International J.Math. Combin. Vol.3 (2011), 88-106

Lucas Gracefulness of Almost and Nearly for Some Graphs

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Abstract: Let G be a (p, q) - graph. An injective function $f : V(G) \rightarrow \{l_0, l_1, l_2, \cdots, l_a\}, (a \in N)$, is said to be Lucas graceful labeling if an induced edge labeling $f_1(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{l_1, l_2, \cdots, l_q\}$ with the assumption of $l_0 = 0, l_1 = 1, l_2 = 3, l_3 = 4, l_4 = 7, l_5 = 11$, etc.. If G admits Lucas graceful labeling, then G is said to be Lucas graceful graph. An injective function $f : V(G) \rightarrow \{l_0, l_1, l_2, \cdots, l_{a-1}, l_{a+1}\}$, $(a \in N)$, is said to be almost Lucas graceful labeling if the induced edge labeling $f_1(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{l_1, l_2, \cdots, l_q\} or \{l_1, l_2, \cdots, l_{q-1}, l_{q+1}\}$ with the assumption of $l_0 = 0, l_1 = 1, l_2 = 3, l_3 = 4, l_4 = 7, l_5 = 11$, etc.. Then G is called almost Lucas graceful graph if it admits almost Lucas graceful labeling. Also, an injective function $f : V(G) \rightarrow \{l_0, l_1, l_2, \cdots, l_a\}$, $(a \in N)$, is said to be nearly Lucas graceful labeling if the induced edge labeling $f_1(u, v) = |f(u) - f(v)|$ onto the set $\{l_1, l_2, \cdots, l_{i-1}, l_{i+1}, l_{i+2}, \cdots, l_{j-1}, l_{j+1}, l_{j+2}, \cdots, l_{k-1}, l_{k+1}, l_{k+2}, \cdots, l_b$ ($b \in N$ and $b \leq a$) with the assumption of $l_0 = 0, l_1 = 1, l_2 = 3, l_3 = 4, l_4 = 7, l_5 = 11$, etc.. If G admits nearly Lucas graceful labeling, then G is said to be nearly Lucas graceful graph. In this paper, we show that the graphs $S_{m,n}, S_{m,n}@P_t$ and $F_m@P_n$ are almost Lucas graceful graphs. Also we show that the graphs $S_{m,n}@P_t$ and C_n are nearly Lucas graceful graphs.

Key Words: Smarandache-Fibonacci triple, super Smarandache-Fibonacci graceful graph, graceful labeling, Lucas graceful labeling, almost Lucas graceful labeling and nearly Lucas graceful labeling.

AMS(2010): 05C78

§1. Introduction

By a graph, we mean a finite undirected graph without loops or multiple edges. A cycle of length n is denoted by C_n $\cdot G^+$ is a graph obtained from the graph G by attaching pendant vertex to each vertex of G. The concept of graceful labeling was introduced by Rosa [3] in 1967. A function f is called a graceful labeling of a graph G with q edges if f is an injection from the vertices of G to the set $\{1, 2, 3, \cdots, q\}$ such that when each edge uv is assigned the label

[1]Received May 26, 2011. Accepted September 6, 2011.

$|f(u) - f(v)|$, the resulting edge labels are distinct. The notion of Fibonacci graceful labeling was introduced by K.M.Kathiresan and S.Amutha [4]. We call a function f, a Fibonacci graceful label labeling of a graph G with q edges if f is an injection from the vertices of G to the set $\{0, 1, 2, \cdots, F_q\}$, where F_q is the q^{th} Fibonacci number of the Fibonacci series $F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, \cdots$ and each edge uv is assigned the label $|f(u) - f(v)|$. Based on the above concept we define the following.

A *Smarandache-Fibonacci triple* is a sequence $S(n)$, $n \geq 0$ such that $S(n) = S(n-1) + S(n-2)$, where $S(n)$ is the Smarandache function for integers $n \geq 0$. Clearly, it is a generalization of *Fibonacci sequence* and *Lucas sequence*. Let G be a (p, q)-graph and $\{S(n)|n \geq 0\}$ a Smarandache-Fibonacci triple. An bijection $f\colon V(G) \rightarrow \{S(0), S(1), S(2), \ldots, S(q)\}$ is said to be a *super Smarandache-Fibonacci graceful graph* if the induced edge labeling $f^*(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{S(1), S(2), \ldots, S(q)\}$. Particularly, if $S(n), n \geq 0$ is just the Lucas sequence, such a labeling $f : V(G) \rightarrow \{l_0, l_1, l_2, \cdots, l_a\}$ $(a \in N)$ is said to be *Lucas graceful labeling* if the induced edge labeling $f_1(uv) = |f(u) - f(v)|$ is a bijection on to the set $\{l_1, l_2, \cdots, l_q\}$. If G admits Lucas graceful labeling, then G is said to be *Lucas graceful graph*. An injective function $f : V(G) \rightarrow \{l_0, l_1, l_2, \cdots, l_{a-1}, l_{a+1}\}$, $(a \in N)$, is said to be *almost Lucas graceful labeling* if the induced edge labeling $f_1(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{l_1, l_2, \cdots, l_q\}$ or $\{l_1, l_2, \cdots, l_{q-1}, l_{q+1}\}$ with the assumption of $l_0 = 0, l_1 = 1, l_2 = 3, l_3 = 4, l_4 = 7, l_5 = 11$, etc.. Then G is called *almost Lucas graceful graph* if it admits almost Lucas graceful labeling. Also, an injective function $f : V(G) \rightarrow \{l_0, l_1, l_2, \cdots, l_a\}$, $(a \in N)$, is said to be *nearly Lucas graceful labeling* if the induced edge labeling $f_1(u, v) = |f(u) - f(v)|$ onto the set $\{l_1, l_2, \cdots, l_{i-1}, l_{i+1}, l_{i+2}, \cdots, l_{j-1}, l_{j+1}, l_{j+2}, \cdots, l_{k-1}, l_{k+1}, l_{k+2}, \cdots, l_b$ $(b \in N$ and $b \leq a)$ with the assumption of $l_0 = 0, l_1 = 1, l_2 = 3, l_3 = 4, l_4 = 7, l_5 = 11$, etc.. If G admits nearly Lucas graceful labeling, then G is said to be *nearly Lucas graceful graph*. In this paper, we show that the graphs $S_{m,n}, S_{m,n}@P_t$ and $F_m@P_n$ are almost Lucas graceful graphs. Also we show that the graphs $S_{m,n}@P_t$ and C_n are nearly Lucas graceful graphs.

§2. Almost Lucas Graceful Graphs

In this section, we show that some graphs namely $S_{m,n}, S_{m,n}@P_t$ and $F_m@P_n$ are almost Lucas graceful graphs.

Definition 2.1 *Let G be a (p, q) - graph. An injective function $f : V(G) \rightarrow \{l_0, l_1, l_2, \cdots, l_{a-1}, l_{a+1}\}$, $a \in N$, is said to be almost Lucas graceful labeling if the induced edge labeling $f_1(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{l_1, l_2, \cdots, l_q\}$ or $\{l_1, l_2, \cdots, l_{q-1}, l_{q+1}\}$. Then G is called almost Lucas graceful graph if it admits almost Lucas graceful labeling.*

Definition 2.2 ([2]) *$S_{m,n}$ denotes a star with n spokes in which each spoke is a path of length m.*

Theorem 2.3 *$S_{m,n}$ is an almost Lucas graceful graph when $m \equiv 1(mod\ 2)$ and $n \equiv 0(mod\ 3)$*

Proof Let $G = S_{m,n}$. Let $V(G) = \{u_{i,j} : 1 \leq i \leq m$ and $1 \leq j \leq n\}$ be the vertex set of

G. Let $E(G) = \{u_0u_{i,1} : 1 \leq i \leq m\} \cup \{u_{i,j}u_{i,j+1} : 1 \leq i \leq m,\ 1 \leq j \leq n-1\}$ be the edge set of G. So, $|V(G)| = mn + 1$ and $|E(G)| = mn$. Define $f : V(G) \to \{l_0, l_1, l_2, \cdots, l_a\}, a \in N$ by $f(u_0) = l_0$. For $i = 1, 2, \cdots, m-2$ and $i \equiv 1(mod\ 2)$, $f(u_{i,j}) = l_{n(i-1)+2j-1}, 1 \leq j \leq n$. For $i = 1, 2, \cdots, m-1$ and $i \equiv 0(mod\ 2)$, $f(u_{i,j}) = l_{ni+2-2j},\ 1 \leq j \leq n$. For $s = 1, 2, \cdots, \dfrac{n-3}{3}$ $f(u_{m,j}) = l_{(m-1)n+2(j+1)-3s},\ 3s-2 \leq j \leq 3s$. and for $s = \dfrac{n}{3}$, $f(u_{m,j}) = l_{(m-1)n+2(j+1)-3s},\ 3s-2 \leq j \leq 3s-1$. We claim that the edge labels are distinct. Let

$$\begin{aligned} E_1 &= \bigcup_{\substack{i=1 \\ i\equiv 1(mod\ 2)}}^{m} \{f_1(u_0u_{i,1})\} = \bigcup_{\substack{i=1 \\ i\equiv 1(mod\ 2)}}^{m} \{|f(u_0) - f(u_{i,1})|\} \\ &= \bigcup_{\substack{i=1 \\ i\equiv 1(mod\ 2)}}^{m} \left\{\left|l_0 - l_{n(i-1)+1}\right|\right\} = \bigcup_{\substack{i=1 \\ i\equiv 1(mod\ 2)}}^{m} \left\{l_{n(i-1)+1}\right\} \\ &= \left\{l_1, l_{2n+1}, l_{4n+1}, \cdots, l_{(m-1)n+1}\right\}, \end{aligned}$$

$$\begin{aligned} E_2 &= \bigcup_{\substack{i=1 \\ i\equiv 0(mod\ 2)}}^{m} \{f_1(u_0u_{i,1})\} = \bigcup_{\substack{i=1 \\ i\equiv 0(mod\ 2)}}^{m} \{|f(u_0) - f(u_{i,1})|\} \\ &= \bigcup_{\substack{i=1 \\ i\equiv 0(mod\ 2)}}^{m} \{|l_0 - l_{ni}|\} = \bigcup_{\substack{i=1 \\ i\equiv 0(mod\ 2)}}^{m} \{l_{ni}\} \\ &= \left\{l_{2n}, l_{4n}, ..., l_{(m-1)n}\right\}, \end{aligned}$$

$$\begin{aligned} E_3 &= \bigcup_{\substack{i=1 \\ i\equiv 1(mod\ 2)}}^{m-2} \bigcup_{j=1}^{n-1} \{f_1(u_{i,j}u_{i,j+1})\} = \bigcup_{\substack{i=1 \\ i\equiv 1(mod\ 2)}}^{m-2} \bigcup_{j=1}^{n-1} \{|f(u_{i,j}) - f(u_{i,j+1}|\} \\ &= \bigcup_{\substack{i=1 \\ i\equiv 1(mod\ 2)}}^{m-2} \bigcup_{j=1}^{n-1} \left\{\left|l_{n(i-1)+2j-1} - l_{n(i-1)+2j+1}\right|\right\} = \bigcup_{\substack{i=1 \\ i\equiv 1(mod\ 2)}}^{m-2} \bigcup_{j=1}^{n-1} \left\{l_{n(i-1)+2j}\right\} \\ &= \bigcup_{\substack{i=1 \\ i\equiv 1(mod\ 2)}}^{m-2} \left\{l_{n(i-1)+2}, l_{n(i-1)+4}, \cdots, l_{n(i-1)+2n-2}\right\} \\ &= \{l_2, l_4, \cdots, l_{2n-2}\} \bigcup \{l_{2n+2}, l_{2n+4}, \cdots, l_{4n-2}\} \bigcup \\ &\quad \cdots \bigcup \left\{l_{(m-3)n+2}, l_{(m-3)n+4}, \cdots, l_{(m-3)n+2n-2}\right\} \\ &= \left\{l_2, l_4, \cdots, l_{2n-2}, l_{2n+4}, \cdots, l_{4n-2}, \cdots, l_{(m-3)n+2}, l_{(m-3)n+4}, \cdots, l_{mn-n-2}\right\}, \end{aligned}$$

$$\begin{aligned}
E_4 &= \bigcup_{\substack{i=1\\ i\equiv 0(mod\ 2)}}^{m-1} \bigcup_{j=1}^{n-1} \{f_1(u_{i,j}u_{ij+1})\} = \bigcup_{\substack{i=1\\ i\equiv 0(mod\ 2)}}^{m-1} \bigcup_{j=1}^{n-1} \{|f(u_{i,j}) - f(u_{i,j+1})|\} \\
&= \bigcup_{\substack{i=1\\ i\equiv 0(mod\ 2)}}^{m-1} \bigcup_{j=1}^{n-1} \{|l_{ni-2j+2} - l_{ni-2j}|\} = \bigcup_{\substack{i=1\\ i\equiv 0(mod\ 2)}}^{m-1} \bigcup_{j=1}^{n-1} \{l_{ni-2j+1}\} \\
&= \bigcup_{\substack{i=1\\ i\equiv 0(mod\ 2)}}^{m-1} \bigcup_{j=1}^{n-1} \{l_{ni-1}, l_{ni-3}, \cdots, l_{ni-2n+3}\} \\
&= \{l_{2n-1}, l_{2n-3}, \cdots, l_3\} \bigcup \{l_{4n-1}, l_{4n-3}, \cdots, l_{2n+3}\} \bigcup \\
&\quad \cdots \bigcup \{l_{(m-1)n-1}, l_{(m-1)n-3}, \cdots, l_{mn-3n+3}\} \\
&= \left\{l_{2n-1}, l_{2n-3}, \cdots, l_3, l_{4n-1}, l_{4n-3}, \cdots, l_{2n+3}, \cdots, l_{(m-1)n-1}, l_{(m-1)n-3}, \cdots, l_{mn-3n+3}\right\},
\end{aligned}$$

$$\begin{aligned}
E_5 &= \bigcup_{s=1}^{\frac{n-3}{3}} \{f_1(u_{m,j}u_{m,j+1}) : 3s-2 \le j \le 3s-1\} \\
&= \bigcup_{s=1}^{\frac{n-3}{3}} \{|f(u_{m,j}) - f(u_{m,j+1})| : 3s-2 \le j \le 3s-1\} \\
&= \bigcup_{s=1}^{\frac{n-3}{3}} \left\{\left|l_{n(m-1)+2j-3s+2} - l_{n(m-1)+2j-3s+4}\right| : 3s-2 \le j \le 3s-1\right\} \\
&= \bigcup_{s=1}^{\frac{n-3}{3}} \left\{l_{n(m-1)+2j-3s+3} : 3s-2 \le j \le 3s-1\right\} \\
&= \left\{l_{n(m-1)+2}, l_{n(m-1)+4}\right\} \bigcup \left\{l_{n(m-1)+5}, l_{n(m-1)+7}\right\} \bigcup \\
&\quad \cdots \bigcup \left\{l_{n(m-1)+2n-10-n+3+3}, l_{n(m-1)+2n-8-n+3+3}\right\} \\
&= \left\{l_{n(m-1)+2}, l_{n(m-1)+4}, l_{n(m-1)+5}, l_{n(m-1)+7}, \cdots, l_{n(m-1)+n-4}, l_{n(m-1)+n-2}\right\} \\
&= \left\{l_{n(m-1)+2}, l_{n(m-1)+4}, l_{n(m-1)+5}, l_{n(m-1)+7}, \cdots, l_{mn-4}, l_{mn-2}\right\}.
\end{aligned}$$

We find the edge labeling between the end vertex of s^{th} loop and the starting vertex of $(s+1)^{th}$ loop and $s = 1, 2, \cdots, \dfrac{n-3}{3}$. Let

$$\begin{aligned}
E_6 &= \bigcup_{s=1}^{\frac{n-3}{3}} \{f_1(u_{m,j}u_{m,j+1}) : j = 3s\} = \bigcup_{s=1}^{\frac{n-3}{3}} \{|f(u_{m,j}) - f(u_{m,j+1}) : j = 3s|\} \\
&= \{|f(u_{m,3}) - f(u_{m,4})|, |f(u_{m,6}) - f(u_{m,7})|, \cdots, |f(u_{m,n-3}) - f(u_{m,n-2})|\} \\
&= \left\{\left|l_{(m-1)n+8-3} - l_{(m-1)n+10-6}\right|, \left|l_{(m-1)n+14-6} - l_{(m-1)n+16-9}\right|,\right. \\
&\quad \left. \cdots, \left|l_{(m-1)n+2n-4-n+3} - l_{(m-1)n+2n-2-n}\right|\right\} \\
&= \left\{\left|l_{(m-1)n+5} - l_{(m-1)n+4}\right|, \left|l_{(m-1)n+8-l}l_{(m-1)}n+7\right|, \cdots, \left|l_{(m-1)n+n-1} - l_{(m-1)n+n-2}\right|\right\} \\
&= \{|l_{(m-1)n+3}, l_{(m-1)n+6}, \cdots, l_{(m-1)n+n-3}|\} \\
&= \left\{l_{(m-1)n+3}, l_{(m-1)n+6}, \cdots, l_{mn-3}\right\}.
\end{aligned}$$

For $s=\dfrac{n}{3}$, let

$$\begin{aligned}
E_7 &= \{f_1(u_{m,j}u_{m,j+1}) : 3s-2 \le j \le 3s-1\} = \{|f(u_{m,j}) - f(u_{m,j+1})| : 3s-2 \le j \le 3s-1\} \\
&= \left\{\left|l_{(m-1)n+2n-2-n} - l_{(m-1)n+2n-n}\right|, \left|l_{(m-1)n+2n-n} - l_{(m-1)n+2n+2-n}\right|\right\} \\
&= \left\{\left|l_{(m-1)n+n-2} - l_{(m-1)n+n}\right|, \left|l_{(m-1)n+n} - l_{(m-1)n+n+2}\right|\right\} \\
&= \left\{l_{(m-1)n+n-1}, l_{(m-1)n+n+1}\right\} = \{l_{mn-1}, l_{mn+1}\}.
\end{aligned}$$

Now, $E=\bigcup_{i=1}^{7} E_i = \{l_1, l_2, ..., l_{mn-1}, l_{mn+1}\}$. So, the edge labels of G are distinct. Therefore, f is an almost Lucas graceful labeling. Thus $G = S_{m,n}$ is an almost Lucas graceful graph, when $m \equiv 1(mod\ 2)$ and $n \equiv 0(mod\ 3)$. □

Example 2.4 An almost Lucas graceful labeling of $S_{7,9}$ is shown in Fig.2.1.

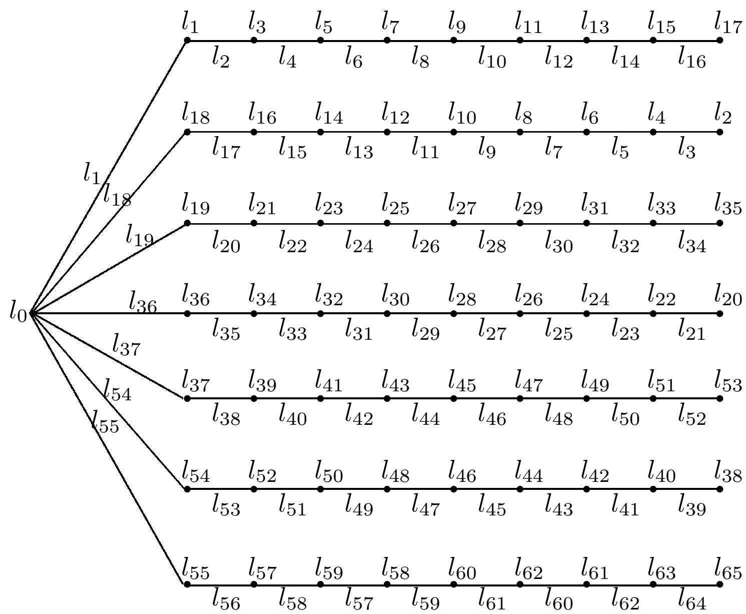

Fig.2.1 $S_{7,9}$

Definition 2.5([2]) *The graph $G = S_{m,n}@P_t$ consists of $S_{m,n}$ and a path P_t of length t which is attached with the maximum degree of the vertex of $S_{m,n}$.*

Theorem 2.6 *$S_{m,n}@P_t$ is an almost Lucas graceful graph when $m \equiv 0(mod\ 2)$ and $t \equiv 0(mod\ 3)$.*

Proof Let $G = S_{m,n}@P_t$ with $m \equiv 0(mod\ 3)$ and $t \equiv 0(mod\ 3)$. Let

$$V(G) = \{u_0, u_{i,j} : 1 \le i \le m \text{ and } 1 \le j \le n\} \bigcup \{v_k : 1 \le k \le t\},$$

$$\begin{aligned}
E(G) &= \{u_0u_{i,1} : 1 \le i \le m\} \bigcup \{u_{i,j}u_{ij+1} : 1 \le i \le m \text{ and } 1 \le j \le n-1\} \\
&\bigcup \{u_0v_1\} \bigcup \{v_kv_{k+1} : 1 \le k \le t-1\}
\end{aligned}$$

be the vertex set and edge set of G, respectively. Thus $|V(G)| = mn+t+1$ and $|E(G)| = mn+t$. Define $f : V(G) \to \{l_0, l_1, l_2, \cdots, l_a\}, a \epsilon N$ by $f(u_0) = l_0$. For $i = 1, 2, \cdots, m$ and for $i \equiv 1(mod\ 2)$, $f(u_{i,j}) = l_{n(i-1)+2j-1}, 1 \leq j \leq n$. For $i = 1, 2, \cdots, m$ and for $i \equiv 1(mod\ 2)$, $f(u_{i,j} = l_{ni-2j+2},\ 1 \leq j \leq n$. For $s = 1, 2, \cdots, \frac{t-3}{3}$, $f(v_k) = l_{mn+2k-3s+2},\ 3s-2 \leq k \leq 3s$ and for $s = \frac{t}{3}$, $f(v_k) = l_{mn+2k-3s+2},\ 3s-2 \leq k \leq 3s-1$. We claim that the edge labels are distinct. Let

$$\begin{aligned} E_1 &= \bigcup_{\substack{i=1\\ i\equiv 1(mod\ 2)}}^{m} \{f_1(u_0u_{i,1})\} \\ &= \bigcup_{\substack{i=1\\ i\equiv 1(mod\ 2)}}^{m} \{|f(u_0) - f(u_{i,1})|\} = \bigcup_{\substack{i=1\\ i\equiv 1(mod\ 2)}}^{m} \left\{\left|l_0 - l_{n(i-1)+1}\right|\right\} \\ &= \bigcup_{\substack{i=1\\ i\equiv 1(mod\ 2)}}^{m} \left\{l_{n(i-1)+1}\right\} = \left\{l_1, l_{2n+1}, l_{4n+1}, \cdots, l_{n(m-1)+1}\right\}, \end{aligned}$$

$$\begin{aligned} E_2 &= \bigcup_{\substack{i=1\\ i\equiv 1(mod\ 2)}}^{m} \{f_1(u_0u_{i,1})\} = \bigcup_{\substack{i=1\\ i\equiv 1(mod\ 2)}}^{m} \{|f(u_0) - f(u_{i,1})|\} \\ &= \bigcup_{\substack{i=1\\ i\equiv 1(mod\ 2)}}^{m} \{|l_0 - l_{ni}|\} = \bigcup_{\substack{i=1\\ i\equiv 1(mod\ 2)}}^{m} \{l_{ni}\} = \{l_{2n}, l_{4n}, \cdots, l_{mn}\}, \end{aligned}$$

$$\begin{aligned} E_3 &= \bigcup_{\substack{i=1\\ i\equiv 1(mod\ 2)}}^{m} \bigcup_{j=1}^{n-1} \{f_1(u_{i,j}u_{i,j+1})\} = \bigcup_{\substack{i=1\\ i\equiv 1(mod\ 2)}}^{m} \bigcup_{j=1}^{n-1} \{|f(u_{i,j}) - f(u_{i,j+1})|\} \\ &= \bigcup_{\substack{i=1\\ i\equiv 1(mod\ 2)}}^{m} \bigcup_{j=1}^{n-1} \left\{\left|l_{n(i-1)+2j-1} - l_{n(i-1)+2j+1}\right|\right\} = \bigcup_{\substack{i=1\\ i\equiv 1(mod\ 2)}}^{m} \bigcup_{j=1}^{n-1} \left\{l_{n(i-1)+2j}\right\} \\ &= \bigcup_{\substack{i=1\\ i\equiv 1(mod\ 2)}}^{m} \left\{l_{n(i-1)+2}, l_{n(i-1)+4}, \cdots, l_{n(i-1)+2n-2}\right\} \\ &= \{l_2, l_4, \cdots, l_{2n-2}\} \bigcup \{l_{2n+2}, l_{2n+4}, \cdots, l_{4n-2}\} \bigcup \\ &\quad \cdots \bigcup \left\{l_{n(m-2)+2}, l_{n(m-2)+4}, \cdots, l_{mn-2}\right\} \\ &= \{l_2, l_4, \cdots, l_{2n-2}, l_{2n+2}, l_{2n+4}, \cdots, l_{4n-2}, \cdots, l_{n(m-2)+2}, l_{n(m-2)+4}, \cdots, l_{mn-2}\}, \end{aligned}$$

$$
\begin{aligned}
E_4 &= \bigcup_{\substack{i=1\\ i\equiv 1(mod\ 2)}}^{m} \bigcup_{j=1}^{n-1} \{f_1(u_{i,j}u_{i,j+1})\} = \bigcup_{\substack{i=1\\ i\equiv 1(mod\ 2)}}^{m} \bigcup_{j=1}^{n-1} \{|f(u_{i,j}) - f(u_{i,j+1})|\} \\
&= \bigcup_{\substack{i=1\\ i\equiv 1(mod\ 2)}}^{m} \bigcup_{j=1}^{n-1} \{|l_{ni-2j+2} - l_{ni-2j}|\} \\
&= \bigcup_{\substack{i=1\\ i\equiv 1(mod\ 2)}}^{m} \bigcup_{j=1}^{n-1} \{l_{ni-2j+1}\} = \bigcup_{\substack{i=1\\ i\equiv 0(mod\ 2)}}^{m} \{l_{ni-1}, l_{ni-3}, \cdots, l_{ni-2n+3}\} \\
&= \{l_{2n-1}, l_{2n-3}, \cdots, l_3\} \cup \{l_{4n-1}, l_{4n-3}, ..., l_{2n+3}\} \cup ... \cup \{l_{mn-1}, l_{mn-3}, \cdots, l_{mn-2n+3}\} \\
&= \{l_{2n-1}, l_{2n-3}, \cdots, l_3, l_{4n-1}, l_{4n-3}, \cdots, l_{2n+3}, \cdots, l_{mn-1}, l_{mn-3}, \cdots, l_{mn-2n+3}\},
\end{aligned}
$$

$$
E_1^{'} = \{f_1(u_0v_1)\} = \{|f(u_0) - f(v_1)|\} = \{|l_0 - l_{mn+1}|\} = \{l_{mn+1}\},
$$

$$
\begin{aligned}
E_2^{'} &= \bigcup_{s=1}^{\frac{t-3}{3}} \{f_1(v_kv_{k+1}) : 3s-2 \leq k \leq 3s-1\} \\
&= \bigcup_{s=1}^{\frac{t-3}{3}} \{|f(v_k) - f(v_{k+1}) : 3s-2 \leq k \leq 3s-1|\} \\
&= \bigcup_{s=1}^{\frac{t-3}{3}} \{|l_{mn+2k+2-3s} - l_{mn+2k+4-3s}| : 3s-2 \leq k \leq 3s-1\} \\
&= \bigcup_{s=1}^{\frac{t-3}{3}} \{l_{mn+2k+3-3s} : 3s-2 \leq k \leq 3s-1\} \\
&= \{l_{mn+2}, l_{mn+4}\} \cup \{l_{mn+5}, l_{mn+7}\} \cup \cdots \cup \{l_{mn+t-4}, l_{mn+t-2}\} \\
&= \{l_{mn+2}, l_{mn+4}, l_{mn+5}, l_{mn+7}, \cdots, l_{mn+t-4}, l_{mn+t-2}\}
\end{aligned}
$$

We find the edge labeling between the end vertex of s^{th} loop and the starting vertex of $(s+1)^{th}$ loop for integers $s = 1, 2, \cdots, \dfrac{t-3}{3}$. Let

$$
\begin{aligned}
E_3^{'} &= \bigcup_{s=1}^{\frac{t-3}{3}} \{f_1(u_{3s}u_{3s+1})\} \\
&= \{|f(u_{3s}) - f(u_{3s+1})|\} \\
&= \{|f(u_3) - f(u_4)|, |f(u_6) - f(u_7)|, ..., |f_{(t-3)} - f_{(t-2)}|\} \\
&= \{|l_{mn+8-3} - l_{mn+10-6}|, |l_{mn+14-6} - l_{mn+16-9}|, ..., |l_{mn+2t-4-t+3} - l_{mn+2t-2-t}|\} \\
&= \{|l_{mn+5} - l_{mn+4}|, |l_{mn+8} - l_{mn+7}|, ..., |l_{mn+t-1} - l_{mn+t-2}|\} \\
&= \{l_{mn+3}, l_{mn+6}, ..., l_{mn+t-3}\}.
\end{aligned}
$$

For $s = \dfrac{t}{3}$, let

$$\begin{aligned} E_4^{'} &= \{f_1(v_k v_{k+1}) : 3s-2 \le k \le 3s-1\} \\ &= \{|f(v_k) - f(v_{k+1})| : 3s-2 \le k \le 3s-1\} \\ &= \{|l_{mn+2t-4+2-t} - l_{mn+2t-2+2-t}|, |l_{mn+2t-2+2-t} - l_{mn+2t+2-t}|\} \\ &= \{|l_{mn+t-2} - l_{mn+t+2}| |l_{mn+t} - l_{mn+t+1}|\} = \{l_{mn+t-1}, l_{mn+t+1}\}. \end{aligned}$$

Now, $E = \bigcup_{i=1}^{4}(E_i \bigcup E_i^{'}) = \{l_1, l_2, \cdots, l_{mn}, \cdots, l_{mn+t-1}, l_{mn+t+1}\}$. So, the edge labels of G are distinct. Therefore, f is an almost Lucas graceful graph. Thus $G = S_{m,n}@P_t$ is an almost Lucas graceful graph when $m \equiv 0(mod\ 2)$ and $t \equiv 0(mod\ 3)$.

Example 2.7 An almost Lucas graceful labeling on $S_{4,7}@P_6$ is shown in Fig.2.2.

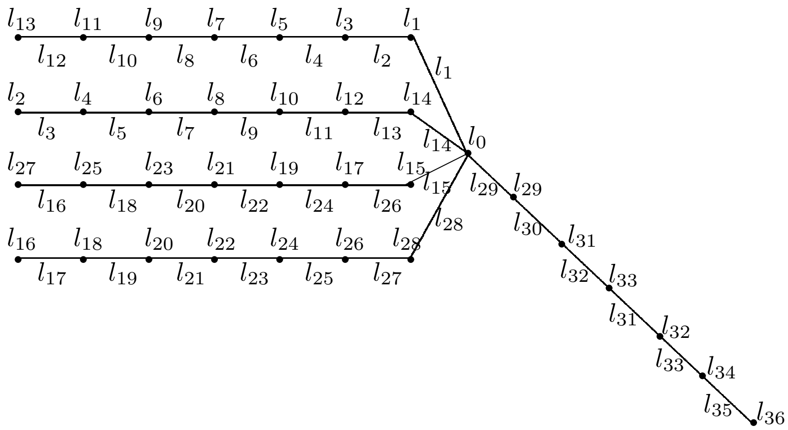

Fig.2.2 $S_{4,7}@P_6$

Definition 2.8([2]) *The graph $G = F_m@P_n$ consists of a fan F_m and a path P_n of length n which is attached with the maximum degree of the vertex of F_m.*

Theorem 2.9 *$F_m@P_n$ is almost Lucas graceful graph when $n \equiv 0(mod\ 3)$.*

Proof Let $v_1, v_2, \cdots, v_{m+1}$ and u_0 be the vertices of a Fan F_m. Let $u_1, u_2, \cdots, u_n$ be the vertices of a path P_n. Let $G = F_m@P_n$, $|V(G)| = m+n+2$ and $|E(G)| = 2m+n+1$. Define $f : V(G) \to \{l_0, l_1, l_2, \cdots, l_{q+2}$ by $f(u_0) = l_0$; $f(v_i) = l_{2i-1}$; $f(u_j) = l_{2m+2j-3s+3}$, $3s-2 \le j \le 3s$. We claim that the edge labels are distinct. Let

$$\begin{aligned} E_1 &= \bigcup_{i=1}^{m} \{f_1(v_i v_{i+1})\} = \bigcup_{i=1}^{m} \{|f(v_i) - f(v_{i+1})|\} \\ &= \bigcup_{i=1}^{m} \{|l_{2i-1} - l_{2i+1}|\} \\ &= \bigcup_{i=1}^{m} \{l_{2i}\} = \{l_2, l_4, ..., l_{2m}\}, \end{aligned}$$

$$\begin{aligned} E_2 &= \bigcup_{i=1}^{m+1}\{f_1(u_0v_i)\} = \bigcup_{i=1}^{m+1}\{|f(u_0)-f(v_i)|\} \\ &= \bigcup_{i=1}^{m+1}\{|l_0-l_{2i-1}|\} = \bigcup_{i=1}^{m+1}\{l_{2i-1}\} = \{l_1, l_3, ..., l_{2m+1}\}, \end{aligned}$$

$$E_3 = \{f_1(u_0u_1)\} = \{|f(u_0)-f(u_1)|\} = \{|l_0-l_{2m+2}|\} = \{l_{2m+2}\},$$

$$\begin{aligned} E_4 &= \bigcup_{s=1}^{\frac{n-3}{3}}\{f_1(u_ju_{j+1}) : 3s-2 \le j \le 3s-1\} \\ &= \bigcup_{s=1}^{\frac{n-3}{3}}\{|f(u_j)-f(u_{j+1})| : 3s-2 \le j \le 3s-1\} \\ &= \{|f(u_1)-f(u_2)|, |f(u_2)-f(u_3)|\}\bigcup\{|f(u_4)-f(u_5)|, |f(u_5)-f(u_6)|\}\bigcup \\ &\quad \cdots\bigcup\{|f(u_{n-5})-f(u_{n-4})|, |f(u_{n-4})-f(u_{n-3})|\} \\ &= \{|l_{2m+2}-l_{2m+4}|, |l_{2m+4}-l_{2m+6}|\}\bigcup\{|l_{2m+5}-l_{2m+7}|, |l_{2m+7}-l_{2m+9}|\}\bigcup \\ &\quad \cdots\bigcup\{|l_{2m+2n-10+3-n+3}-l_{2m+2n-8+3-n+3}|, |l_{2m+2n-8+3-n+3}l_{2m+2n-6+3-n+3}|\} \\ &= \{l_{2m+3}, l_{2m+5}\} \cup \{l_{2m+6}, l_{2m+8}\} \cup \cdots \cup \{l_{2m+n-3}, l_{2m+n-1}\} \\ &= \{l_{2m+3}, l_{2m+5}, l_{2m+6}, l_{2m+8}, \cdots, l_{2m+n-3}, l_{2m+n-1}\}. \end{aligned}$$

We find the edge labeling between the end vertex of s^{th} loop and the starting vertex of $(s+1)^{th}$ loop for $s = 1, 2, \cdots, \frac{n}{3}-1$. Let

$$\begin{aligned} E_5 &= \bigcup_{s=1}^{\frac{n}{3}-1}\{f_1(u_ju_{j+1}) : j = 3s\} = \bigcup_{s=1}^{\frac{n}{3}-1}\{|f(u_j)-f(u_{j+1})| : j = 3s\} \\ &= \{|l_{2m+6+3-3}-l_{2m+8+3-6}|, |l_{2m+12+3-6}-l_{2m+14+3-9}|, \\ &\quad \cdots, |l_{2m+2n-6+3-n+3}-l_{2m+2n-4+3-n}|\} \\ &= \{|l_{2m+6}-l_{2m+5}|, |l_{2m+9}-l_{2m+8}|, |l_{2m+n}-l_{2m+n-1}|\} \\ &= \{l_{2m+4}, l_{2m+7}, \cdots, l_{2m+n-2}\}. \end{aligned}$$

For $s = \frac{n}{3}$, let

$$\begin{aligned} E_6 &= \{f_1(u_ju_{j+1}) : 3s-2 \le j \le 3s-1\} \\ &= \{|f(u_j)-f(u_{j+1})| : 3s-2 \le j \le 3s-1\} \\ &= \{|f(u_{n-2})-f(u_{n-1})|, |f(u_{n-1})-f(u_n)|\} \\ &= \{|l_{2m+2n-4+3-n}-l_{2m+2n-2+3-n}|, |l_{2m+2n-2+3-n}-l_{2m+2n+3-n}|\} \\ &= \{|l_{2m+n-1}-l_{2m+n+1}|, |l_{2m+n+1}-l_{2m+n+3}|\} \\ &= \{l_{2m+n}, l_{2m+n+2}\}. \end{aligned}$$

Now, $E = \bigcup_{i=1}^{6} E_i = \{l_1, l_2, \cdots, l_{2m}, l_{2m+1}, l_{2m+2}, \cdots, l_{2m+n-2}, l_{2m+n-1}, l_{2m+n}, l_{2m+n+2}\}$. So, the edge labels of G are distinct. Therefore, f is an almost Lucas graceful labeling.
Thus $G = F_m@P_n$ is an almost Lucas graceful graph when $n \equiv 0(mod\ 3)$. □

Example 2.10 An almost Lucas graceful labeling on $F_5@P_6$ is shown in Fig.2.3.

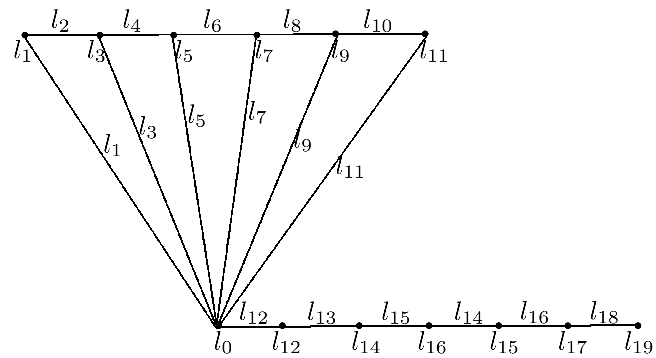

Fig.2.3 $F_5@P_6$

§3. Nearly Lucas Graceful Graphs

In this section, we show that the graphs $S_{m,n}@P_t$ and C_n are nearly Lucas graceful graphs.

Definition 3.1 *Let G be a (p,q) - graph. An injective function $f : V(G) \to \{l_0, l_1, l_2, \cdots, l_a\}$, $(a \in N)$, is said to be nearly Lucas graceful labeling if the induced edge labeling $f_1(u,v) = |f(u) - f(v)|$ onto the set $\{l_1, l_2, \cdots, l_{i-1}, l_{i+1}, l_{i+2}, \cdots, l_{j-1}, l_{j+1}, l_{j+2}, \cdots, l_{k-1}, l_{k+1}, l_{k+2}, \cdots, l_b$ $(b \in N$ and $b \le a)$ with the assumption of $l_0 = 0, l_1 = 1, l_2 = 3, l_3 = 4, l_4 = 7, l_5 = 11$, etc.. If G admits nearly Lucas graceful labeling, then G is said to be nearly Lucas graceful graph.*

Theorem 3.2 *$S_{m,n}@P_t$ is a nearly Lucas graceful graph when $n \equiv 1, 2(mod\ 3)$ $m \equiv 1(mod\ 2)$ and $t = 1, 2(mod\ 3)$*

Proof Let $G = S_{m,n}@P_t$ with $V(G) = \{u_0, u_{i,j} : 1 \le i \le m$ and $1 \le j \le n\} \cup \{v_k : 1 \le k \le t\}$. Let $E(G) = \{u_0u_{i,j} : 1 \le i \le m\} \bigcup \{u_{i,j}u_{i,j+1} : 1 \le i \le m$ and $1 \le j \le n\} \bigcup \{u_0v_1\} \cup \{v_kv_{k+1} : 1 \le k \le t-1\}$ be the edge set of G. So, $|V(G)| = mn + t + 1$ and $|E(G)| = mn + t$. Define $f : V(G) \to \{l_0, l_1, \cdots, l_a\}$, $a \in N$ by $f(u_0) = l_0$. For $i = 1, 2, \cdots, m$ and for $i \equiv 1(mod\ 2)$ $f(u_{i,j}) = l_{n(i-1)+2j-1}$, $1 \le j \le n$. For $i = 1, 2, \cdots, m$ and for $i \equiv 0(mod\ 2)$, $f(u_{i,j}) = l_{in-2j+2}$, $1 \le j \le n$. For $s = 1, 2, \cdots, \dfrac{n-2}{3} - 1$ or $s = 1, 2, \cdots, \dfrac{n-1}{3} - 1$ or $s = 1, 2, 3, \cdots, \dfrac{n}{3} - 1$, $f(u_{m,j}) = l_{mn+2(j+1)-3s}$, $3s - 2 \le j \le 3s$. For $s = \dfrac{n-2}{3}$ or $\dfrac{n-1}{3}$ or $\dfrac{n}{3}$, $f(u_{m,j} = l_{mn+2(j+1)-3s}$, $3s - 2 \le j \le 3s - 1$. For $r = 1, 2, \cdots, \dfrac{t-2}{3}$ or $r = 1, 2, \cdots, \dfrac{t-1}{3}$ or $r = 1, 2, 3, \cdots, \dfrac{t}{3}$, $f(v_k) = l_{mn+2k+3-3r}$, $3r - 2 \le j \le 3r - 1$. We claim that the edge labels are distinct. Let

$$\begin{aligned} E_1 &= \bigcup_{\substack{i=1\\ i\equiv 1(mod\ 2)}}^{m} \{f_1(u_0u_{i,1}\} = \bigcup_{\substack{i=1\\ i\equiv 1(mod\ 2)}}^{m} \{|f(u_0)-f(u_{i,j})|\} \\ &= \bigcup_{\substack{i=1\\ i\equiv 1(mod\ 2)}}^{m} \left\{\left|l_0 - l_{(i-1)n+1}\right|\right\} = \bigcup_{\substack{i=1\\ i\equiv 1(mod\ 2)}}^{m} \left\{l_{(i-1)n+1}\right\} = \left\{l_1, l_{2n+1}, \cdots, l_{(m-1)n+1}\right\}, \end{aligned}$$

$$\begin{aligned} E_2 &= \bigcup_{\substack{i=1\\ i\equiv 0(mod\ 2)}}^{m} \{f_1(u_0u_{i,1})\} = \bigcup_{\substack{i=1\\ i\equiv 0(mod\ 2)}}^{m} \{|f(u_0)-f(u_{i,1})|\} \\ &= \bigcup_{\substack{i=1\\ i\equiv 0(mod\ 2)}}^{m} \{l_0 - l_{in}\} = \bigcup_{\substack{i=1\\ i\equiv 0(mod\ 2)}}^{m} \{l_{in}\} = \left\{l_{2n}, l_{4n}, \cdots, l_{(m-1)n}\right\}, \end{aligned}$$

$$\begin{aligned} E_3 &= \bigcup_{\substack{i=1\\ i\equiv 1(mod\ 2)}}^{m-2} \bigcup_{j=1}^{n-1} \{f_1(u_{i,j}u_{i,j+1})\} = \bigcup_{\substack{i=1\\ i\equiv 1(mod\ 2)}}^{m-2} \bigcup_{j=1}^{n-1} \{|f(u_{i,j})-f(u_{i,j+1})|\} \\ &= \bigcup_{\substack{i=1\\ i\equiv 1(mod\ 2)}}^{m-2} \bigcup_{j=1}^{n-1} \left\{\left|l_{(i-1)n+2j-1} - l_{(i-1)n+2j+1}\right|\right\} = \bigcup_{\substack{i=1\\ i\equiv 1(mod\ 2)}}^{m-2} \bigcup_{j=1}^{n-1} \left\{l_{(i-1)n+2j}\right\} \\ &= \bigcup_{\substack{i=1\\ i\equiv 1(mod\ 2)}}^{m-2} \left\{l_{(i-1)n+2}, l_{(i-1)n+4}, \cdots, l_{(i-1)n+2n-2}\right\} \\ &= \{l_2, l_4, \cdots, l_{2n-2}\} \bigcup \{l_{2n+2}, l_{2n+4}, \cdots, l_{4n-2}\} \bigcup \\ &\quad \cdots \bigcup \left\{l_{(m-3)n+2}, l_{(m-3)n+4}, \cdots, l_{mn-n-2}\right\} \\ &= \left\{l_2, l_4, \cdots, l_{2n-2}, l_{2n+2}, l_{2n+4}, \cdots, l_{4n-2}, \cdots, l_{(m-3)+2}, l_{(m-3)n+4}, \cdots, l_{mn-n-2}\right\}, \end{aligned}$$

$$\begin{aligned} E_4 &= \bigcup_{\substack{i=1\\ i\equiv 0(mod\ 2)}}^{m-1} \bigcup_{j=1}^{n-1} \{f_1(u_{i,j}u_{i,j+1})\} = \bigcup_{\substack{i=1\\ i\equiv 0(mod\ 2)}}^{m-1} \bigcup_{j=1}^{n-1} \{|f(u_{i,j})-f(u_{i,j+1})|\} \\ &= \bigcup_{\substack{i=1\\ i\equiv 0(mod\ 2)}}^{m-1} \bigcup_{j=1}^{n-1} \{|l_{ni-2j+2} - l_{ni-2j}|\} = \bigcup_{\substack{i=1\\ i\equiv 0(mod\ 2)}}^{m-1} \bigcup_{j=1}^{n-1} \{l_{ni-2j+1}\} \\ &= \bigcup_{\substack{i=1\\ i\equiv 0(mod\ 2)}}^{m-1} \{l_{in-1}, l_{in-3}, \cdots, l_{in-2n+3}\} \\ &= \{l_{2n-1}, l_{2n-3}, \cdots, l_3\} \bigcup \{l_{4n-1}, l_{4n-3}, \cdots, l_{2n+3}\} \bigcup \\ &\quad \cdots \bigcup \left\{l_{(m-1)n-1}, l_{(m-1)n-3}, \cdots, l_{mn-3n+3}\right\} \\ &= \left\{l_{2n-1}, l_{2n-3}, \cdots, l_{4n-1}, l_{4n-3}, \cdots, l_{2n+3}, \cdots, l_{(m-1)n-1}, l_{(m-1)n-3}, \cdots, l_{mn-3n+3}\right\}. \end{aligned}$$

For $n \equiv 1(mod\ 3)$ and $s = 1, 2, \cdots, \dfrac{n-4}{3}$, let

$$\begin{aligned} E_5 &= \bigcup_{s=1}^{\frac{n-4}{3}} \{f_1(u_{m,j}u_{m,j+1}) : 3s-2 \le j \le 3s-1\} \\ &= \bigcup_{s=1}^{\frac{n-4}{3}} \{|f(u_{m,j}) - f(u_{m,j+1})| : 3s-2 \le j \le 3s-1\} \\ &= \bigcup_{s=1}^{\frac{n-4}{3}} \left\{\left|l_{(m-1)n+2j-3s+2} - l_{(m-1)n+2j-3s+4}\right| : 3s-2 \le j \le 3s-1\right\} \end{aligned}$$

$$\begin{aligned} &= \bigcup_{s=1}^{\frac{n-4}{3}} \left\{l_{(m-1)n+2j-3s+3} : 3s-2 \le j \le 3s-1\right\} \\ &= \left\{l_{(m-1)n+2}, l_{(m-1)n+4}\right\} \cup \left\{l_{(m-1)n+5}, l_{(m-1)n+7}\right\} \cup ... \cup \left\{l_{(m-1)n+n-4}, l_{(m-1)n+n-2}\right\} \\ &= \left\{l_{(m-1)n+2}, l_{(m-1)n+4}, l_{(m-1)n+5}, l_{(m-1)n+7}, ..., l_{mn-4}, l_{mn-2}\right\}. \end{aligned}$$

We find the edge labeling between the end vertex of s^{th} loop and the starting vertex of $(s+1)^{th}$ loop for integers $s = 1, 2, \cdots, \dfrac{n-4}{3}$. Let

$$\begin{aligned} E_6 &= \bigcup_{s=1}^{\frac{n-1}{3}} \{f_1(u_{m,j}u_{m,j+1}) : j = 3s\} = \bigcup_{s=1}^{\frac{n-1}{3}} \{|f(u_{m,j}) - f(u_{m,j+1})| : j = 3s\} \\ &= \{|f(u_{m,3}) - f(u_{m,4})|, |f(u_{m,6}) - f(u_{m,7})|, \cdots, |f(u_{m,n-1}) - f(u_{m,n})|\} \\ &= \left\{\left|l_{(m-1)n+5} - l_{(m-1)n+4}\right|, \left|l_{(m-1)n+7}\right|\ , \cdots, \left|l_{(m-1)n+2n-2-n+1} - l_{(m-1)n+2n+2-n-2}\right|\right\} \\ &= \left\{l_{(m-1)n+3}, l_{(m-1)n+6}, \cdots, l_{mn-1}\right\}. \end{aligned}$$

For $s = \dfrac{n-1}{3}$, Let

$$\begin{aligned} E_7 &= \{f_1(u_{m,j}u_{m,j+1}) : 3s-2 \le j \le 3s-1\} \\ &= \{|f(u_{m,j}) - f(u_{m,j+1})| : 3s-2 \le j \le 3s-1\} \\ &= \left\{\left|l_{(m-1)n+2n-6+2-n+1} - l_{(m-1)n+2n-4+2-n+1}\right|,\right. \\ &\qquad \left.\left|l_{(m-1)n+2n-4+2-n+1} - l_{(m-1)n+2n-2+2-n+1}\right|\right\} \\ &= \{|l_{mn-3} - l_{mn-1}|, |l_{mn-1} - l_{mn+1}|\} = \{l_{mn-2}, l_{mn}\} \end{aligned}$$

Now, $E=\bigcup\limits_{i=1}^{7}E_i=\{l_1,l_2,\cdots,l_{mn}\}$. For $n\equiv 2(mod\ 3)$ and integers $s=1,2,\cdots,\dfrac{n-2}{3}$,

$$\begin{aligned}
E_1^{'} &= \bigcup_{s=1}^{\frac{n-2}{3}}\{f_1(u_{m,j}u_{m,j+1}):3s-2\le j\le 3s-1\}\\
&= \bigcup_{s=1}^{\frac{n-2}{3}}\{|f(u_{m,j})-f(u_{m,j+1})|:3s-2\le j\le 3s-1\}\\
&= \bigcup_{s=1}^{\frac{n-2}{3}}\left\{\left|l_{(m-1)n+2j+2-3s}-l_{(m-1)n+2j+4-3s}\right|:3s-2\le j\le 3s-1\right\}\\
&= \bigcup_{s=1}^{\frac{n-2}{3}}\left\{l_{(m-1)n+2j+3-3s}:3s-2\le j\le 3s-1\right\}\\
&= \left\{l_{(m-1)n+2},l_{(m-1)n+4}\right\}\cup\left\{l_{(m-1)n+5},l_{(m-1)n+7}\right\}\cup...\cup\left\{l_{(m-1)n+n-3},l_{(m-1)n+n-1}\right\}\\
&= \left\{l_{(m-1)n+2},l_{(m-1)n+4},l_{(m-1)n+5},l_{(m-1)n+7},\cdots,l_{mn-3},l_{mn-1}\right\}
\end{aligned}$$

We find the edge labeling between the end vertex of s^{th} loop and the starting vertex of $s+1^{th}$ loop for integers $s=1,2,\cdots,\dfrac{n-2}{3}$. Let

$$\begin{aligned}
E_2^{'} &= \bigcup_{s=1}^{\frac{n-2}{3}}\{f_1(u_{m,j}u_{m,j+1})j=3s\}=\bigcup_{s=1}^{\frac{n-2}{3}}\{|f(u_{m,j})-f(u_{m,j+1})|:j=3s\}\\
&= \{|f(u_{m,3})-f(u_{m,4})|,|f(u_{m,6})-f(u_{m,7})|,\cdots,|f(u_{m,n-2})-f(u_{m,n-1}|\}\\
&= \left\{\left|l_{(m-1)n+8-3}-l_{(m-1)n+10-6}\right|,\left|l_{(m-1)n+14-6}-l_{(m-1)n+16-9}\right|,\right.\\
&\qquad \left.\cdots,\left|l_{(m-1)n+2n-2-n+2}-l_{(m-1)n+2n-n-1}\right|\right\}\\
&= \left\{\left|l_{(m-1)n+5}-l_{(m-1)n+4}\right|,\left|l_{(m-1)n+8}-l_{(m-1)n+7}\right|,\right.\\
&\qquad \left.\cdots,\left|l_{(m-1)n+n}-l_{(m-1)n+n-1}\right|\right\}\\
&= \left\{l_{(m-1)n+3},l_{(m-1)n+6},\cdots,l_{mn-2}\right\}.
\end{aligned}$$

For $s=\dfrac{n+1}{3}$, let

$$\begin{aligned}
E_3^{'} &= \{f_1(u_{m,j}u_{m,j+1}):j=3s-2\}=\{|f(u_{m,j})-f(u_{m,j+1})|:j=n-1\}\\
&= \{|f(u_{m,n-1})-f(u_{m,n})|\}=\left\{\left|l_{(m-1)n+2n-n-1}-l_{(m-1)n+2n+2-n-1}\right|\right\}\\
&= \{|l_{mn-1}-l_{mn+1}|\}=\{l_{mn}\}.
\end{aligned}$$

Therefore, $E^{'}=\bigcup\limits_{i=1}^{3}E_i^{'}$. Let

$$E_0=\{f_1(u_0v_1)\}=\{|f(u_0)-f(v_1)|\}=\{|l_0-l_{mn+2}|\}=\{l_{mn+2}\}.$$

For $t \equiv 2(mod\ 3)$ and $r = 1, 2, \cdots, \dfrac{t-2}{3}$, let

$$\begin{aligned}
E_1^{"} &= \bigcup_{r=1}^{\frac{t-2}{3}} \{f_1(v_k v_{k+1}) : 3r-2 \le k \le 3r-1\} \\
&= \bigcup_{r=1}^{\frac{t-2}{3}} \{|f(v_k) - f(v_{k+1})| : 3r-2 \le k \le 3r-1\} \\
&= \{|f(v_1) - f(v_2)|, |f(v_2) - f(v_3)|\} \bigcup \{|f(v_4) - f(v_5)|, |f(v_5) - f(v_6)|\} \bigcup \\
&\quad \cdots \bigcup \{|f(v_{t-4}) - f(v_{t-3})|, |f(v_{t-3}) - f(v_{t-2})|\} \\
&= \{|l_{mn+3+2-3} - l_{mn+3+4-3}|, |l_{mn+3+4-3} - l_{mn+3+6-3}|\} \bigcup \\
&\quad \{|l_{mn+8+3-6} - l_{mn+10+3-6}|, |l_{mn+10+3-6} - l_{mn+12+3-6}|\} \bigcup \\
&\quad \cdots \bigcup \{|l_{mn+3+2t-8-t+2} - l_{mn+3+2t-6-t+2}|, |l_{mn+3+2t-6-t+2} - l_{mn+3+2t-4-t+2}|\} \\
&= \{|l_{mn+2} - l_{mn+4}|, |l_{mn+4} - l_{mn+6}|\} \bigcup \{|l_{mn+5} - l_{mn+7}|, |l_{mn+7} = l_{mn+9}|\} \bigcup \\
&\quad \cdots \bigcup \{|l_{mn+t-3} - l_{mn+t-1}|, |l_{mn+t-1} - l_{mn+t+1}|\} \\
&= \{l_{mn+3}, l_{mn+5}\} \bigcup \{l_{mn+6}, l_{mn+8}\} \bigcup \cdots \bigcup \{l_{mn+t-2}, l_{mn+t}\} \\
&= \{l_{mn+3}, l_{mn+5}, l_{mn+6}, l_{mn+8}, \cdots, l_{mn+t-2}, l_{mn+t}\}.
\end{aligned}$$

We find the edge labeling between the end vertex of r^{th} loop and the starting vertex of $(r+1)^{th}$ loop for integers $r = 1, 2, \cdots, \dfrac{t-2}{3}$. Let

$$\begin{aligned}
E_2^{"} &= \bigcup_{r=1}^{\frac{t-2}{3}} \{f_1(v_k v_{k+1}) : k = 3r\} = \bigcup_{r=1}^{\frac{t-2}{3}} \{|f(v_k) - f(v_{k+1})| : k = 3r\} \\
&= \{|f(v_3) - f(v_4)|, |f(v_6) - f(v_7)|, \cdots, |f(v_{t-2}) - f(v_{t-1})|\} \\
&= \{|l_{mn+3+6-3} - l_{mn+3+8-6}|, |l_{mn+3+12-6} - l_{mn+3+14-9}|, \\
&\quad \cdots, |l_{mn+3+2t-4-t+2} - l_{mn+3+2t-2-t-1}|\} \\
&= \{|l_{mn+6} - l_{mn+5}|, |l_{mn+9} - l_{mn+8}|, ..., |l_{mn+t+1} - l_{mn+t}|\} \\
&= \{l_{mn+4}, l_{mn+7}, \cdots, l_{mn+t-1}\}.
\end{aligned}$$

For $s = \dfrac{t+1}{3}$, let

$$\begin{aligned}
E_3^{"} &= \{f_1(v_k v_{k+1}) : k = 3r-2\} = \{|f(v_k) - f(v_{k+1})| : k = 3r-2\} \\
&= \{|l_{mn+3+2t-2-t-1} - l_{mn+3+2t-t-1}|\} = \{|l_{mn+t} - l_{mn+t+2}|\} = \{l_{mn+t+1}\}
\end{aligned}$$

Therefore, $E" = E_0 \bigcup E_1^{"} \bigcup E_2^{"} \bigcup E_3^{"} = \{l_{mn+2}, l_{mn+3}, l_{mn+5}, l_{mn+6}, l_{mn+8}, \cdots, l_{mn+t-2}, l_{mn+t}, l_{mn+t+1}, l_{mn+4}, l_{mn+7}, \cdots, l_{mn+t-1}\}$. Now, $E \bigcup E^{"} = \bigcup_{i=1}^{7} E_i \cup E_0 \cup E_1^{"} \cup E_2^{"} \cup E_3^{"} = \{l_1, l_2, \cdots, l_{mn}, l_{mn+2}, l_{mn+3}, l_{mn+4}, \cdots, l_{mn+t-2}, l_{mn+t-1}, l_{mn+t}, l_{mn+t+1}\}$. So, the edge labels of G are distinct. For $t \equiv 1(mod\ 3)$ and integers $r = 1, 2, \cdots, \dfrac{t-1}{3}$, let

$$\begin{aligned}
E_1^{'''} &= \bigcup_{r=1}^{\frac{t-1}{3}} \{f_1(v_k v_{k+1}) : 3r-2 \leq k \leq 3r-1\} \\
&= \bigcup_{r=1}^{\frac{t-1}{3}} \{|f(v_k) - f(v_{k+1})| : 3r-2 \leq k \leq 3r-1\} \\
&= \{|f(v_1) - f(v_2)|, |f(v_2) - f(v_3)|\} \bigcup \{|f(v_4) - f(v_5)|, |f(v_5) - f(v_6)|\} \bigcup \\
&\quad \cdots \bigcup \{|f(v_{t-3}) - f(v_{t-2})|, |f(v_{t-2}) - f(v_{t-1})|\} \\
&= \{|l_{mn+3+2-3} - l_{mn+3+4-3}|, |l_{mn+3+4-3} - l_{mn+3+6-3}|\} \\
&\quad \bigcup \{|l_{mn+3+8-6} - l_{mn+3+10-6}|, |l_{mn+3+10-6} - l_{mn+3+12-6}|\} \bigcup \\
&\quad \cdots \bigcup \{|l_{mn+3+2t-6-t+1} - l_{mn+3+2t-4-t+1}|, |l_{mn+3+2t-4-t+1} - l_{mn+3+2t-2-t+1}|\} \\
&= \{|l_{mn+2} - l_{mn+4}|, |l_{mn+4} - l_{mn+6}|\} \bigcup \{|l_{mn+5} - l_{mn+7}|, |l_{mn+7} - l_{mn+9}|\} \bigcup \\
&\quad \cdots \bigcup \{|l_{mn+t-2} - l_{mn+t}|, |l_{mn+t} - l_{mn+t+2}|\} \\
&= \{l_{mn+3}, l_{mn+5}, l_{mn+6}, l_{mn+8}, \cdots, l_{mn+t-1}, l_{mn+t+1}\}.
\end{aligned}$$

We find the edge labeling between the end vertex of r^{th} loop and the starting vertex of $(r+1)^{th}$ loop for integers $r = 1, 2, \cdots, \dfrac{t-1}{3}$. Let

$$\begin{aligned}
E_2^{'''} &= \bigcup_{r=1}^{\frac{t-1}{3}} \{f_1(v_k v_{k+1}) : k = 3r\} \\
&= \bigcup_{r=1}^{\frac{t-1}{3}} \{|f(v_k) - f(v_{k+1})| : k = 3r\} \\
&= \{|f(v_3) - f(v_4)|, |f(v_6) - f(v_7)|, \cdots, |f(v_{t-1}) - f(v_t)|\} \\
&= \{|l_{mn+3+6-3} - l_{mn+3+8-6}|, \cdots, |l_{mn+3+2t-2-t+1} - l_{mn+3+2t-t-2}|\} \\
&= \{|l_{mn+6} - l_{mn+5}|, |l_{mn+9} - l_{mn+8}|, ..., |l_{mn+t+2} - l_{mn+t+1}|\} \\
&= \{l_{mn+4}, l_{mn+7}, \cdots, l_{mn+t}\}
\end{aligned}$$

Therefore $E^{'''} = E_0 \bigcup E_1^{'''} \bigcup E_2^{'''} = \{l_{mn+2}, l_{mn+3}, \cdots, l_{mn+t-1}, l_{mn+t+1}, l_{mn+4}, l_{mn+7}, \cdots, l_{mn+t}\} = \{l_{mn+2}, l_{mn+3}, l_{mn+4}, \cdots, l_{mn+t-1}, l_{mn+t}, l_{mn+t+1}\}$. Now, $E \cup E^{'} \cup E^{'''} = \bigcup\limits_{i=1}^{4} E_i \bigcup \left\{\bigcup\limits_{i=1}^{3} E_i^{'}\right\} \bigcup \left\{E_0 \bigcup E_1^{'''} \bigcup E_2^{'''}\right\} = \{l_1, l_2, \cdots, l_{mn}, l_{mn+2}, l_{mn+3}, \cdots, l_{mn+t-1}, l_{mn+t}, l_{mn+t+1}\}$. So, the edge labels of G are distinct. In both cases, f is a nearly Lucas graceful labeling. Thus $G = S_{m,n}@P_t$ is a nearly Lucas graceful graph when $m \equiv 1(mod\ 2)$, $n \equiv 1, 2(mod\ 3)$ and $t \equiv 1, 2, (mod\ 3)$.

Example 3.3 A nearly Lucas graceful labeling of $S_{5,7}@P_7$ is shown in Fig.3.1.

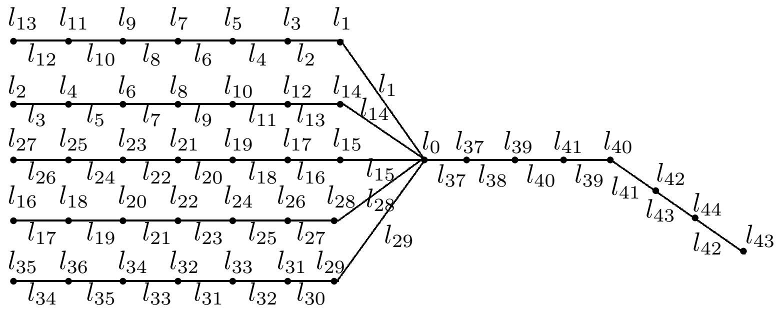

Fig.3.1 $S_{5,7}@P_7$

Theorem 3.4 *C_n is a nearly Lucas graceful graph. when $n \equiv 1, 2(mod\ 3)$.*

Proof Let $G = C_n$ with $V(G) = \{u_i : 1 \leq i \leq n\}$. Let $E(G) = \{u_i u_{i+1} : 1 \leq i \leq n-1\} \cup \{u_n u_1\}$ be the edge set of G. So, $|V(G)| = n$ and $|E(G)| = n$.

Case 1 $n \equiv 1(mod\ 3)$.

Define $f : V(G) \to \{l_0, l_1, l_2, \cdots, l_a\},\ a\ \epsilon\ N$ by $f(u_1) = l_0$. For $s = 1, 2, \cdots, \frac{n-4}{3}$, $f(u_i) = l_{2i-3s},\ 3s-1 \leq i \leq 3s+1$ and for $s = \frac{n-1}{3}$, $f(u_i) = l_{2i-3s},\ 3s-1 \leq i \leq 3s$. We claim that the edge labels are distinct. Let

$$\begin{aligned} E_1 &= \{f_1(u_1u_2), f_1(u_nu_1)\} = \{|f(u_1) - f(u_2)|, |f(u_n - f(u_1)|\} \\ &= \{|l_0 - l_1|, |l_{2n-n+1} - l_0|\} = \{l_1, l_{n+1}\}. \end{aligned}$$

For $s = 1, 2, \cdots, \frac{n-1}{3}$, let

$$\begin{aligned} E_2 &= \bigcup_{s=1}^{\frac{n-1}{3}} \{f_1(u_iu_{i+1})\ 3s-1 \leq i \leq 3s\} \\ &= \bigcup_{s=1}^{\frac{n-1}{3}} \{|f(u_i) - f(u_{i+1})| : 3s-1 \leq i \leq 3s\} \\ &= \{|f(u_2) - f(u_3)|, |f(u_3) - f(u_4)|\} \bigcup \{|f(u_5) - f(u_6)|, |f(u_6) - f(u_7)|\} \bigcup \\ &\quad \cdots \bigcup \{|f(u_{n-2}) - f(u_{n-1})|, |f(u_{n-1}) - f(u_n)|\} \\ &= \{|l_1 - l_3|, |l_3 - l_5|\} \bigcup \{|l_4 - l_6|, |l_6 - l_8|\} \bigcup \\ &\quad \cdots \bigcup \{|l_{2n-4-n+1} - l_{2n-2-n+1}|, |l_{2n-2-n+1} - l_{2n-n+1}|\} \\ &= \{l_2, l_4\} \bigcup \{l_5, l_7\} \bigcup \{l_{n-2}, l_n\}. \end{aligned}$$

We find the edge labeling between the end vertex of s^{th} loop and the starting vertex of $(s+1)^{th}$

loop for integers $s = 1, 2, \cdots, \dfrac{n-1}{3} - 1$. Let

$$\begin{aligned}
E_3 &= \bigcup_{s=1}^{\frac{n-4}{3}} \{f_1(u_iu_{i+1}) : i = 3s+1\} \\
&= \bigcup_{s=1}^{\frac{n-4}{3}} \{|f(u_i) - f(u_{i+1})| : i = 3s+1\} \\
&= \{|f(u_4) - f(u_5)|, |f(u_7) - f(u_8)|, \cdots, |f(u_{n-3} - f(u_{n-2})|\} \\
&= \{|l_{8-3} - l_{10-6}|, |l_{14-6} - l_{16-9}|, \cdots, |l_{2n-6-n+4} - l_{2n-4-n+1}|\} \\
&= \{|l_5 - l_4|, |l_8 - l_7|, \cdots, |l_{n-2} - l_{n-3}|\} = \{l_3, l_6, \cdots, l_{n-4}\}
\end{aligned}$$

Now, $E = \bigcup_{i=1}^{3} E_i = \{l_1, l_2, l_3, l_4, \cdots, l_{n-2}, l_n, l_{n+1}\}$.

Case 2 $n \equiv 2(mod\ 3)$.

Define $f : V(G) \to \{l_0, l_1, l_2, \cdots, l_a\}$, $a \in N$ by $f(u_1) = l_0$, $f(u_n) = l_{n+2}$. For $s = 1, 2, \cdots, \dfrac{n-2}{3} - 1$, $f(u_i) = l_{2i-3s}$, $3s-1 \le i \le 3s+1$ and for $s = \dfrac{n-2}{3}$, $f(u_i) = l_{2i-3s}$, $3s-1 \le i \le 3s$. We claim that the edge labels are distinct. Let

$$\begin{aligned}
E_1 &= \{f_1(u_1u_2), f_1(u_{n-1}u_n), f_1(u_nu_1)\} \\
&= \{|f(u_1) - f(u_2)|, |f(u_{n-1}) - f(u_n)|, |f(u_n) - f(u_1)|\} \\
&= \{|l_0 - l_1|, |l_{2n-2-n+2} - l_{n+2}|, |l_{n+2} - l_0|\} = \{l_1, l_{n+1}, l_{n+2}\},
\end{aligned}$$

$$\begin{aligned}
E_2 &= \bigcup_{s=1}^{\frac{n-2}{3}} \{f_1(u_iu_{i+1}) : 3s-1 \le i \le 3s\} \\
&= \bigcup_{s=1}^{\frac{n-2}{3}} \{|f(u_i) - f(u_{i+1})| : 3s-1 \le i \le 3s\} \\
&= \{|f(u_2) - f(u_3)|, |f(u_3) - f(u_4)|\} \bigcup \{|f(u_5) - f(u_6)|, |f(u_6) - f(u_7)|\} \bigcup \\
&\quad \cdots \bigcup \{|f(u_{n-3}) - f(u_{n-2})|, |f(u_{n-2}) - f(u_{n-1})|\} \\
&= \{|l_{4-3} - l_{6-3}|, |l_{6-3} - l_{8-3}|\} \bigcup \{|l_{10-6} - l_{12-6}|, |l_{12-6} - l_{14-6}|\} \bigcup \\
&\quad \cdots \bigcup \{|l_{2n-6-n+2} - l_{2n-4-n+2}|\} \\
&= \{|l_1 - l_3|, |l_3 - l_5|\} \bigcup \{|l_4 - l_6|, |l_6 - l_8|\} \bigcup \\
&\quad \cdots \bigcup \{|l_{n-4} - l_{n-2}|, |l_{n-2} - l_n|\} \\
&= \{l_2, l_4, l_5, l_7, ..., l_{n-3}, l_{n-1}\}.
\end{aligned}$$

We find the edge labeling between the end vertex of $(s-1)^{th}$ loop and the starting vertex of

s^{th} loop for integers $s = 1, 2, \cdots, \dfrac{n-5}{3}$. Let

$$\begin{aligned} E_3 &= \bigcup_{s=1}^{\frac{n-5}{3}} \{f_1(u_i u_{i+1}) : i = 3s+1\} \\ &= \bigcup_{s=1}^{\frac{n-5}{3}} \{|f(u_i) - f(u_{i+1})| : i = 3s+1\} \\ &= \{|f(u_4) - f(u_5)|, |f(u_7) - f(u_8)|, ..., |f(u_{n-4}) - f(u_{n-3})|\} \\ &= \{|l_5 - l_4|, |l_8 - l_7|, \cdots, |l_{2n-8-n+5} - l_{2n-6-n+2}|\} = \{l_3, l_6, \cdots, l_{n-2}\} \end{aligned}$$

Now, $E = \bigcup_{i=1}^{3} E_i = \{l_1, l_2, l_3, l_4, \cdots, l_{n-3}, l_{n-2}, l_{n-1}, l_{n+1}, l_{n+2}\}$ So, all these edge labels of G are distinct. In both the cases, f is a nearly Lucas graceful graph. Thus $G = C_n$ is a nearly Lucas graceful graph when $n \equiv 1, 2(mod\ 3)$. □

Example 3.5 A nearly Lucas graceful labeling on C_{13} in Case 1 is shown in Fig.3.2.

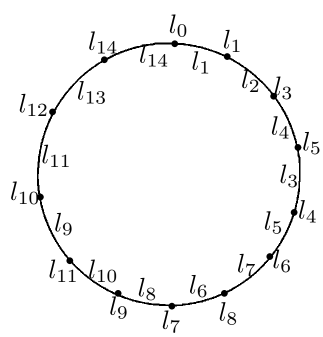

Fig.3.2 C_{13}

Example 3.6 A nearly Lucas graceful labeling on C_{14} in Case 2 is shown in Fig.3.3.

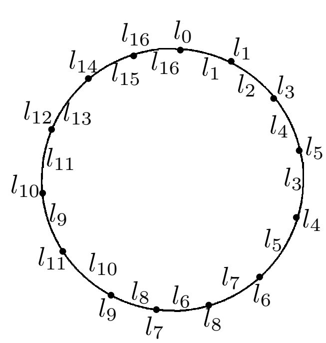

Fig.3.3 C_{14}

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