DIVISIBILITY TESTS FOR SMARANDACHE SEMIGROUPS.

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Abstract

Two Divisibility Tests for Smarandache semigroups are given . Further, the notion of divisibility of elements in a semigroup is applied to characterize the Smarandache semigroups. Examples are provided for justification.

Key words : Semigroup , Divisibility, Right Divisor , Left Divisor , Group , Smarandache semigroup.

1.**Introduction**

Padilla Raul introduced the notion of Smarandache semigroups in the year 1998 in the paper entitled Smarandache algebraic structures[2]. Since groups are the prefect Structures under a single closed associative binary operation, it has become infeasible to define Smarandache groups. Smarandache semigroups are the analog in the Smarandache ideologies of the groups. The Smarandache notions in groups and the concept of Smarandache semigroups have been studied in [9].

In[7] , the notion of divisibility of elements in a semigroup is introduced and the properties of the elements which are both left and right divisiors of every element of a semigroup are studied. Further , the properties of the elements which are divisible both on the right and on the left by all elements of the semigroup are studied as well. Such elements were first considered in a paper by Clifford and Miller, where they were termed zeroid elements.

The concept of divisibility of elements in a semigroup is very much useful to study the Smarandache semigroups. In this paper we give two tests called Divisibility tests for Smarandache semigroups , and a characterization of Smarandache semigroups by applying the notion of divisibility of elements in a semigroup. Examples are provided for justification.

In section 2 we give the basic definitions and properties of divisibility of elements in a semigroup. The definition and example of Smarandache semigroups are given as well. In section 3 we present our theorems and in section 4 we provide examples for justification. For , more basic definitions and concepts please refer [7]and [9]. In this paper , we denote the operation multiplication (product)of elements in a semigroup by jux-ta-position.

2.Preliminaries

Definition 2.1([7]). A semigroup is a nonempty set S, in which for every ordered pair of elements x , y in S there is defined a new element called their product $xy \in S$, where for all x , y , z $\in S$, we have (xy)z = x(yz).

Definition 2.2([7]). An element b of the semigroup S is called a right divisor of the element a of the semigroup if there exists in S an element x such that xb = a. b is called a left divisor of a if there exists in S an element y such that by = a.

If b is a right divisor of a , we say that a is divisible on the right by b. If b is a left divisor of a , we say that a is divisible on the left by b.

The following observations in the divisibility of elements in a semigroup are well known in [7].

2.2.1 An element a of a semigroup S will be a right (left) divisor of every element of S if and only if in the column (row) of the multiplication table corresponding to the element a , all elements of S occur.

2.2.2 The element a is divisible on the left(right)by every element of S if and only if a occurs in every row(column) of the multiplication table.

Definition 2.3.([7]). A nonempty subset H of the semigroup S is called a subsemigroup of S if $HH \subset S$.

Definition 2.4.([6]). A nonempty set G, together with an associative binary operation * on G such that equations $a * x=b$ and $y * a= b$ have solutions in G for all $a, b \in G$, is a group.

Several examples of Semigroups, subsemigroups and groups can be found in the literature.

Definition 2.5([9]). The Smarandache semigroup defined to be a semigroup S such that a proper subset B of S is a group with respect to the same operation on S.

Example 2.6. Let us consider the semigroup S={0,1,2,3,4,5} under the operation multiplication modulo 6. The multiplication table is as follows :

	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

Table 1.

S is a Smarandache semigroup as the proper subsets { 2 , 4 } , { 1 , 5 } of S are groups.

3. PROOFS OF THE THEOREMS

In this section we present Divisibility test 1 , Divisibility test 2 for Smarandache semigroups. Further, a Characterization of Smarandache semigroups, using the notion of divisibility of elements in a semigroup, is given as well .

Theorem 3.1 (Divisibility test – 1) A semigroup S (not a group) containing the elements that are both left and right divisors of every element of S is always a Smarandache semigroup.

Proof : Let G be the set of all elements that are both left and right divisors of every element of S. Cleary G $\subset$S. (if S = G , then G is a group as every element of S is a left and right divisors of every element of S.). To show that S is a Smarandache semigroup, it is sufficient to show that G is a group under the operation on S.

Suppose that $b_1, b_2 \in$ G. For every a $\in$ S there exists x_2 in S such that $x_2b_2 = a$. Corresponding to $x_2 \in$ S there exists in S an element x_1 such that $x_1b_1 = x_2$. Therefore, $x_1 b_1b_2 = x_2 b_2 = a$. Accordingly , b_1b_2 is a right divisor of a . Similarly , we can show that b_1b_2 is a left divisor of a. . Therefore, $b_1b_2 \in$ G. The multiplication in G is associative as G $\subset$ S.

Suppose that $b_1, b_2 \in$ S . There exist in S elements x and y such that $xb_2 = b_1$, $b_2 y= b_1$. If we show that x , y $\in$ G the proof that G is a group will be completed.

Let α be an arbitrary element of S. Since, for some t $\in$ S, we have $b_1t = \alpha$; $xb_2t = b_1t = \alpha$. i.e., x is a left divisor of α

There exist also in S elements e , c_1 , c_2 such that $b_2e = b_2$, $c_1b_1 = e$, $b_2c_2 = e$. Since for arbitrary z in S there exists an element $z^1 \in$ S such that $z = z^1b_2$, we have

$ze = z^1 b_2 e = z^1 b_2 = z$ i.e., e is such that $ze = z$ for arbitrary $z \in S$. Making use of this property of e, we obtain,

$\alpha = \alpha e = \alpha b_2 c_2 = \alpha b_2 e c_2 = \alpha b_2 c_1 b_1 c_2 = \alpha b_2 c_1 x b_2 c_2 = \alpha b_2 c_1 x e = \alpha b_2 c_1 x$.

i.e., x turns to be also a right divisor of α. Therefore, $x \in G$. Analogously, we can show that $y \in G$. Hence, S is a Smarandache semigroup.

Theorem 3.2 (Divisibility test – 2) A semigroup S (not a group) containing the elements that are divisible both on the right and on the left by every element of S is always a Smarandache semigroup.

Proof : Let G be the set of all elements of S that are divisible both on the right and on the left by every element of S. Clearly, $G \subset S$.

Let $c_1, c_2 \in G$, for every $a \in S$, there is some x in S such that $c_1 = ax$. Therefore, $c_1 c_2 = a(xc_2)$, i.e., $c_1 c_2$ is divisible on the left by an arbitrary element a. Analogously, we can show that $c_1 c_2$ is divisible on the right by a. Therefore, $c_1 c_2 \in G$.

The associativity of the operation in G follows from the fact that $G \subset S$. Since c_1 and c_2 belong to G, there exist elements u, v, w, in S such that

$uc_2^2 = c_2$, $c_2^2 v = c_2$, $c_1 = c_2 w$, We have then $uc_2 = uc_2 c_2 v = c_2 v$. This yields, $c_2(c_2 v^2 c_1) = (c_2^2 v)(vc_1) = c_2 v c_1 = uc_2 c_1 = uc_2 c_2 w = c_2 w = c_1$.

We have shown that the element $y = c_2 v^2 c_1$ is a solution of the equation $c_2 y = c_1$.

Let us show that $y \in G$. In fact, for arbitrary z in S there exist u^1 and v^1 in S such that $c_1 = u^1 z$, $c_2 = zv^1$. Therefore, $y = c_2 v^2 c_1 = zv^1 v^2 u^1 z$. i.e., y is divisible by z both on the right and on the left. Analogously, we may find a solution x of the equation $xc_2 = c_1$, such that $x \in G$. Therefore, G is a group. Hence, the semigroup S is a Smarandache semigroup.

Theorem 3.3 . Let S be a semigroup. S is a Smarandache semigroup if and only if there exists a proper subsemigroup T of S such that T possesses an element which is both a right and left divisor of every element of T and is at the same time itself divisible both on the left and on the right by every element of T.

Proof : Let S be a semigroup. If S is a Smarandache semigroup then there exists a proper subset T of S such that T is a group with respect to the same operation on S. T possesses an element which is both a right and a left divisor of every element of T and is at the same time itself divisible both on the left and on the right by every element of T as T is a group with respect to the same operation on S.

On the other hand suppose that there exists a subsemigroup T of S such that T possesses an element which is both a right and a left divisor of every element of T and is at the same time itself divisible both on the left and on the right by every element of T. Now we show that S is a Smarandache semigroup . For, it is sufficient to prove that the proper subsemigroup T is a group.

Let a and b be arbitrary elements of T and let d be an element having the properties postulated in the hypothesis of the theorem. There exist elements x_1 , y_1 , x_2 , y_2 in T such that $a = x_1 d$, $a = dy_1$, $d = x_2 b$, $d = by_2$

It follows that $x = x_1x_2$ and $y = y_1y_2$ are solutions of the equations $a = xb$, $a = by$ in T. Therefore, T is a group and hence S is a Smarandache semigroup.

4. Examples

In this section we provide examples to justify our Divisibility test 1 , Divisibility test 2 for Smarandache semigroups , and the characterization of Smarandache semigroups as well . Further , we give examples to show that the conditions stated in theorem (3.1) and theorem (3.2) are sufficient conditions but not necessary conditions .

Example 4.1 Let z_8 = {0,1,2,3,4,5,6,7} be a semigroup under multiplication mod 8 . The composition table is as follows.

X_8	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	0	2	4	6
3	0	3	6	1	4	7	2	5
4	0	4	0	4	0	4	0	4
5	0	5	2	7	4	1	6	3
6	0	6	4	2	0	6	4	2
7	0	7	6	5	4	3	2	1

Table 2

In view of (2.2.1) the elements 1 and 7 in z_8 are both left and right divisors of every element of the semigroup z_8. So, in view of the theorem(3.1), the proper subset G = {1,7} is a group under the multiplication mod 8. Hence, the semigroup (z_8 , x_8)is a Smarandache semigroup.

Example 4.2 Let z_7={0,1,2,3,4,5,6,}be a semigroup under multiplication mod 7. The composition table is as follows.

X_7	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

Table 3

In view of (2.2.1) the elements 1,2,3,4,5 and 6 in z_7 are both left and right divisors of every element of the semigroup z_7. So, in view of the theorem (3.1), the proper subset G={1,2,3,4,5,6} is a group under the multiplication mod 7. Hence, the semigroup (z_7 , x_7)is a Smarandache semigroup.

Example 4.3 Let z_4={0,1,2,3} be a semigroup under the multiplication mod 4. The composition table is as follows.

X_4	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

Table 4

In view of (2.2.1) the elements 1 and 3 in z_4 are both left and right divisors of every element of the semigroup z_4. So, in view of the theorm (3.1) , the proper subset G={1,3} is a group under the multiplication mod 4. Hence, the semigroup (z_4,x_4) is a Smarandache semigroup.

Example 4.4 Let S={e,a,b,c} be a semigroup under the operation defined by the following composition table.

	e	a	b	c
e	e	a	b	c
a	a	e	b	c
b	b	b	c	b
c	c	c	b	c

Table 5

In view of (2.2.2) the elements b and c are divisible both on the right and on the left by every element of the semigroup S. In view of the theorm (3.2), the proper subset

G={b,c} of S is a group under the operation on S. Hence, S is a Smarandache semigroup.

Example 4.5 Let S = { 0 ,1,2,3,4,5,6,7,8} be a semigroup under multiplication mod 9. Consider the proper subset T = {1,2,4,5,7,8} of S. The composition table on T of the multiplication mod 9 is as follows:

X_9	1	2	4	5	7	8
1	1	2	4	5	7	8
2	2	4	8	1	5	7
4	4	8	7	2	1	5
5	5	1	2	7	8	4
7	7	5	1	8	4	2
8	8	7	5	4	2	1

Table 6.

From table 6 it is evident that T is a subsemigroup possessing the properties postulated in the hypothesis of the theorem (3.3). Therefore, T is a group and hence S is a Smarandache semigroup.

Finally, we show by providing examples that the conditions stated in theorem (3.1) and theorem (3.2) are sufficient but not necessary .

Example 4.6 Let S={1,2,3,4,5,6} be a semigroup under the operation defined by xy = the great common divisor of x,y for all x, y $\in$ S. The composition table is as follows.

	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	2	1	2	1	2
3	1	1	3	1	1	3
4	1	2	1	4	1	2
5	1	1	1	1	5	1
6	1	2	3	2	1	6

Table 7.

This semigroup is not satisfying the conditions stated in theorem (3.1) but this semigroup S is Smarandache semigroup . (see [1]).

Example 4.7 Let us consider the example 2.6 . From the table 1 it is evident that the semigroup S in the example 2.6 is not satisfying the conditions stated in theorem (3.2) but this semigroup S is Smarandahce semigroup.

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