International J.Math. Combin. Vol.1 (2011), 119-122

A Note On Line Graphs

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Abstract: In this note we define two generalizations of the line graph and obtain some results. Also, we mark some open problems.

Key Words: Smarandachely (k, r) line graph, (k, r)-graphs, line graphs

AMS(2010): 05C75, 05C45

For standard terminology and notion in graph theory we refer the reader to Harary [2]; the non-standard will be given in this paper as and when required. We treat only finite simple graphs without self loops and isolates.

The line graph $L(G)$ of a graph G is defined to have as its vertices the edges of G, with two being adjacent if the corresponding edges share a vertex in G. Line graphs have a rich history. The name line graph was first used by Harary and Norman [3] in 1960. But line graphs were the subject of investigation as far back as 1932 in Whitney s paper [7], where he studied edge isomorphism and showed that for connected graphs, edge-isomorphism implies isomorphism except for K_3 and $K_{1,3}$. The first characterization (partition into complete subgraphs) was given by Krausz [5]. Instead, we refer the interested reader to a somewhat older but still an excellent survey on line graphs and line digraphs by Hemminger and Beineke [4]. An excellent book by Prisner [6] describes many interesting generalizations of line graphs. In this note we generalize the line graph $L(G)$ of G as follows:

Let $G = (V, E)$ be a graph of order $p \geq 3$, k and r be integers with $1 \leq r < k \leq p$. Let $U = \{S_1, S_2, ..., S_n\}$ be the set of all distinct connected acyclic subgraphs of G of order k and $U' = \{T_1, T_2, ..., T_m\}$ be the set of all distinct connected subgraphs of G with size k.

The *vertex* (k, r)-*graph* $L^v_{(k,r)}(G)$, where $1 \leq r < k \leq p$ is the graph has the vertex set U where two vertices S_i and S_j, $i \neq j$ are adjacent if, and only if, $S_i \cap S_j$ has a connected subgraph of order r.

The *edge* (k, r)-*graph* $L^e_{(k,r)}(G)$, where $0 \leq r < k \leq q$ is the graph has the vertex set U' where two vertices T_i and T_j, $i \neq j$ are adjacent if, and only if, $T_i \cap T_j$ has a connected subgraph of size r.

The *Smarandachely* (k, r) *line graph* $L^S_{(k,r)}(G)$ of a graph G is such a graph with vertex set U' and two vertices T_i and T_j, $i \neq j$ are adjacent if and only if, $T_i \cap T_j$ has a connected subgraph with order or size r. Clearly, $L^v_{(k,r)}(G) \leq L^S_{(k,r)}(G)$ and $L^e_{(k,r)}(G) \leq L^S_{(k,r)}(G)$. In

[1]Received December 22, 2010. Accepted March 3, 2011.

Figure 1, we depicted $L(G)$, $L^v_{(k,r)}(G)$ and $L^e_{(k,r)}(G)$ for the graph G.

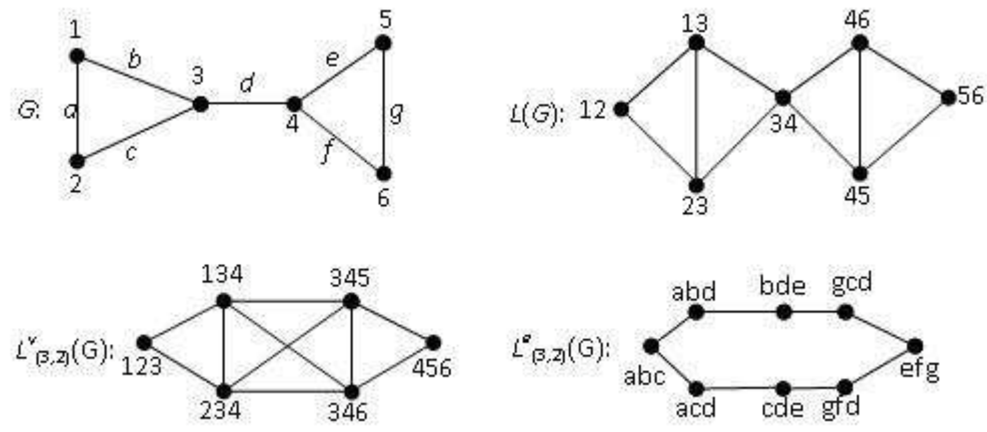

Figure.1

One can easily verify that: $L_3(K_{1,3}) = K_3$, $L_3(C_n) = C_n$, for $n \geq 3$, $L_3(P_n) = P_{n-1}$ and $L_3(K_4) = L(K_4 - e) = K_4$.

For any positive integer k, the k^{th} iterated line graph $L(G)$ of G is defined as follows: $L^0(G) = G$, $L^k(G) = L(L^{k-1}(G))$.

A graph G is a $(3,2)$-*graph* if there exists a graph H such that $L_{(3,2)}(H) \cong G$. First we prove the following result:

Proposition 1 *For any graph G, $L_{(3,2)} \cong L^2(G)$.*

Proof First we show that $L_{(3,2)(G)}$ and $L^2(G)$ have the same number of vertices. Let S be a vertex in $L_{(3,2)}(G)$. Then S corresponds to a subgraph of order 3 in G. Say, S consists of two adjacent edges ab and bc. Then corresponding to S we have an edge in $L(G)$ with end vertices ab and bc, and corresponding this edge, we have a vertex say abc in $L^2(G)$. Similarly, we can show that very vertex in $L^2(G)$ corresponds to a connected subgraph of order 3. This proves that $L_{(3,2)}(G)$ and $L^2(G)$ have the same number of vertices. Now, let S_1 and S_2 be two adjacent edges in $L_{(3,2)}(G)$. Then S_1 and S_2 correspond to two connected subgraphs of order 3 each, having a common edge. These in turn will give two adjacent edges, say $e(S_1)$ and $e(S_2)$ in $L(G)$ and this will give an edge in $L^2(G)$ with end vertices $e(S_1)$ and $e(S_2)$. This proves the result. □

In general one can establish the following result.

Proposition 2 *Let G be a graph of order p and $2 \geq r < k \leq p$. If $\triangle(G) \leq 2$ then $L_{(k,k-1)}(G) \cong L^{k-1}(G)$.*

Note that this is true only when $\triangle(G) \leq 2$. For example, we find $L^{n-1}_{(n,n-1)}(K_{1,n}) = K_n = L(K_{1,n})$.

Proposition 3 *The star $K_{1,3}$ is not a 3-line graph.*

Proof Let $K_{1,3}$ be a 3-line graph. Then there exists a graph G such that $L(K_{1,3}) = G$. Since $K_{1,3}$ has four vertices, G should have exactly four connected subgraphs each of order three. All connected graphs having exactly four induced subgraphs are as follows: i) C_4, ii) K_4, iii) $K_4 - e$ and iv) P_6. None of these graphs have $K_{1,3}$ as its 3-line graph. □

Clearly, any graph having $K_{1,3}$ as an induced subgraph is not a 3-line graph. Hence, we have

Corollary 4 *$K_{1,n}$, $n \geq 3$ is not a 3-line graph.*

It is not true in general that the line graph $L(G)$ of a graph G is a subgraph of $L_3(G)$. For example, $L_3(K_{1,4})$ does not contain K_4, the line graph of $K_{1,4}$ as a subgraph.

Problem 5 *Characterize 3-line graphs.*

A graph G is a *self 3-line graph*, if it is isomorphic to its 3-line graph.

Problem 6 *Characterize self 3-line graphs.*

Proposition 7 *Let $L_3(G)$ be the 3-line graph of a graph G of order $p \geq 3$. The degree of a vertex s in $L_3(G)$ is denoted by degs and is defined as follows:*

Let S be the subgraph of G corresponding to the vertex s in $L_3(G)$. For an edge $x = uv$ in S, let $d(x) = (deg_G u + deg_G v) - (deg_S u + deg_S v)$, where $deg_G u$ and $deg_S u$ are the degrees of u in G and S respectively. Then $d(s) = \sum_{x \in S} d(x)$.

Proof Consider an edge uv in S. Suppose $y = uz$ is an edge of G at u which is not in S. Then y belongs to a connected subgraph S_1 of cardinality three containing the edge uv which is distinct form S. Since S and S_1 have common edge, ss_1 is an edge in $L_3(G)$, where s_1 is the vertex in $L_3(G)$ corresponding to the subgraph S_1 in G. Similarly, for any edge $y_1 = vz_1$ at v in G which is not in S, we have an edge ss_2 in $L_3(G)$. This implies that corresponding to the edge $x = uv$ in S, we have $(deg_G u - deg_S u) + (deg_G v - deg_S v)$ edges in $L_3(G)$, and hence the result follows. □

Acknowledgement

The authors are grateful to Sri. B. Premnath Reddy, Chairman, Acharya Institutes, for his constant support and encouragement for research and development.

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