\sum_{d^k m\le x} \omega(d)\omega(m)\right) \\
&= \left(\frac{x(\ln\ln x)^2}{\zeta(k)} + O\left(x\ln\ln x\right)\right) + 2\left(Cx\ln\ln x + O(x)\right) + O(x\ln\ln x) \\
&= \frac{x(\ln\ln x)^2}{\zeta(k)} + O\left(x\ln\ln x\right).
\end{aligned}$$

This completes the proof of the Theorem .

References

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