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Labeling, Covering and Decomposing of Graphs
— Smarandache's Notion in Graph Theory

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Abstract: This paper surveys the applications of Smarandache's notion to graph theory appeared in *International J.Math.Combin.* from Vol.1,2008 to Vol.3,2009. In fact, many problems discussed in these papers are generalized in this paper. Topics covered in this paper include: (1)What is a Smarandache System? (2)Vertex-Edge Labeled Graphs with Applications: (i)Smarandachely k-constrained labeling of a graph; (ii)Smarandachely super m-mean graph; (iii)Smarandachely uniform k-graph; (iv)Smarandachely total coloring of a graph; (3)Covering and Decomposing of a Graph: (i)Smarandache path k-cover of a graph; (ii)Smarandache graphoidal tree d-cover of a graph; (4)Furthermore.

Key Words: Smarandache system, labeled graph, Smarandachely k-constrained labeling, Smarandachely k-constrained labelingSmarandachely super m-mean graph, Smarandachely uniform k-graph, Smarandachely total coloring of a graph, Smarandache path k-cover of a graph, Smarandache graphoidal tree d-cover of a graph.

AMS(2000): 05C12, 05C70, 05C78

§1. What is a Smarandache System?

A *Smarandache System* first appeared in [1] is defined in the following.

Definition 1.1([1]) *A rule in a mathematical system* $(\Sigma;\mathcal{R})$ *is said to be Smarandachely denied if it behaves in at least two different ways within the same set* Σ, *i.e., validated and invalided, or only invalided but in multiple distinct ways.*

A Smarandache system $(\Sigma;\mathcal{R})$ *is a mathematical system which has at least one Smarandachely denied rule in* $\mathcal{R}$.

Definition 1.2([2]) *For an integer* $m \geq 2$, *let* $(\Sigma_1;\mathcal{R}_1)$, $(\Sigma_2;\mathcal{R}_2)$, $\cdots$, $(\Sigma_m;\mathcal{R}_m)$ *be* m *mathematical systems different two by two. A Smarandache multi-space is a pair* $(\widetilde{\Sigma};\widetilde{\mathcal{R}})$ *with*

$$\widetilde{\Sigma} = \bigcup_{i=1}^{m} \Sigma_i, \quad \textit{and} \quad \widetilde{\mathcal{R}} = \bigcup_{i=1}^{m} \mathcal{R}_i.$$

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[2]Reported at Beijing Jiaotong University, November 18, 2009.

Definition 1.3([3]) *An axiom is said to be Smarandachely denied if the axiom behaves in at least two different ways within the same space, i.e., validated and invalided, or only invalided but in multiple distinct ways.*

*A Smarandache geometry is a geometry which has at least one Smarandachely denied axiom(*1969*).*

Example 1.1 Let us consider an Euclidean plane $\mathbf{R}^2$ and three non-collinear points A, B and C. Define s-points as all usual Euclidean points on $\mathbf{R}^2$ and s-lines any Euclidean line that passes through one and only one of points A, B and C, such as those shown in Fig.1.1.

(i) The axiom (A5) that through a point exterior to a given line there is only one parallel passing through it is now replaced by two statements: *one parallel*, and *no parallel*. Let L be an s-line passes through C and is parallel in the Euclidean sense to AB. Notice that through any s-point not lying on AB there is one s-line parallel to L and through any other s-point lying on AB there is no s-lines parallel to L such as those shown in Fig.1(a).

(ii) The axiom that through any two distinct points there exist one line passing through them is now replaced by; *one s-line*, and *no s-line*. Notice that through any two distinct s-points D, E collinear with one of A, B and C, there is one s-line passing through them and through any two distinct s-points F, G lying on AB or non-collinear with one of A, B and C, there is no s-line passing through them such as those shown in Fig.1(b).

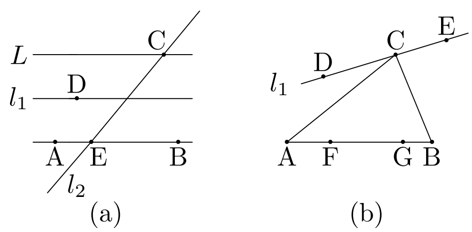

Fig.1

Definition 1.4 *A combinatorial system* $\mathscr{C}_G$ *is a union of mathematical systems* $(\Sigma_1; \mathcal{R}_1)$,$(\Sigma_2; \mathcal{R}_2)$, $\cdots$, $(\Sigma_m; \mathcal{R}_m)$ *for an integer* m*, i.e.,*

$$\mathscr{C}_G = (\bigcup_{i=1}^{m} \Sigma_i;\ \bigcup_{i=1}^{m} \mathcal{R}_i)$$

with an underlying connected graph structure G*, where*

$$V(G) = \{\Sigma_1, \Sigma_2, \cdots, \Sigma_m\},$$

$$E(G) = \{\ (\Sigma_i, \Sigma_j) \mid \Sigma_i \bigcap \Sigma_j \neq \emptyset, 1 \leq i, j \leq m\}.$$

§2. Vertex-Edge Labeled Graphs with Applications

2.1 Application to Principal Fiber Bundles

Definition 2.1 *A labeling on a graph* $G = (V, E)$ *is a mapping* $\theta_L : V \cup E \to L$ *for a label set* L*, denoted by* G^L*.*

If $\theta_L : E \to \emptyset$ or $\theta_L : V \to \emptyset$, then G^L is called a vertex labeled graph or an edge labeled graph, denoted by G^V or G^E, respectively. Otherwise, it is called a vertex-edge labeled graph.

Example:

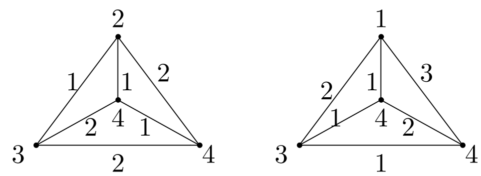

Fig.2

Definition 2.2([4]) *For a given integer sequence $0 < n_1 < n_2 < \cdots < n_m$, $m \geq 1$, a combinatorial manifold $\widetilde{M}$ is a Hausdorff space such that for any point $p \in \widetilde{M}$, there is a local chart (U_p, φ_p) of p, i.e., an open neighborhood U_p of p in $\widetilde{M}$ and a homoeomorphism $\varphi_p : U_p \to \widetilde{\mathbf{R}}(n_1(p), n_2(p), \cdots, n_{s(p)}(p))$, a combinatorial fan-space with $\{n_1(p), n_2(p), \cdots, n_{s(p)}(p)\} \subseteq \{n_1, n_2, \cdots, n_m\}$, and $\bigcup\limits_{p\in\widetilde{M}} \{n_1(p), n_2(p), \cdots, n_{s(p)}(p)\} = \{n_1, n_2, \cdots, n_m\}$, denoted by $\widetilde{M}(n_1, n_2, \cdots, n_m)$ or $\widetilde{M}$ on the context and*

$$\widetilde{\mathcal{A}} = \{(U_p, \varphi_p)|p \in \widetilde{M}(n_1, n_2, \cdots, n_m))\}$$

an atlas on $\widetilde{M}(n_1, n_2, \cdots, n_m)$.

A combinatorial manifold $\widetilde{M}$ is *finite* if it is just combined by finite manifolds with an underlying combinatorial structure G without one manifold contained in the union of others. Certainly, a finitely combinatorial manifold is indeed a combinatorial manifold. Examples of combinatorial manifolds can be seen in Fig.3.

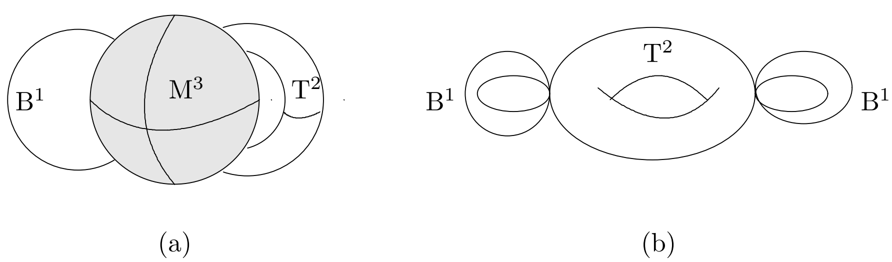

Fig.3

Let $\widetilde{M}(n_1, n_2, \cdots, n_m)$ be a finitely combinatorial manifold and $d, d \geq 1$ an integer. We construct a vertex-edge labeled graph $G^d[\widetilde{M}(n_1, n_2, \cdots, n_m)]$ by

$$V(G^d[\widetilde{M}(n_1, n_2, \cdots, n_m)]) = V_1 \bigcup V_2,$$

where $V_1 = \{n_i-\text{manifolds } M^{n_i} \text{ in } \widetilde{M}(n_1, \cdots, n_m)|1 \leq i \leq m\}$ and $V_2 = \{$isolated intersection points $O_{M^{n_i}, M^{n_j}}$ of M^{n_i}, M^{n_j} in $\widetilde{M}(n_1, n_2, \cdots, n_m)$ for $1 \leq i, j \leq m\}$. Label n_i for each

n_i-manifold in V_1 and 0 for each vertex in V_2 and

$$E(G^d[\widetilde{M}(n_1, n_2, \cdots, n_m)]) = E_1 \bigcup E_2,$$

where $E_1 = \{(M^{n_i}, M^{n_j})$ labeled with $\dim(M^{n_i} \bigcap M^{n_j}) \mid \dim(M^{n_i} \bigcap M^{n_j}) \geq d, 1 \leq i, j \leq m\}$ and $E_2 = \{(O_{M^{n_i}, M^{n_j}}, M^{n_i}), (O_{M^{n_i}, M^{n_j}}, M^{n_j})$ labeled with $0|M^{n_i}$ tangent M^{n_j} at the point $O_{M^{n_i}, M^{n_j}}$ for $1 \leq i, j \leq m\}$.

Now denote by $\mathcal{H}(n_1, n_2, \cdots, n_m)$ all finitely combinatorial manifolds $\widetilde{M}(n_1, n_2, \cdots, n_m)$ and $\mathcal{G}[0, n_m]$ all vertex-edge labeled graphs G^L with $\theta_L : V(G^L) \cup E(G^L) \to \{0, 1, \cdots, n_m\}$ with conditions following hold.

(1)Each induced subgraph by vertices labeled with 1 in G is a union of complete graphs and vertices labeled with 0 can only be adjacent to vertices labeled with 1.

(2)For each edge $e = (u, v) \in E(G)$, $\tau_2(e) \leq \min\{\tau_1(u), \tau_1(v)\}$.

Then we know a relation between sets $\mathcal{H}(n_1, n_2, \cdots, n_m)$ and $\mathcal{G}([0, n_m], [0, n_m])$ following.

Theorem 2.1([1]) *Let $1 \leq n_1 < n_2 < \cdots < n_m, m \geq 1$ be a given integer sequence. Then every finitely combinatorial manifold $\widetilde{M} \in \mathcal{H}(n_1, n_2, \cdots, n_m)$ defines a vertex-edge labeled graph $G([0, n_m]) \in \mathcal{G}[0, n_m]$. Conversely, every vertex-edge labeled graph $G([0, n_m]) \in \mathcal{G}[0, n_m]$ defines a finitely combinatorial manifold $\widetilde{M} \in \mathcal{H}(n_1, n_2, \cdots, n_m)$ with a $1-1$ mapping $\theta : G([0, n_m]) \to \widetilde{M}$ such that $\theta(u)$ is a $\theta(u)$-manifold in $\widetilde{M}$, $\tau_1(u) = \dim\theta(u)$ and $\tau_2(v, w) = \dim(\theta(v) \bigcap \theta(w))$ for $\forall u \in V(G([0, n_m]))$ and $\forall (v, w) \in E(G([0, n_m]))$.*

Definition 2.3([4]) *A principal fiber bundle consists of a manifold P action by a Lie group $\mathscr{G}$, which is a manifold with group operation $\mathscr{G} \times \mathscr{G} \to \mathscr{G}$ given by $(g, h) \to g \circ h$ being C^∞ mapping, a projection $\pi : P \to M$, a base pseudo-manifold M, denoted by $(P, M, \mathscr{G})$, seeing Fig.4 (where $V = \pi^{-1}(U)$) such that conditions (1), (2) and (3) following hold.*

(1) there is a right freely action of $\mathscr{G}$ on P,, i.e., for $\forall g \in \mathscr{G}$, there is a diffeomorphism $R_g : P \to P$ with $R_g(p) = pg$ for $\forall p \in P$ such that $p(g_1 g_2) = (pg_1)g_2$ for $\forall p \in P$, $\forall g_1, g_2 \in \mathscr{G}$ and $pe = p$ for some $p \in P$, $e \in \mathscr{G}$ if and only if e is the identity element of $\mathscr{G}$.

(2) the map $\pi : P \to M$ is onto with $\pi^{-1}(\pi(p)) = \{pg | g \in \mathscr{G}\}$.

(3) for $\forall x \in M$ there is an open set U with $x \in U$ and a diffeomorphism $T_U : \pi^{-1}(U) \to U \times \mathscr{G}$ of the form $T_U(p) = (\pi(p), s_U(p))$, where $s_U : \pi^{-1}(U) \to \mathscr{G}$ has the property $s_U(pg) = s_U(p)g$ for $\forall g \in \mathscr{G}, p \in \pi^{-1}(U)$.

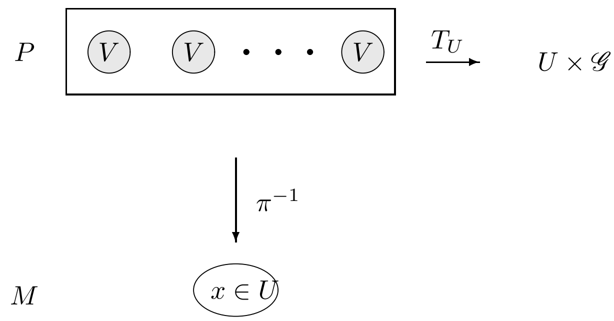

Fig.4

Question *For a family of k principal fiber bundles $P_1(M_1, \mathscr{G}_1)$, $P_2(M_2, \mathscr{G}_2), \cdots$, $P_k(M_k, \mathscr{G}_k)$ over manifolds M_1, M_2, $\cdots$, M_k, how can we construct principal fiber bundles on a smoothly combinatorial manifold consisting of $M_1, M_2, \cdots, M_k$ underlying a connected graph G?*

The answer is YES. The technique is by voltage assignment on labeled graphs defined as follows.

Definition 2.4([4]) *A voltage labeled graph on a vertex-edge labeled graph G^L is a 2-tuple $(G^L; \alpha)$ with a voltage assignments $\alpha : E(G^L) \to \Gamma$ such that*

$$\alpha(u, v) = \alpha^{-1}(v, u), \quad \forall (u, v) \in E(G^L),$$

with its labeled lifting G^{L_α} defined by

$$V(G^{L_\alpha}) = V(G^L) \times \Gamma, \ \ (u, g) \in V(G^L) \times \Gamma \ \textit{abbreviated to}\ u_g;$$

$$E(G^L_\alpha) = \{ \ (u_g, v_{g \circ h}) \mid \text{for} \ \ \forall (u, v) \in E(G^L) \ \text{with} \ \alpha(u, v) = h \ \}$$

with labels $\Theta_L : G^{L_\alpha} \to L$ following:

$$\Theta_L(u_g) = \theta_L(u), \ \ \text{and} \ \ \Theta_L(u_g, v_{g \circ h}) = \theta_L(u, v)$$

for $u, v \in V(G^L)$, $(u, v) \in E(G^L)$ with $\alpha(u, v) = h$ and $g, h \in \Gamma$.

For a voltage labeled graph (G^L, α) with its lifting G^L_α, a *natural projection* $\pi : G^{L_\alpha} \to G^L$ is defined by $\pi(u_g) = u$ and $\pi(u_g, v_{g \circ h}) = (u, v)$ for $\forall u, v \in V(G^L)$ and $(u, v) \in E(G^L)$ with $\alpha(u, v) = h$. Whence, (G^{L_α}, π) is a covering space of the labeled graph G^L. A voltage labeled graph with its labeled lifting are shown in Fig.4.4, in where, $G^L = C_3^L$ and $\Gamma = Z_2$.

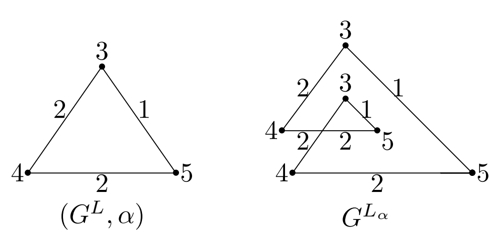

Fig.5

Now we show how to construct principal fiber bundles over a combinatorial manifold $\widetilde{M}$.

Construction 2.1 *For a family of principal fiber bundles over manifolds* $M_1, M_2, \cdots, M_l$, *such as those shown in Fig.*6,

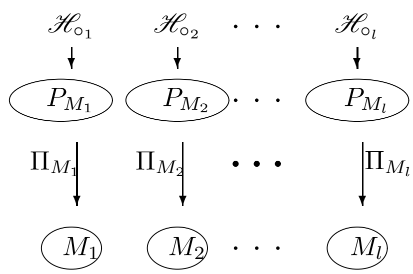

Fig.6

where $\mathscr{H}_{\circ_i}$ *is a Lie group acting on* P_{M_i} *for* $1 \leq i \leq l$ *satisfying conditions PFB1-PFB3, let* $\widetilde{M}$ *be a differentiably combinatorial manifold consisting of* M_i, $1 \leq i \leq l$ *and* $(G^L[\widetilde{M}], \alpha)$ *a voltage graph with a voltage assignment* $\alpha : G^L[\widetilde{M}] \to \mathfrak{G}$ *over a finite group* $\mathfrak{G}$, *which naturally induced a projection* $\pi : G^L[\widetilde{P}] \to G^L[\widetilde{M}]$. *For* $\forall M \in V(G^L[\widetilde{M}])$, *if* $\pi(P_M) = M$, *place* P_M *on each lifting vertex* M^{L_α} *in the fiber* $\pi^{-1}(M)$ *of* $G^{L_\alpha}[\widetilde{M}]$, *such as those shown in Fig.*7.

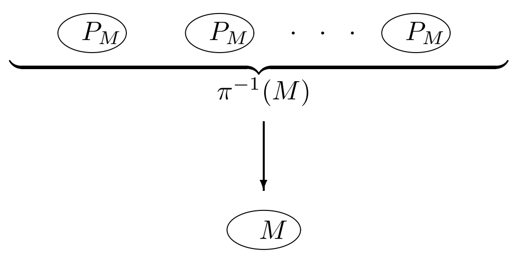

Fig.7

Let $\Pi = \pi \Pi_M \pi^{-1}$ *for* $\forall M \in V(G^L[\widetilde{M}])$. *Then* $\widetilde{P} = \bigcup\limits_{M \in V(G^L[\widetilde{M}])} P_M$ *is a smoothly combinatorial manifold and* $\mathscr{L}_G = \bigcup\limits_{M \in V(G^L[\widetilde{M}])} \mathscr{H}_M$ *a Lie multi-group by definition. Such a constructed combinatorial fiber bundle is denoted by* $\widetilde{P}^{L_\alpha}(\widetilde{M}, \mathscr{L}_G)$.

For example, let $\mathfrak{G} = Z_2$ and $G^L[\widetilde{M}] = C_3$. A voltage assignment $\alpha : G^L[\widetilde{M}] \to Z_2$ and its induced combinatorial fiber bundle are shown in Fig.8.

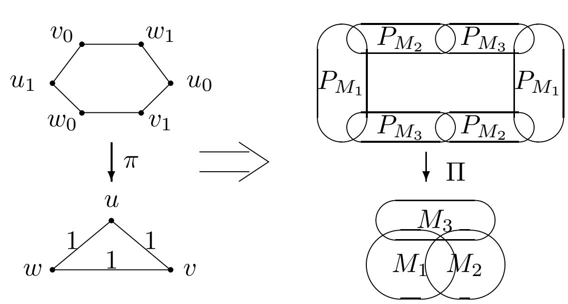

Fig.8

Then we know the existence result following.

Theorem 2.2([4]) *A combinatorial fiber bundle $\widetilde{P}^{\alpha}(\widetilde{M}, \mathscr{L}_G)$ is a principal fiber bundle if and only if for $\forall (M', M'') \in E(G^L[\widetilde{M}])$ and $(P_{M'}, P_{M''}) = (M', M'')^{L_\alpha} \in E(G^L[\widetilde{P}])$, $\Pi_{M'}|_{P_{M'} \cap P_{M''}} = \Pi_{M''}|_{P_{M'} \cap P_{M''}}$.*

2.2 Smarandachely k-constrained labeling of a graph

In references [5]-[6], the Smarandachely k-constrained labeling on some graph families are discussed.

Definition 2.5 *A Smarandachely k-constrained labeling of a graph $G(V, E)$ is a bijective mapping $f : V \cup E \to \{1, 2, .., |V| + |E|\}$ with the additional conditions that $|f(u) - f(v)| \geq k$ whenever $uv \in E$, $|f(u) - f(uv)| \geq k$ and $|f(uv) - f(vw)| \geq k$ whenever $u \neq w$, for an integer $k \geq 2$. A graph G which admits a such labeling is called a Smarandachely k-constrained total graph, abbreviated as $k - CTG$.*

An example for $k = 5$:

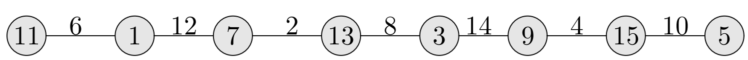

Fig.9: A 5-constrained labeling of a path P_7.

Definition 2.6 *The minimum positive integer n such that the graph $G \cup \overline{K}_n$ is a $k - CTG$ is called k-constrained number of the graph G and denoted by $t_k(G)$, the corresponding labeling is called a minimum k-constrained total labeling of G.*

Problem 2.1 *Determine $t_k(G)$ for $\forall k \in \mathbf{Z}^+$ and a graph G.*

≫Update Results for Problem 2.1 obtained in [5]-[6]:

Case 1. $k = 1$

In fact, $t_1(G) = 0$ for any graph G since any bijective mapping $f : V \cup E \to \{1, 2, .., |V| + |E|\}$ satisfies that $|f(u) - f(v)| \geq 1$ whenever $uv \in E$, $|f(u) - f(uv)| \geq 1$ and $|f(uv) - f(vw)| \geq 1$ whenever $u \neq w$.

Case 2. $k = 2$

$$(1)\ t_2(P_n) = \begin{cases} 0 & if \quad n = 2, \\ 1 & if \quad n = 3, \\ 0 & else. \end{cases}$$

Proof Let $V(P_n) = \{v_1, v_2, ..., v_n\}$ and $E(P_n) = \{v_i v_{i+1} | 1 \leq i \leq n-1\}$. Consider a total labeling $f : V \cup E \longrightarrow \{1, 2, 3, ..., 2n-1\}$ defined as $f(v_1) = 2n-3$; $f(v_2) = 2n-1$; $f(v_1 v_2) = 2$; $f(v_2 v_3) = 4$; and $f(v_k) = 2k - 5$, $f(v_k v_{k+1}) = 2k$, for all $k \geq 3$. This function f serves as a Smarandachely 2-constrained labeling for P_n, for $n \geq 4$. Further, the cases $n = 2$ and $n = 3$

are easy to prove. $\square$

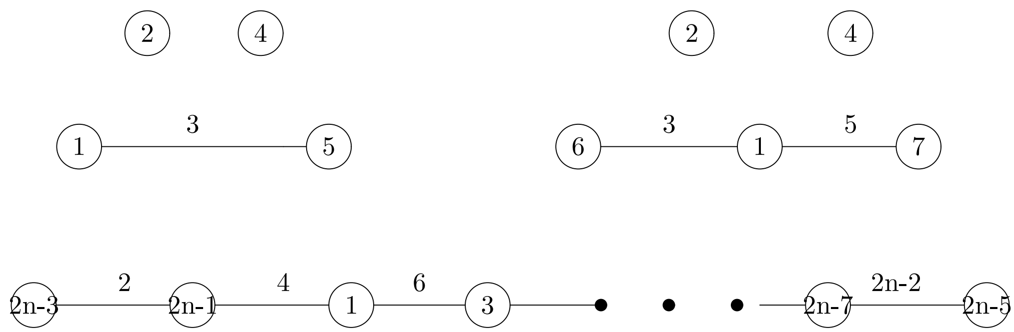

Fig.10

(2) $t_2(C_n) = 0$ if $n \geq 4$ and $t_2(C_3) = 2$.

Proof If $n \geq 4$, then the result follows immediately by joining end vertices of P_n by an edge $v_1 v_n$, and, extending the total labeling f of the path as in the proof of the Theorem 2.4 above to include $f(v_1 v_2) = 2n$.

Consider the case $n = 3$. If the integers a and $a + 1$ are used as labels, then one of them is assigned for a vertex and other is to the edge not incident with that vertex. But then, $a + 2$ can not be used to label the vertex or an edge in C_3. Therefore, for each three consecutive integers we should leave at least one integer to label C_3. Hence the span of any Smarandachely 2-constrained labeling of C_3 should be at least 8. So $t_2(C_3) \geq 2$. Now from the Figure 3 it is clear that $t_2(C_3) \leq 2$. Thus $t_2(C_3) = 2$. $\square$

(3) $t_2(K_n) = 0$ if $n \geq 4$.

(4) $t_2(W_{1,n}) = 0$ if $n \geq 3$.

(6) $t_2(K_{m,n}) = \begin{cases} 2 \ \ if \ \ n = 1 \ \ and \ \ m = 1, \\ 1 \ \ if \ \ n = 1 \ \ and \ \ m \geq 2, \\ 0 \ \ else. \end{cases}$

Case 3. $k \geq 3$

(1) $t_k(K_{1,n}) = \begin{cases} 3k - 6, \ \ if \ \ n = 3, \\ n(k-2), \ \ otherwise. \end{cases}$ if $k.n \geq 3$.

Proof For any Smarandachely k-constrained labeling f of a star $K_{1,n}$, the span of f, after labeling an edge by the least positive integer a is at least $a + nk$. Further, the span is minimum only if $a = 1$. Thus, as there are only $n + 1$ vertices and n edges, for any minimum total labeling we require at least $1 + nk - (2n + 1) = n(k - 2)$ isolated vertices if $n \geq 4$ and at least $1 + nk - 2n = n(k - 2) + 1$ if $n = 3$. In fact, for the case $n = 3$, as the central vertex is incident with each edge and edges are mutually adjacent, by a minimum k-constrained total labeling, the edges as well the central vertex can be labeled only by the set $\{1, 1 + k, 1 + 2k, 1 + 3k\}$.

Suppose the label 1 is assigned for the central vertex, then to label the end vertex adjacent to edge labeled $1+2k$ is at least $(1+3k)+1$ (since it is adjacent to 1, it can not be less than $1+k$). Thus at most two vertices can only be labeled by the integers between 1 and $1+3k$. Similar argument holds for the other cases also.

Therefore, $t(K_{1,n}) \geq n(k-2)$ for $n \geq 4$ and $t(K_{1,n}) \geq n(k-2)+1$ for $n=3$.

To prove the reverse inequality, we define a k-constrained total labeling for all $k \geq 3$, as follows:

(1) When $n=3$, the labeling is shown in the Fig.11 below

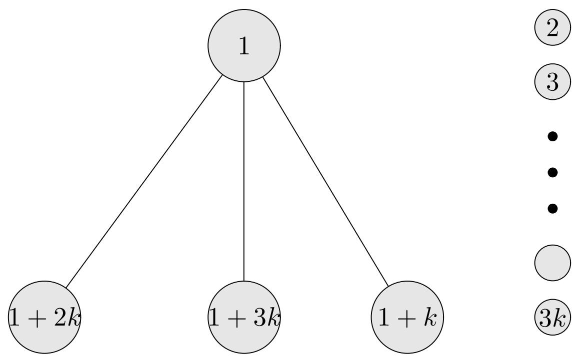

Fig.11

(2) When $n \geq 4$, define a total labeling f as $f(v_0v_j) = 1+(j-1)k$ for all $j, 1 \leq j \leq n$. $f(v_0) = 1+nk$, $f(v_1) = 2+(n-2)k$, $f(v_2) = 3+(n-2)k$,and for $3 \leq i \leq (n-1)$,

$$f(v_{i+1}) = \begin{cases} f(v_i)+2, & if \quad f(v_i) \equiv 0 (mod \quad k), \\ f(v_i)+1, & otherwise. \end{cases}$$

and the rest all unassigned integers between 1 and $1+nk$ to the $n(k-2)$ isolated vertices, where v_0 is the central vertex and $v_1, v_2, v_3, ..., v_n$ are the end vertices.

The function so defined is a Smarandachely k-constrained labeling of $K_{1,n} \cup \bar{K}_{n(k-2)}$, for all $n \geq 4$. $\square$

(2) Let P_n be a path on n vertices and $k_0 = \lfloor \frac{2n-1}{3} \rfloor$. Then

$$t_k(P_n) = \begin{cases} 0 \quad if \quad k \leq k_0, \\ 2(k-k_0)-1 \quad if \quad k > k_0 \quad and \quad 2n \equiv 0 (mod\ 3), \\ 2(k-k_0) \quad if \ k > k_0 \quad and \quad 2n \equiv 1 \ or \ 2 (mod\ 3). \end{cases}$$

(3) Let C_n be a cycle on n vertices and $k_0 = \lfloor \frac{2n-1}{3} \rfloor$. Then

$$t_k(C_n) = \begin{cases} 0 \quad if \quad k \leq k_0, \\ 2(k-k_0) \quad if \quad k > k_0 \quad and \quad 2n \equiv 0 \ (mod\ 3), \\ 3(k-k_0) \quad if \ k > k_0 \quad and \quad 2n \equiv 1 \ or \ 2 (mod\ 3). \end{cases}$$

2.3 Smarandachely Super m-Mean Graph

The conception of Smarandachely edge m-labeling on a graph was introduced in [7].

Definition 2.7 *Let G be a graph and $f : V(G) \to \{1, 2, 3, \cdots, |V| + |E(G)|\}$ be an injection. For each edge $e = uv$ and an integer $m \geq 2$, the induced Smarandachely edge m-labeling f_S^* is defined by*

$$f_S^*(e) = \left\lceil \frac{f(u) + f(v)}{m} \right\rceil.$$

Then f is called a Smarandachely super m-mean labeling if $f(V(G)) \cup \{f^*(e) : e \in E(G)\} = \{1, 2, 3, \cdots, |V| + |E(G)|\}$. A graph that admits a Smarandachely super mean m-labeling is called Smarandachely super m-mean graph.

Particularly, if $m = 2$, we know that

$$f^*(e) = \begin{cases} \frac{f(u)+f(v)}{2} & \text{if } f(u) + f(v) \text{ is even;} \\ \frac{f(u)+f(v)+1}{2} & \text{if } f(u) + f(v) \text{ is odd.} \end{cases}$$

Example: A Smarandache super 2-mean graph P_6^2

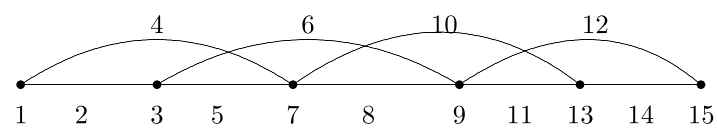

Fig.12

Problem 2.2 *Find integers m and graphs G such that G is a Smarandachely super m-mean graph.*

≫Update Results for Problem 2.2 **Obtained in [7]:**

Now all results is on the case of Smarandache super 2-mean graphs.

(1) A *H-graph* of a path P_n is the graph obtained from two copies of P_n with vertices $v_1, v_2, \ldots, v_n$ and $u_1, u_2, \ldots, u_n$ by joining the vertices $v_{\frac{n+1}{2}}$ and $u_{\frac{n+1}{2}}$ if n is odd and the vertices $v_{\frac{n}{2}+1}$ and $u_{\frac{n}{2}}$ if n is even. Then

A H-graph G is a Smarandache super 2-mean graph.

(2) The corona of a graph G on p vertices $v_1, v_2, \ldots, v_p$ is the graph obtained from G by adding p new vertices $u_1, u_2, \ldots, u_p$ and the new edges u_iv_i for $1 \leq i \leq p$, denoted by $G \odot K_1$.

If a H-graph G is a Smarandache super 2-mean graph, then $G \odot K_1$ is a Smarandache super 2-mean graph.

(3) For a graph G, the 2-corona of G is the graph obtained from G by identifying the center vertex of the star S_2 at each vertex of G, denoted by $G \odot S_2$.

If a H-graph G is a Smarandache super 2-mean graph, then $G \odot S_2$ is a Smarandache super 2-mean graph.

(4) *Cycle C_{2n} is a Smarandache super 2-mean graph for $n \geq 3$.*

(5) *Corona of a cycle C_n is a Smarandache super 2-mean graph for $n \geq 3$.*

(6) A cyclic snake mC_n is the graph obtained from m copies of C_n by identifying the vertex $v_{(k+2)_j}$ in the j^{th} copy at a vertex $v_{1_{j+1}}$ in the $(j+1)^{th}$ copy if $n = 2k+1$ and identifying the vertex $v_{(k+1)_j}$ in the j^{th} copy at a vertex $v_{1_{j+1}}$ in the $(j+1)^{th}$ copy if $n = 2k$.

The graph mC_n-snake, $m \geq 1, n \geq 3$ and $n \neq 4$ has a Smarandache super 2-mean labeling.

(7) A $P_n(G)$ is a graph obtained from G by identifying an end vertex of P_n at a vertex of G.

If G is a Smarandache super 2-mean graph then $P_n(G)$ is also a Smarandache super 2-mean graph.

(8) *$C_m \times P_n$ for $n \geq 1, m = 3, 5$ are Smarandache super 2-mean graphs.*

Problem 2.3 *For what values of m (except 3,5) the graph $C_m \times P_n$ is a Smarandache super 2-mean graph?*

2.4 **Smarandachely Uniform k-Graphs**

The conception of Smarandachely Uniform k-Graph was introduced in the reference [8].

Definition 2.7 *For an non-empty subset M of vertices in a graph $G = (V, E)$, each vertex u in G is associated with the set $f^o_M(u) = \{d(u, v) \ : \ v \in M, \ u \neq v\}$, called its open M-distance-pattern.*

A graph G is called a Smarandachely uniform k-graph if there exist subsets $M_1, M_2, \cdots, M_k$ for an integer $k \geq 1$ such that $f^o_{M_i}(u) = f^o_{M_j}(u)$ and $f^o_{M_i}(u) = f^o_{M_j}(v)$ for $1 \leq i, j \leq k$ and $\forall u, v \in V(G)$. Such subsets $M_1, M_2, \cdots, M_k$ are called a k-family of open distance-pattern uniform (odpu-) set of G and the minimum cardinality of odpu-sets in G, if they exist, is called the Smarandachely odpu-number of G, denoted by $od^S_k(G)$.

Usually, a Smarandachely uniform 1-graph G is called an *open distance-pattern uniform (odpu-) graph.* In this case, its odpu-number $od^S_k(G)$ of G is abbreviated to $od(G)$.

Problem 2.4 *Determine which graph G is Smarandachely uniform k-graph for an integer $k \geq 1$.*

≫Update Results for Problem 2.4 Obtained in [8]:

(1) *A connected graph G is an odpu-graph if and only if the center $Z(G)$ of G is an odpu-set.*

(2) *Every self-centered graph is an odpu-graph.*

(3) *A tree T has an odpu-set M if and only if T is isomorphic to P_2.*

(4) *If G is a unicyclic odpu-graph, then G is isomorphic to a cycle.*

(5) *A block graph G is an odpu-graph if and only if G is complete.*

(6) *A graph with radius 1 and diameter 2 is an odpu-graph if and only if there exists a subset $M \subset V(G)$ with $|M| \geq 2$ such that the induced subgraph $\langle M \rangle$ is complete, $\langle V - M \rangle$ is not complete and any vertex in $V - M$ is adjacent to all the vertices of M.*

Problem 2.5 *Determine the Smarandachely odpu-number $od_k^S(G)$ of G for an integer $k \geq 1$.*

$\gg$**Update Results for Problem** 2.5 **obtained in [8]:**

(1) *For every positive integer $k \neq 1, 3$, there exists a graph G with odpu-number k.*

(2) *If a graph G has odpu-number 4, then $r(G) = 2$.*

(3) *The number 5 cannot be the odpu-number of a bipartite graph.*

(4) *Let G be a bipartite odpu-graph. Then $od(G) = 2$ if and only if G is isomorphic to P_2.*

(5) $od(C_{2k+1}) = 2k$.

(6) $od(K_n) = 2$ *for all* $n \geqslant 2$.

2.5 Smarandachely Total Coloring of a graph

The conception of Smarandachely total k-coloring of a graph following is introduced by Zhongfu Zhang et al. in [9].

Definition 2.8 *Let f be a total $k-$coloring on G. Its total-color neighbor of a vertex u of G is denoted by $C_f(x) = \{f(x)|x \in T_N(u)\}$. For any adjacent vertices x and y of $V(G)$, if $C_f(x) \neq C_f(y)$, say f a k AVSDT-coloring of G (the abbreviation of adjacent-vertex-strongly-distinguishing total coloring of G).*

The AVSDT-coloring number of G, denoted by $\chi_{ast}(G)$ is the minimal number of colors required for an AVSDT-coloring of G

Definition 2.9 *A Smarandachely total k-coloring of a graph G is an AVSDT-coloring with $|C_f(x)\backslash C_f(y)| \geq k$ and $|C_f(y)\backslash C_f(x)| \geq k$.*

The minimum Smarandachely total k-coloring number of a graph G is denoted by $\chi_{ast}^k(G)$.

Obviously, $\chi_{ast}(G) = \chi_{ast}^1(G)$ and

$$\cdots \leq \chi_{ast}^{k+1}(G) \leq \chi_{ast}^k(G) \leq \chi_{ast}^{k-1}(G) \leq \cdots \leq \chi_{ast}^1(G)$$

by definition.

Problem 2.6 Determine $\chi_{ast}^k(G)$ for a graph G.

$\gg$**Update Results for Problem** 2.6 **obtained in [9]:**

$\chi_{ast}^1(S_m + W_n) = m + n + 3$ *if* $min\{m, n\} \geq 5$.

It should be noted that the number $\chi^k_{ast}(G)$ of graph families following are determined for integers $k \geq 1$ by Zhongfu Zhang et al. in references [10]-[15].

(1) 3-regular Halin graphs;

(2) $2P_n,\ 2C_n,\ 2K_{1,n}$ and double fan graphs for integers $n \geq 1$;

(3) $P_m + P_n$ for integers $m, n \geq 1$;

(4) $P_m \vee P_n$ for integers $m, n \geq 1$;

(5) Generalized Petersen $G(n, k)$;

(6) k-cube graphs.

§3. Covering and Decomposing of a Graph

Definition 3.1 *Let $\mathscr{P}$ be a graphical property. A Smarandache graphoidal $\mathscr{P}$ (k, d)-cover of a graph G is a partition of edges of G into subgraphs $G_1, G_2, \cdots, G_l \in \mathscr{P}$ such that $E(G_i) \cap E(G_j) \leq k$ and $\Delta(G_i) \leq d$ for integers $1 \leq i, j \leq l$.*

The minimum cardinality of Smarandache graphoidal $\mathscr{P}$ (k, d)-cover of a graph G is denoted by $\Pi^{(k,d)}_{\mathscr{P}}(G)$.

Problem 3.1 *determine $\Pi^{(k,d)}_{\mathscr{P}}(G)$ for a graph G.*

3.1 Smarandache path k-cover of a graph

The Smarandache path k-cover of a graph was discussed by S. Arumugam and I.Sahul Hamid in [16].

Definition 3.2 *A Smarandache path k-cover of a graph G is a Smarandache graphoidal $\mathscr{P}$ $(k, \Delta(G))$-cover of G with $\mathscr{P}$=path for an integer $k \geq 1$.*

A Smarandache path 1-cover of G such that its every edge is in exactly one path in it is called a simple path cover.

The minimum cardinality of simple path covers of G is called the simple path covering number of G and is denoted by $\Pi^{(1,\Delta(G))}_{\mathscr{P}}(G)$.

If do not consider the condition $E(G_i) \cap E(G_j) \leq 1$, then a simple path cover is called path cover of G, its minimum number of path cover is denoted by $\pi(G)$ in reference. For examples, $\pi_s(K_n) = \lceil \frac{n}{2} \rceil$ and $\pi_s(T) = \frac{k}{2}$, where k is the number of odd degree in tree T.

Problem 3.2 *determine $\Pi^{(k,d)}_{\mathscr{P}}(G)$ for a graph G.*

$\gg$Update Results for Problem 3.2 Obtained in [10]:

(1) $\Pi^{(1,\Delta(G))}_{\mathscr{P}}(T) = \pi(T) = \frac{k}{2}$, *where k is the number of vertices of odd degree in T.*

(2) *Let G be a unicyclic graph with cycle C. Let m denote the number of vertices of degree greater than 2 on C. Let k be the number of vertices of odd degree. Then*

$$\Pi_{\mathscr{P}}^{(1,\Delta(G))}(G) = \begin{cases} 3 & \text{if } m = 0 \\ \frac{k}{2} + 2 & \text{if } m = 1 \\ \frac{k}{2} + 1 & \text{if } m = 2 \\ \frac{k}{2} & \text{if } m \geq 3 \end{cases}$$

(3) *For a wheel* $W_n = K_1 + C_{n-1}$, *we have*

$$\Pi_{\mathscr{P}}^{(1,\Delta(G))}(W_n) = \begin{cases} 6 & \text{if } n = 4 \\ \left\lfloor \frac{n}{2} \right\rfloor + 3 & \text{if } n \geq 5 \end{cases}$$

Proof Let $V(W_n) = \{v_0, v_1, \ldots, v_{n-1}\}$ and $E(W_n) = \{v_0v_i : 1 \leq i \leq n-1\} \cup \{v_iv_{i+1} : 1 \leq i \leq n-2\} \cup \{v_1v_{n-1}\}$.

If $n = 4$, then $W_n = K_4$ and hence $\Pi_{\mathscr{P}}^{(1,\Delta(G))}(W_n)(W_n) = 6$.

Now, suppose $n \geq 5$. Let $r = \left\lfloor \frac{n}{2} \right\rfloor$

If n is odd, let

$P_i = (v_i, v_0, v_{r+i})$, $i = 1, 2, \ldots, r$.

$P_{r+1} = (v_1, v_2, \ldots, v_r)$,

$P_{r+2} = (v_1, v_{2r}, v_{2r-1}, \ldots, v_{r+2})$ and

$P_{r+3} = (v_r, v_{r+1}, v_{r+2})$.

If n is even, let

$P_i = (v_i, v_0, v_{r-1+i})$, $i = 1, 2, \ldots, r-1$.

$P_r = (v_0, v_{2r-1})$,

$P_{r+1} = (v_1, v_2, \ldots, v_{r-1})$,

$P_{r+2} = (v_1, v_{2r-1}, \ldots, v_{r+1})$ and

$P_{r+3} = (v_{r-1}, v_r, v_{r+1})$.

Then $\Pi_{\mathscr{P}}^{(1,\Delta(G))}(W_n) = \{P_1, P_2, \ldots, P_{r+3}\}$ is a simple path cover of W_n. Hence $\pi_s(W_n) \leq r + 3 = \left\lfloor \frac{n}{2} \right\rfloor + 3$. Further, for any simple path cover ψ of W_n at least three vertices on $C = (v_1, v_2, \ldots, v_{n-1})$ are terminal vertices of paths in ψ. Hence $t \leq q - \frac{k}{2} - 3$, so that $\Pi_{\mathscr{P}}^{(1,\Delta(G))}(W_n) = q - t \geq \frac{k}{2} + 3 = \left\lfloor \frac{n}{2} \right\rfloor + 3$. Thus $\Pi_{\mathscr{P}}^{(1,\Delta(G))}(W_n) = \left\lfloor \frac{n}{2} \right\rfloor + 3$. □

A. Nagarajan, V. Maheswari and S. Navaneethakrishnan discussed Smarandache path 1-cover in [17].

Definition 3.3 *A Smarandache path* 1*-cover of* G *such that its every edge is in exactly two path in it is called a path double cover.*

Define $G * H$ with vertex set $V(G) \times V(H)$ in which (g_1, h_1) is joined to (g_2, h_2) whenever $g_1g_2 \in E(G)$ or $g_1 = g_2$ and $h_1h_2 \in E(H)$; $G \circ H$, the weak product of graphs G, H with vertex set $V(G)\times V(H)$ in which two vertices (g_1,h_1) and (g_2,h_2) are adjacent whenever $g_1g_2\in E(G)$ and $h_1h_2\in E(H)$ and

$$\gamma_2(G)=\min\{\ |\psi|:\psi \text{ is a path double cover of } G\ \}.$$

(4) *Let $m\geq 3$.*

$$\gamma_2(C_m\circ K_2)=\begin{cases} 3 & \text{if m is odd;}\\ 6 & \text{if m is even.}\end{cases}$$

(5) *Let $m,n\geq 3$. $\gamma_2(C_m\circ C_n)=5$ if at least one of the numbers m and n is odd.*

(6) *Let $m,n\geq 3$.*

$$\gamma_2(P_m\circ C_n)=\begin{cases} 4 & \text{if n } \equiv 1 \text{ or } 3(\text{mod } 4)\\ 8 & \text{if n } \equiv 0 \text{ or } 2(\text{mod } 4)\end{cases}$$

(7) $\gamma_2(C_m*K_2)=6$ *if $m\geq 3$ is odd.*

(8) $\gamma_2(P_m*K_2)=4$ *for $m\geq 3$.*

(9) $\gamma_2(P_m*K_2)=5$ *for $m\geq 3$.*

(10) $\gamma_2(C_m\times P_3)=5$ *if $m\geq 3$ is odd.*

(11) $\gamma_2(P_m\circ K_2)=4$ *for $m\geq 2$.*

(12) $\gamma_2(K_{m,n})=\max\{m,n\}$.

(13)

$$\gamma_2(P_m\times P_n)=\begin{cases} 3 & \text{if m=2 or n=2;}\\ 4 & \text{if } m,n\geq 2.\end{cases}$$

(14) $\gamma_2(C_m\times C_n)=5$ *if $m\geq 3$, $n\geq 3$ and at least one of the numbers m and n is odd.*

(15) $\gamma_2(C_m\times K_2)=4$ *for $m\geq 3$.*

3.2 Smarandache graphoidal tree d-cover of a graph

S.Somasundaram, A.Nagarajan and G.Mahadevan discussed Smarandache graphoidal tree d-cover of a graph in references [18]-[19].

Definition 3.4 *A Smarandache graphoidal tree d-cover of a graph G is a Smarandache graphoidal $\mathscr{P}$ $(|G|,d)$-cover of G with $\mathscr{P}$=tree for an integer $d\geq 1$.*

The minimum cardinality of Smarandache graphoidal tree d-cover of G is denoted by $\gamma_T^{(d)}(G)=\Pi_{\mathscr{P}}^{(|G|,d)}(G)$. If $d=\Delta(G)$, then $\gamma_T^{(d)}(G)$ is abbreviated to $\gamma_T(G)$.

Problem 3.3 *determine $\gamma_T(G)$ for a graph G, particularly, $\gamma_T(G)$.*

≫Update Results for Problem 3.3 **Obtained in [12-13]:**

Case 1: $\gamma_T(G)$

(1) $\gamma_T(K_p)=\lceil\frac{p}{2}\rceil$;

(2) $\gamma_T(K_{m,n})=\lceil\frac{m+n}{3}\rceil$ *if $m\leq n<2m-3$.*

(3) $\gamma_T(K_{m,n})=m$ *if $n>2m-3$.*

(4) $\gamma_T(P_m \times P_n) = 2$ *for integers* $m, n \geq 2$.

(5) $\gamma_T(P_n \times C_m) = 2$ *for integers* $m \geq 3$, $n \geq 2$.

(6) $\gamma_T(C_m \times C_n) = 3$ *if* $m, n \geq 3$.

Case 2: $\gamma_T^{(d)}(G)$

(1)

$$\gamma_T^{(d)}(K_p) = \begin{cases} \frac{p(p-2d+1)}{2} & if\ d < \frac{p}{2}, \\ \lceil \frac{p}{2} \rceil & if\ d \geq \frac{p}{2} \end{cases}$$

if $p \geq 4$.

(2) $\gamma_T^{(d)}(K_{m,n}) = p + q - pd = mn - (m+n)(d-1)$ *if* $n, m \geq 2d$.

(3) $\gamma_T^{(d)}(K_{2d-1,2d-1}) = p + q - pd = 2d - 1$.

(4) $\gamma_T^{(d)}(K_{n,n}) = \lceil \frac{2n}{3} \rceil$ *for* $d \geq \lceil \frac{2n}{3} \rceil$ *and* $n > 3$.

(5) $\gamma_T^{(d)}(C_m \times C_n) = 3$ *for* $d \geq 4$ *and* $\gamma_T^{(2)}(C_m \times C_n) = q - p$.

§5. Furthermore

In fact, Smarandache's notion can be used to generalize more and more conceptions and problems in classical graph theory. Some of them will appeared in my books *Automorphism Groups of Maps, Surfaces and Smarandache's Geometries* (Second edition), *Smarandache Multi-Space Theory* (Second edition) published in forthcoming, or my monograph *Graph Theory – A Smarandachely Type* will be appeared in 2012.

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