Topology and its Applications 156 (2009) 3052–3061

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

An extension of the dual complexity space and an application to Computer Science

J. Rodríguez-López [a,1], M.P. Schellekens [b,2], O. Valero [c,*,1]

[a] *Instituto Universitario de Matemática Pura y Aplicada, Universidad Politécnica de Valencia, 46022 Valencia, Spain*
[b] *Centre for Efficiency-Oriented Languages (CEOL), Department of Computer Science, University College Cork (UCC), Cork, Ireland*
[c] *Departamento de Ciencias Matemáticas e Informática, Universidad de las Islas Baleares, 07122 Baleares, Spain*

ARTICLE INFO

Article history:
Received 15 October 2008
Received in revised form 17 February 2009
Accepted 19 February 2009

MSC:
54E50
54F05
54C35
54H99
68Q05

Keywords:
Ordered normed monoid
Dual complexity space
Extended quasi-metric
Right K-sequentially complete
Interval Analysis

ABSTRACT

In 1999, Romaguera and Schellekens introduced the theory of dual complexity spaces as a part of the development of a mathematical (topological) foundation for the complexity analysis of programs and algorithms [S. Romaguera, M.P. Schellekens, Quasi-metric properties of complexity spaces, Topology Appl. 98 (1999) 311–322]. In this work we extend the theory of dual complexity spaces to the case that the complexity functions are valued on an ordered normed monoid. We show that the complexity space of an ordered normed monoid inherits the ordered normed structure. Moreover, the order structure allows us to prove some topological and quasi-metric properties of the new dual complexity spaces. In particular, we show that these complexity spaces are, under certain conditions, Hausdorff and satisfy a kind of completeness. Finally, we develop a connection of our new approach with Interval Analysis.

1. Introduction and preliminaries

Throughout this paper the letters $\mathbb{R}^+$, $\mathbb{N}$ and ω will denote the set of nonnegative real numbers, the set of natural numbers and the set of nonnegative integer numbers, respectively.

Recall that a *monoid* is a semigroup $(X,+)$ with neutral element 0. A strict monoid is a monoid $(X,+)$ such that $x, y \in X$ with $x+y=0$ implies $x=y=0$ (see [24]). An *abelian monoid* is a monoid $(X,+)$ such that $x+y=y+x$ for all $x, y \in X$. An abelian monoid $(X,+)$ is called *cancellative* if for all $x, y, z \in X$, $z+x=z+y$ implies $x=y$. A *submonoid* of a monoid $(X,+)$ is a subset Y of X containing the neutral element 0 such that $(Y,+|_{Y\times Y})$ is a monoid.

Next we give some pertinent concepts on quasi-metric spaces. Our basic references are [9,10].

Following the modern terminology, by a *quasi-metric* on a nonempty set X we mean a function $d: X\times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

* Corresponding author.
E-mail addresses: jrlopez@mat.upv.es (J. Rodríguez-López), m.schellekens@cs.ucc.ie (M.P. Schellekens), o.valero@uib.es (O. Valero).
[1] The first and the third authors thank the support of the Spanish Ministry of Education and Science, and FEDER, grant MTM2006-14925-C02-01.
[2] The second author acknowledges the support of Science Foundation Ireland, grant number 07/IN.1/I977.

0166-8641/$ – see front matter
doi:10.1016/j.topol.2009.02.009

(i) $d(x, y) = d(y, x) = 0 \Leftrightarrow x = y$;
(ii) $d(x, z) \leqslant d(x, y) + d(y, z)$.

We will also consider extended quasi-metrics. They satisfy the two above axioms, except that we allow $d(x, y) = +\infty$ whenever $x \neq y$.

Each (extended) quasi-metric d on a nonempty set X induces a T_0 topology $\tau(d)$ on X which has as a base the family of open d-balls $\{B_d(x, r)\colon x \in X,\ r > 0\}$, where $B_d(x, r) = \{y \in X\colon d(x, y) < r\}$ for all $x \in X$ and $r > 0$.

A(n extended) quasi-metric space is a pair (X, d) such that X is a nonempty set and d is a(n extended) quasi-metric on X.

If d is a(n extended) quasi-metric on a nonempty set X, then the function d^s defined on $X \times X$ by $d^s(x, y) = \max\{d(x, y), d(y, x)\}$ is a(n extended) metric on X.

A(n extended) quasi-metric is called *bicomplete* if d^s is a(n extended) complete metric.

Let us recall that a sequence $(x_n)_{n\in\mathbb{N}}$ in a(n extended) quasi-metric space (X, d) is *right K-Cauchy* provided that for each $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon$ for all $m \geqslant n \geqslant n_0$. (X, d) is said to be *right K-sequentially complete* if every right K-Cauchy sequence is convergent with respect to $\tau(d)$. Right K-sequential completeness is an appropriate notion of (extended) quasi-metric completeness in the study of functions spaces and hyperspaces (see, for instance, [11,12]). Some computational interpretation of right K-sequential completeness has been given in [17].

A *norm* on a monoid $(X, +)$ is a function $\|\cdot\|: X \to \mathbb{R}^+$ such that for all $x, y \in X$:

(i) $\|x\| = 0 \Leftrightarrow x = 0$;
(ii) $\|x + y\| \leqslant \|x\| + \|y\|$.

A *normed monoid* is a pair $(X, \|\cdot\|)$ where X is an abelian cancellative monoid and $\|\cdot\|$ is a norm on X.

As usual a *partial order* (or simply an order) on a nonempty set X is a reflexive, antisymmetric and transitive binary relation $\leqslant$ on X. A nonempty set X equipped with an order is said to be an ordered set. An ordered set $(X, \leqslant)$ is said to be *linear* if, for all $x, y \in X$, either $x \leqslant y$ or $y \leqslant x$. If a binary relation $\leqslant$ on a set X only satisfies reflexivity and transitivity, we will call it a *preorder*.

Let us recall that a nonnegative real valued function f defined on an ordered set $(X, \preccurlyeq)$ is called *order-preserving* if $x \preccurlyeq y$ implies $f(x) \leqslant f(y)$.

In [22], Schellekens introduced the (quasi-metric) complexity space as a part of the development of a topological foundation for the complexity analysis of programs and algorithms. In particular, he presented some applications of this theory to the complexity analysis of Divide & Conquer algorithms. More concretely, he gave a novel proof, based on fixed point arguments, of the well known fact that the mergesort algorithm has optimal asymptotic average running time.

Later on, Romaguera and Schellekens [19] introduced the so-called dual complexity space in order to obtain a more robust mathematical structure for the complexity analysis of programs and algorithms. In the same reference they studied several quasi-metric properties of the complexity space, such as Smyth completeness and total boundedness, via the analysis of its dual. Recall that the *dual complexity space* is the pair $(\mathcal{C}^*, d_{\mathcal{C}^*})$, where

$$\mathcal{C}^* = \left\{ f\colon \omega \to \mathbb{R}^+ \;\Big|\; \sum_{n=0}^{\infty} 2^{-n} f(n) < +\infty \right\},$$

and $d_{\mathcal{C}^*}$ is the quasi-metric on $\mathcal{C}^*$ defined by

$$d_{\mathcal{C}^*}(f, g) = \sum_{n=0}^{\infty} 2^{-n}\big[\big(g(n) - f(n)\big) \vee 0\big].$$

A motivation for the use of the dual complexity space is given by the fact that Romaguera and Schellekens proved that the complexity analysis of algorithms can be carried out by means of techniques based on the dual complexity space when the considered complexity measure is the running time of computing. Thus, the intuition behind the complexity distance between two functions $f, g \in \mathcal{C}^*$ is that $d_{\mathcal{C}^*}(f, g)$ measures the relative progress made in lowering the complexity by replacing any program Q with complexity function f by any program P with complexity function g. Therefore, if $f \neq g$, the condition $d_{\mathcal{C}^*}(f, g) = 0$ can be interpreted as f is "more efficient" than g.

Furthermore, the dual complexity space has another advantage with respect to the original one. In the dual context, and contrary to the case of the complexity space, there is a minimum which corresponds to the minimum of semantic domains. Let us recall that the minimum plays a central role in domain theory in order to model in a suitable way the mathematical meaning of recursive definitions of procedures. Furthermore, the dual complexity space admits a more robust mathematical structure than the original complexity space. In particular, the dual complexity space has a cancellative abelian monoid structure while that the complexity space is only a semigroup without neutral element. However, the use of the complexity distance $d_{\mathcal{C}^*}$ presents a handicap. Indeed, if we have the condition $d_{\mathcal{C}^*}(f, g) \neq 0$ then it is not possible to establish which complexity function of both, f or g, is more efficient.

Motivated by this disadvantage Romaguera, Sánchez-Pérez and Valero introduced in [17,18] a slight modification of the "complexity distance" $d_{\mathcal{C}^*}$. In particular the new complexity distance was constructed on $\mathcal{C}^*$ as follows:

$$e_{\mathcal{C}^*}(f,g)=\begin{cases}\sum_{n=0}^{\infty}2^{-n}[g(n)-f(n)] & \text{if } f\leqslant_{\mathcal{C}^*} g,\\ +\infty & \text{otherwise},\end{cases}$$

where $\leqslant_{\mathcal{C}^*}$ denotes the natural pointwise order, i.e. $f\leqslant_{\mathcal{C}^*} g \Leftrightarrow f(n)\leqslant g(n)$ for all $n\in\omega$. The space $(\mathcal{C}^*, e_{\mathcal{C}^*})$ is called the extended dual complexity space.

The complexity distance $e_{\mathcal{C}^*}$ (an extended quasi-metric) can be also used for the analysis of the relative progress made in lowering the complexity (running time of computing) when an algorithm is replaced by another one and, in addition, it is a useful tool for the quantitative analysis of algorithms (see [17] for a deeper discussion). Moreover, this new complexity distance was also applied to the measurement of distances between infinite words over the decimal alphabet and some advantages of these computational techniques with respect to the ones provided by the classical Baire metric were discussed in [17]. Besides all these advantages, the new complexity distance $e_{\mathcal{C}^*}$ has rich quasi-metric properties such as Hausdorffness and right K-sequential completeness.

In this paper, motivated in part by the methods of successive approximations in interval computation, we generalize the construction presented in [17] to the case that the complexity functions are valued on an ordered normed monoid. We give conditions under which these new complexity spaces are Hausdorff and right K-sequentially complete in Section 2.2. Section 2.3 is devoted to present a connection of the developed theory with Interval Analysis. In particular we show the correctness of iterative schemes to solve systems of equations in Interval Analysis via interval valued complexity functions.

A related work on generalized complexity spaces can be found in [20].

2. Generalized dual complexity spaces

2.1. Ordered normed monoids

In this subsection we introduce the basic tools of the theory that we will develop.

In the literature it is well known that a monoid can be endowed with a natural preorder which is generated by a submonoid [13]. In particular if we consider a monoid X and a submonoid M of X, then a preorder $\leqslant_M$ can be defined in the following way: $x\leqslant_M y \Leftrightarrow y=x+z$ for some $z\in M$.

However, under more restrictive assumptions the natural preorder $\leqslant_M$ is in fact an order.

Proposition 1. *Let X be a cancellative abelian monoid X and let M be a strict submonoid of X. Then $(X,\leqslant_M)$ is an ordered set.*

Notice that if there exists $z\in M$ such that $y=x+z$, then, by the cancellativity of X, z is the unique element in X satisfying the preceding equality.

On the other hand, and following [18], given a normed monoid $(X,\|\cdot\|)$ we can construct a bicomplete extended quasi-metric $d_{\|\cdot\|}$ on X given by

$$d_{\|\cdot\|}(x,y)=\begin{cases}\|z\| & \text{if there exists } z\in X \text{ with } y=x+z,\\ +\infty & \text{otherwise.}\end{cases}$$

Note that the induced topology $\tau(d_{\|\cdot\|})$ is T_1.

In the following result, whose easy proof we omit, we give a technique for generating a bicomplete extended quasi-metric from a norm by means of a slight modification of the preceding one.

Proposition 2. *Let $(X,\|\cdot\|)$ be a normed monoid and let M be a strict submonoid of X. Then the function $d_{\|\cdot\|}: X\times X\to\mathbb{R}^+$ defined by*

$$d_{\|\cdot\|}(x,y)=\begin{cases}\|z\| & \textit{if there exists } z\in M \textit{ with } y=x+z,\\ +\infty & \textit{otherwise},\end{cases}$$

is a bicomplete extended quasi-metric whose induced topology $\tau(d_{\|\cdot\|})$ is T_1.

The method introduced in the preceding proposition will play a central role for our purpose of constructing an extension of the dual complexity space to a more general context.

The following are two interesting examples of this type of structures.

Example 3. Define the function $\|\cdot\|_S:\mathbb{R}^+\to\mathbb{R}^+$ as $\|x\|_S=x$. It is clear that $\mathbb{R}^+$ is a strict cancellative abelian monoid. Moreover, it follows easily that $x\leqslant y\Leftrightarrow x\leqslant_{\mathbb{R}^+} y$ for all $x,y\in\mathbb{R}^+$, where we denote by $\leqslant$ the usual order on the set of real numbers. Obviously $\|\cdot\|_S$ is a norm and, thus, the extended quasi-metric induced by $\|\cdot\|_S$ on $\mathbb{R}^+$ is given by

$$d_{\|\cdot\|_S}(x,y)=\begin{cases}y-x & \text{if } x\leqslant_{\mathbb{R}^+} y.\\ +\infty & \text{otherwise.}\end{cases}$$

Of course the topology induced by $d_{\|\cdot\|_S}$, $\tau(d_{\|\cdot\|_S})$, is the Sorgenfrey topology on $\mathbb{R}^+$.

Example 4. It is well known that the set $\mathrm{I}(\mathbb{R})$ of all nonempty compact intervals of the real line plays a relevant role in the theory of computation and in applied mathematics (see Section 2.3). Following [16], given $[a,b],[c,d] \in \mathrm{I}(\mathbb{R})$, the interval sum $\oplus$ is defined by $[a,b] \oplus [c,d] = [a+c, b+d]$. This algebraic operation lends $\mathrm{I}(\mathbb{R})$ a cancellative abelian monoid structure. Let $\mathrm{I}(\mathbb{R})_0 = \{[a,b] \in \mathrm{I}(\mathbb{R})\colon a \leqslant 0, 0 \leqslant b\}$. Of course $\mathrm{I}(\mathbb{R})_0$ is a strict submonoid of $\mathrm{I}(\mathbb{R})$.

On the other hand, the set $\mathrm{I}(\mathbb{R})$ is ordered under the inclusion order $\sqsubseteq$, i.e. $[a,b] \sqsubseteq [c,d] \Leftrightarrow [a,b] \subseteq [c,d]$. The former order is key to guarantee the finite convergence of iterative numerical processes (see [16] for a detailed discussion). It is not hard to see that the order $\sqsubseteq$ coincides with $\leqslant_{\mathrm{I}(\mathbb{R})_0}$, i.e. $[a,b] \sqsubseteq [c,d] \Leftrightarrow [a,b] \leqslant_{\mathrm{I}(\mathbb{R})_0} [c,d]$.

We can construct an extended quasi-metric via Proposition 2 on $\mathrm{I}(\mathbb{R})$. Define the norm $\|\cdot\|_{\mathrm{I}(\mathbb{R})} : \mathrm{I}(\mathbb{R}) \to \mathbb{R}^+$ by $\|[x,y]\|_{\mathrm{I}(\mathbb{R})} = y - x$. Then the extended bicomplete quasi-metric on $\mathrm{I}(\mathbb{R})$ induced by $\|\cdot\|_{\mathrm{I}(\mathbb{R})}$ is given by

$$d_{\|\cdot\|_{\mathrm{I}(\mathbb{R})}}\big([a,b],[c,d]\big) = \begin{cases} (d-b)+(a-c) & \text{if } [a,b] \leqslant_{\mathrm{I}(\mathbb{R})_0} [c,d], \\ +\infty & \text{otherwise.} \end{cases}$$

In [16] a metric topology is introduced in $\mathrm{I}(\mathbb{R})$ to give stopping criterias for iterative computations when a program obtains successively refined approximations to a desired result. More concretely the set $\mathrm{I}(\mathbb{R})$ becomes a metric space endowed with the Moore metric $D_{\mathrm{I}(\mathbb{R})} : \mathrm{I}(\mathbb{R}) \times \mathrm{I}(\mathbb{R}) \to \mathbb{R}^+$ given by $D_{\mathrm{I}(\mathbb{R})}([a,b],[c,d]) = \max\{|a-c|, |d-b|\}$ for all $[a,b],[c,d] \in \mathrm{I}(\mathbb{R})$. According to [1] we can equip the set $\mathrm{I}(\mathbb{R})$ with a quasi-metric $d_{\mathrm{I}(\mathbb{R})} : \mathrm{I}(\mathbb{R}) \times \mathrm{I}(\mathbb{R}) \to \mathbb{R}^+$ which is defined as follows:

$$d_{\mathrm{I}(\mathbb{R})}\big([a,b],[c,d]\big) = \max\{c-a, b-d, 0\}$$

for all $[a,b],[c,d] \in \mathrm{I}(\mathbb{R})$. This quasi-metric was introduced by Acióly and Bedregal in order to replace the Moore topology $\tau(D_{\mathrm{I}(\mathbb{R})})$ by another one which has the continuous functions as the monotonic ones. In fact, the main properties of the quasi-metric $d_{\mathrm{I}(\mathbb{R})}$ are that the topology $\tau(d_{\mathrm{I}(\mathbb{R})})$ coincides with the Scott topology on $\mathrm{I}(\mathbb{R})$ and that $D_{\mathrm{I}(\mathbb{R})}([a,b],[c,d]) = d^s_{\mathrm{I}(\mathbb{R})}([a,b],[c,d])$ for all $[a,b],[c,d] \in \mathrm{I}(\mathbb{R})$. Note that $[a,b] \leqslant_{\mathrm{I}(\mathbb{R})_0} [c,d] \Leftrightarrow d_{\mathrm{I}(\mathbb{R})}([a,b],[c,d]) = 0$.

Next we show that from a numerical computation point of view the use of our extended quasi-metric $d_{\|\cdot\|_{\mathrm{I}(\mathbb{R})}}$ presents some advantages with respect the use of the Moore metric $D_{\mathrm{I}(\mathbb{R})}$ and the quasi-metric $d_{\mathrm{I}(\mathbb{R})}$. Indeed, let us consider a sequence of intervals generated by an iteration procedure in such a way that each interval represents an approximation to the real number π. For instance, $\ldots [3.14, 3.15] \leqslant_{\mathrm{I}(\mathbb{R})_0} [3.1, 3.2] \leqslant_{\mathrm{I}(\mathbb{R})_0} [3,4]$. It is evident that $[3.14, 3.15]$ is a better approximation of π than $[3.1, 3.2]$. Clearly the quasi-metric $d_{\mathrm{I}(\mathbb{R})}$ reflects the latter fact because $d_{\mathrm{I}(\mathbb{R})}([3.14, 3.15],[3.1, 3.2]) = 0$. But the preceding quasi-metric does not quantify the variation of the approximation to π when the interval $[3.1, 3.2]$ is replaced by $[3.14, 3.15]$, since $d_{\mathrm{I}(\mathbb{R})}([\pi,\pi],[3.14, 3.15]) = d_{\mathrm{I}(\mathbb{R})}([\pi,\pi],[3.1, 3.2]) = 0$. However we have $d_{\|\cdot\|_{\mathrm{I}(\mathbb{R})}}([\pi,\pi],[3.14, 3.15]) = 10^{-2}$ and $d_{\|\cdot\|_{\mathrm{I}(\mathbb{R})}}([\pi,\pi],[3.1, 3.2]) = 10^{-1}$. Therefore the quasi-metric $d_{\|\cdot\|_{\mathrm{I}(\mathbb{R})}}$ quantifies numerically the degree of approximation of each step of the computation to the expected result π. The metric $D_{\mathrm{I}(\mathbb{R})}$ also gives a quantification of the mentioned amount of information when the interval $[3.1, 3.2]$ is replaced by $[3.14, 3.15]$ because $D_{\mathrm{I}(\mathbb{R})}([\pi,\pi],[3.14, 3.15]) = 16 \cdot 10^{-3}$ and $D_{\mathrm{I}(\mathbb{R})}([\pi,\pi],[3.1, 3.2]) = 84 \cdot 10^{-3}$. Nevertheless our extended quasi-metric, contrarily to the case of the Moore metric, gives us the degree of approximation of the intervals $[3.14, 3.15]$ and $[3.1, 3.2]$ to π. In particular the numerical values 10^{-2} and 10^{-1} show that the approximation given by the interval $[3.14, 3.15]$ contains one digit more of π than the given one by the interval $[3.1, 3.2]$.

Motivated by the preceding examples, in the sequel we will say that $(X, M, \leqslant_M, \|\cdot\|)$ is an ordered normed monoid if $(X, \|\cdot\|)$ is a normed monoid and M is a strict submonoid of X which induces on X the order $\leqslant_M$. If $X = M$, we will only write $(X, \leqslant_X, \|\cdot\|)$.

2.2. *The extension of the dual complexity space*

The extended dual complexity space introduced by Romaguera, Sánchez-Pérez and Valero (see Section 1) can be formulated in terms of ordered normed monoids.

Consider the ordered normed monoid $(\mathbb{R}^+, \leqslant_{\mathbb{R}^+}, \|\cdot\|_S)$ introduced in Example 3. The *extended dual complexity space*, such as it has been pointed out in Section 1, is the set $\mathcal{C}^*$ endowed with the complexity distance (extended quasi-metric) $e_{\mathcal{C}^*}$. Obviously $\mathcal{C}^*$ is a strict cancellative abelian monoid endowed with the usual pointwise sum operation, and the natural order $\leqslant_{\mathcal{C}^*}$ given by

$$f \leqslant_{\mathcal{C}^*} g \quad \Leftrightarrow \quad f(n) \leqslant g(n) \quad \text{for all } n \in \omega \quad \Leftrightarrow \quad g = f + h \quad \text{for some } h \in \mathcal{C}^*.$$

Furthermore if $f \leqslant_{\mathcal{C}^*} g$, then $e_{\mathcal{C}^*}(f,g) = \|h\|_{\mathcal{C}^*}$ with

$$\|h\|_{\mathcal{C}^*} = \sum_{n=0}^{\infty} 2^{-n} \big\|h(n)\big\|_S = \sum_{n=0}^{\infty} 2^{-n} h(n) = \sum_{n=0}^{\infty} 2^{-n} \big[g(n) - f(n)\big].$$

It is easy to check that $\|\cdot\|_{\mathcal{C}^*}$ is a norm on $\mathcal{C}^*$ (see [17]). Therefore the extended dual complexity space can be seen as the ordered normed monoid $(\mathcal{C}^*, \leqslant_{\mathcal{C}^*}, \|\cdot\|_{\mathcal{C}^*})$, where $e_{\mathcal{C}^*}$ is exactly the extended quasi-metric on $\mathcal{C}^*$ induced by the norm $\|\cdot\|_{\mathcal{C}^*}$.

It is clear that the set $\mathcal{C}^*$ inherits the strict cancellative abelian structure from the monoid $\mathbb{R}^+$. Moreover the norm $\|\cdot\|_{\mathcal{C}^*}$ is induced by the norm $\|\cdot\|_S$ on $\mathbb{R}^+$, and the order $\leqslant_{\mathcal{C}^*}$ is generated by the order $\leqslant_{\mathbb{R}^+}$ since $f \leqslant_{\mathcal{C}^*} g \Leftrightarrow f(n) \leqslant_{\mathbb{R}^+} g(n)$ for all $n \in \omega$. So it seems natural to consider that the extended dual complexity space $(\mathcal{C}^*, e_{\mathcal{C}^*})$ has as a base structure the ordered normed monoid $(\mathbb{R}^+, \leqslant_{\mathbb{R}^+}, \|\cdot\|_S)$. Motivated by this fact we construct a general dual complexity space from ordered normed monoids which allows us to recuperate as a particular case the extended dual complexity space $(\mathcal{C}^*, e_{\mathcal{C}^*})$.

Let $(X, M, \leqslant_M, \|\cdot\|)$ be an ordered normed monoid. Define

$$\mathcal{C}_X^* = \left\{ f : \omega \to X \;\Big|\; \sum_{n=0}^{\infty} 2^{-n} \| f(n) \| < +\infty \right\} \quad \text{and} \quad \mathcal{C}_M^* = \left\{ f \in \mathcal{C}_X^* \;\big|\; f : \omega \to M \right\}.$$

If for each $f, g \in \mathcal{C}_X^*$ we define $f + g$ in the natural way, then it is easy to see that $(\mathcal{C}_X^*, +)$ has a cancellative abelian monoid structure with $\mathcal{C}_M^*$ as a strict submonoid. Moreover $\|\cdot\|_{\mathcal{C}_X^*}$ is a norm on $\mathcal{C}_X^*$, where $\|f\|_{\mathcal{C}_X^*} = \sum_{n=0}^{\infty} 2^{-n} \|f(n)\|$ for all $f \in \mathcal{C}_X^*$.

From the order $\leqslant_M$ we can define an order $\leqslant_{\mathcal{C}_M^*}$ on $\mathcal{C}_X^*$ as follows: $f \leqslant_{\mathcal{C}_M^*} g \Leftrightarrow f(n) \leqslant_M g(n)$ for all $n \in \omega$. Note, in addition, that the order $\leqslant_{\mathcal{C}_M^*}$ is exactly the order induced by $\mathcal{C}_M^*$ on $\mathcal{C}_X^*$, i.e. $f \leqslant_{\mathcal{C}_M^*} g \Leftrightarrow g = f + h$ for some $h \in \mathcal{C}_M^*$.

Thus, by Proposition 2, a bicomplete extended quasi-metric $d_{\|\cdot\|_{\mathcal{C}_X^*}}$ can be defined on $\mathcal{C}_X^*$ by

$$d_{\|\cdot\|_{\mathcal{C}_X^*}}(f, g) = \begin{cases} \|h\|_{\mathcal{C}_X^*} & \text{if there exists } h \in \mathcal{C}_M^* \text{ with } g = f + h, \\ +\infty & \text{otherwise.} \end{cases}$$

From now on we will say that $(\mathcal{C}_X^*, \mathcal{C}_M^*, \leqslant_{\mathcal{C}_M^*}, \|\cdot\|_{\mathcal{C}_X^*})$ is the *extended dual complexity space of the ordered normed monoid* $(X, M, \leqslant_M, \|\cdot\|)$.

Remark 5. Note that the extended quasi-metric structure of the extended dual complexity space $(\mathcal{C}_X^*, \mathcal{C}_M^*, \leqslant_{\mathcal{C}_M^*}, \|\cdot\|_{\mathcal{C}_X^*})$ can be also derived directly from the extended quasi-metric structure of the ordered normed monoid $(X, M, \leqslant_M, \|\cdot\|)$. Indeed

$$d_{\|\cdot\|_{\mathcal{C}_X^*}}(f, g) = \begin{cases} \sum_{n=0}^{\infty} 2^{-n} d_{\|\cdot\|}(f(n), g(n)) & \text{if } f(n) \leqslant_M g(n) \text{ for all } n \in \omega, \\ +\infty & \text{otherwise.} \end{cases}$$

In [17] the extended dual complexity space $(\mathcal{C}^*, e_{\mathcal{C}^*})$ has been shown to be Hausdorff. However, this is not true in general for the extended dual complexity space associated to an ordered normed monoid.

Example 6. Let us consider the strict cancellative monoid $(\omega, +)$. We endow ω with the following norm: $\|n\| = 1/n$ for all $n \in \mathbb{N}$ and $\|0\| = 0$. It is clear that $(\omega, \leqslant_\omega, \|\cdot\|)$ is an ordered normed monoid. We show that $(\mathcal{C}_\omega^*, d_{\|\cdot\|_{\mathcal{C}_\omega^*}})$ is not Hausdorff. Let us consider the sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{C}_\omega^*$ such that $f_n(k) = n$ for all $k \in \omega$. Define $f(k) = 0$ and $g(k) = 1$ for all $k \in \omega$. Obviously $f, g \in \mathcal{C}_\omega^*$. It is evident that $f \leqslant_{\mathcal{C}_\omega^*} f_n$ and $g \leqslant_{\mathcal{C}_\omega^*} f_n$ for all $n \in \mathbb{N}$. Furthermore,

$$d_{\|\cdot\|_{\mathcal{C}_\omega^*}}(f, f_n) = \sum_{k=0}^{\infty} 2^{-k} \| f_n(k) \| = \sum_{k=0}^{\infty} 2^{-k} \|n\| = \frac{2}{n},$$
$$d_{\|\cdot\|_{\mathcal{C}_\omega^*}}(g, f_n) = \sum_{k=0}^{\infty} 2^{-k} \| f_{n-1}(k) \| = \sum_{k=0}^{\infty} 2^{-k} \|n-1\| = \frac{2}{n-1}$$

for all $n \geqslant 2$. Consequently, $(f_n)_{n\in\mathbb{N}}$ is $\tau(d_{\|\cdot\|_{\mathcal{C}_\omega^*}})$ convergent to f and g.

Remark 7. Nevertheless, given an ordered normed monoid $(X, M, \leqslant_M, \|\cdot\|)$ then $\tau(d_{\|\cdot\|_{\mathcal{C}_X^*}})$ is T_1 by Proposition 2. Observe that the topology induced by the complexity distance $d_{\mathcal{C}^*}$, which has been introduced in Section 1, is T_0 but not T_1.

Now, we study under which conditions we can preserve the Hausdorffness in our more general context.

Order-convex quasi-metric spaces were considered by Schellekens [21] and Romaguera and Schellekens [19,20] in order to obtain an appropriate structure for the development of a consistent theory of the complexity analysis of programs and algorithms. Next we adapt the notion of order-convex quasi-metric space to the ordered normed monoid case.

Definition 8. A convex ordered normed monoid is an ordered normed monoid $(X, M, \leqslant_M, \|\cdot\|)$ such that $\|x + y\| = \|x\| + \|y\|$ for all $x, y \in X$.

Remark 9. Note that if $(X, M, \leqslant_M, \|\cdot\|)$ is a convex ordered normed monoid, then the norm $\|\cdot\|$ is order-preserving on $(X, \leqslant_M)$, and $d_{\|\cdot\|}(x, y) = d_{\|\cdot\|}(x, z) + d_{\|\cdot\|}(z, y)$ whenever $x \leqslant_M z \leqslant_M y$.

Examples of convex ordered normed monoids are given in Examples 3 and 4.

We omit the proof of the following easy but useful result.

Proposition 10. *Let $(X, M, \leqslant_M, \|\cdot\|)$ be a convex ordered normed monoid. Then $(\mathcal{C}_X^*, \mathcal{C}_M^*, \leqslant_{\mathcal{C}_M^*}, \|\cdot\|_{\mathcal{C}^*})$ is a convex ordered normed monoid.*

Corollary 11. *The extended dual complexity space $(\mathcal{C}^*, \leqslant_{\mathcal{C}^*}, \|\cdot\|_{\mathcal{C}^*})$ is a convex ordered normed monoid.*

Theorem 12. *Let $(X, M, \leqslant_M, \|\cdot\|)$ be a convex ordered normed monoid such that $(X, \leqslant_M)$ is linear. Then the extended quasi-metric space $(\mathcal{C}_X^*, d_{\|\cdot\|_{\mathcal{C}_X^*}})$ is Hausdorff.*

Proof. Let $f, g \in \mathcal{C}_X^*$ and $(f_n)_{n\in\mathbb{N}} \subset \mathcal{C}_X^*$ such that $\lim_{n\to+\infty} d_{\|\cdot\|_{\mathcal{C}_X^*}}(f, f_n) = 0$ and $\lim_{n\to+\infty} d_{\|\cdot\|_{\mathcal{C}_X^*}}(g, f_n) = 0$. It follows that given $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $f \leqslant_{\mathcal{C}_M^*} f_n$ and $g \leqslant_{\mathcal{C}_M^*} f_n$ with $d_{\|\cdot\|_{\mathcal{C}_X^*}}(f, f_n) < \varepsilon$ and $d_{\|\cdot\|_{\mathcal{C}_X^*}}(g, f_n) < \varepsilon$ for all $n \geqslant n_0$. Whence there exist $h_n, t_n \in \mathcal{C}_M^*$ such that $f_n = f + h_n$, $f_n = g + t_n$ with $\max\{\|h_n\|_{\mathcal{C}_X^*}, \|t_n\|_{\mathcal{C}_X^*}\} < \varepsilon$ for all $n \geqslant n_0$.

Now we show that $f \leqslant_{\mathcal{C}_M^*} g$ and $g \leqslant_{\mathcal{C}_M^*} f$. To obtain a contradiction, suppose that $f \nleqslant_{\mathcal{C}_M^*} g$. Then from the linearity of $(X, \leqslant_M)$ we deduce that there exists $k \in \omega$ such that $g(k) \leqslant_M f(k)$ with $g(k) \neq f(k)$. So there exists $z \in M$ such that $z \neq 0$ and $f(k) = g(k) + z$. Since $\|h_j\|_{\mathcal{C}_X^*} < \frac{\varepsilon}{2^k}$ and $\|t_j\|_{\mathcal{C}_X^*} < \frac{\varepsilon}{2^k}$ eventually, we obtain that

$$\max\left\{\sum_{n=0}^{\infty} 2^{-n}\|h_j(n)\|, \sum_{n=0}^{\infty} 2^{-n}\|t_j(n)\|\right\} < \frac{\varepsilon}{2^k}$$

and

$$\max\{2^{-k}\|h_j(k)\|, 2^{-k}\|t_j(k)\|\} < \frac{\varepsilon}{2^k}$$

eventually.

On the other hand, by the cancellativity of $(X, +)$ we have that $t_j(k) = h_j(k) + z$. Hence, by the convexity of $\|\cdot\|$, we obtain that $\|z\| = \|t_j(k)\| - \|h_j(k)\| \leqslant \|t_j(k)\| < \varepsilon$. This contradicts the fact that $0 < \|z\|$. Therefore $f \leqslant_{\mathcal{C}_M^*} g$. Similarly we show that $g \leqslant_{\mathcal{C}_M^*} f$. By the antisymmetry of $\leqslant_{\mathcal{C}_M^*}$ we conclude that $f = g$. □

Example 6 shows that the convexity of the norm in the preceding result cannot be omitted. In the next example we show that the linearity of the order induced by the strict submonoid cannot be omitted either.

Example 13. Let $X = \{(x, y) \in \mathbb{R}^2\colon\ x \geqslant 0,\ y > 0\} \cup \{(0, 0)\}$ endowed with the usual sum $+$ on $\mathbb{R}^2$. Then the pair $(X, \leqslant_X)$ is an ordered set but it is not linear. On the other hand, $(X, +)$ has a strict cancellative monoid structure. Consider the norm q defined on X as $q((x, y)) = y$. Then $(X, \leqslant_X, q)$ is a convex ordered normed monoid. Let $f, g \in \mathcal{C}_X^*$ defined by

$$f(k) = (2, 1) \quad \text{and} \quad g(k) = (1, 1)$$

for all $k \in \omega$. Define the sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{C}_X^*$ given by

$$f_n(k) = \left(2, 1 + \frac{1}{n}\right)$$

for all $n \in \mathbb{N}$, $k \in \omega$. Clearly $f \leqslant_X f_n$ and $g \leqslant_X f_n$ for all $n \in \mathbb{N}$. Furthermore

$$d_{\|\cdot\|_{\mathcal{C}_X^*}}(f, f_n) = \sum_{k=0}^{\infty} 2^{-k}\left\|\left(0, \frac{1}{n}\right)\right\| = \frac{2}{n}$$

and

$$d_{\|\cdot\|_{\mathcal{C}_X^*}}(g, f_n) = \sum_{k=0}^{\infty} 2^{-k}\left\|\left(1, \frac{1}{n}\right)\right\| = \frac{2}{n}.$$

So $(f_n)_{n\in\mathbb{N}}$ is $\tau(d_{\|\cdot\|_{\mathcal{C}_X^*}})$ convergent to f and g, and thus $(\mathcal{C}_X^*, \|\cdot\|_{\mathcal{C}_X^*})$ is not Hausdorff.

As we have mentioned previously, the extended dual complexity space $(\mathcal{C}^*, e_{\mathcal{C}^*})$ is right K-sequentially complete. In the rest of this subsection we give conditions under which the new complexity structures satisfy the right K-sequential completeness.

We will say that an ordered normed monoid $(X, M, \leqslant_M, \|\cdot\|)$ is meet complete provided that each lower bounded subset A of X has an infimum $z_A \in X$ with respect to the order $\leqslant_M$ and $\|z_A\| = \inf_{a\in A}\|a\|$. The ordered normed monoids given in Examples 3 and 4 are meet complete. The extended dual complexity space $(\mathcal{C}^*, \leqslant_{\mathcal{C}^*}, \|\cdot\|_{\mathcal{C}^*})$ is another example of ordered normed monoid which is meet complete.

Theorem 14. *Let $(X, M, \leqslant_M, \|\cdot\|)$ be a meet complete convex ordered normed monoid. Then $(\mathcal{C}_X^*, \mathcal{C}_M^*, \leqslant_{\mathcal{C}_M^*}, \|\cdot\|_{\mathcal{C}_X^*})$ is meet complete. Furthermore, every decreasing sequence $(f_n)_{n\in\mathbb{N}}$ bounded below in $(\mathcal{C}_X^*, \leqslant_{\mathcal{C}_M^*})$ has a unique limit point in $(\mathcal{C}_X^*, d_{\|\cdot\|_{\mathcal{C}_X^*}})$.*

Proof. Let $\mathcal{F}$ be a subset of $\mathcal{C}_X^*$ bounded below. Define $F(k) = \inf_{f\in\mathcal{F}} f(k)$ for all $k \in \omega$. It is obvious that this infimum always exists because $(X, M, \leqslant_M, \|\cdot\|)$ is meet complete.

Furthermore, since $(X, M, \leqslant_M, \|\cdot\|)$ is a convex ordered normed monoid we have, by Remark 9, that $\|F(k)\| \leqslant \|f(k)\|$ for all $k \in \omega$ where f is a fixed element of $\mathcal{F}$. Hence

$$\sum_{k=0}^{\infty} 2^{-k}\|F(k)\| \leqslant \sum_{k=0}^{\infty} 2^{-k}\|f(k)\| = \|f\|_{\mathcal{C}_X^*} < +\infty.$$

Thus $F \in \mathcal{C}_X^*$. It is easily seen that F is the infimum of $\mathcal{F}$ with respect to $\leqslant_{\mathcal{C}_M^*}$.

Next we prove that $(f_n)_{n\in\mathbb{N}}$ converges to f in $(\mathcal{C}_X^*, d_{\|\cdot\|_{\mathcal{C}_X^*}})$ where f is the infimum of the set $\{f_n\colon n \in \mathbb{N}\}$. Since $f_m \leqslant_{\mathcal{C}_M^*} f_1$ for all $m \geqslant 1$, $f_1 \in \mathcal{C}_X^*$ and by Remark 9, we obtain that, given $\varepsilon > 0$, there exists $k_\varepsilon \in \omega$ such that

$$\sum_{k=k_\varepsilon+1}^{\infty} 2^{-k}\|f_m(k)\| \leqslant \sum_{k=k_\varepsilon+1}^{\infty} 2^{-k}\|f_1(k)\| < \varepsilon/3$$

for all $m \geqslant 1$.

Since $\|f(k)\| = \inf_{n\in\mathbb{N}} \|f_n(k)\|$ there exists $n_0 \in \mathbb{N}$ such that $\|f_m(k)\| - \|f(k)\| < \varepsilon/3$ for all $k = 0, \dots, k_\varepsilon$ and for all $m \geqslant n_0$. It follows, by Proposition 10, that

$$\begin{aligned} d_{\|\cdot\|_{\mathcal{C}_X^*}}(f, f_m) &= \sum_{k=0}^{\infty} 2^{-k}\big[\|f_m(k) - f(k)\|\big] \\ &\leqslant \sum_{k=0}^{k_\varepsilon} 2^{-k}\big[\|f_m(k)\| - \|f(k)\|\big] + \sum_{k=k_\varepsilon+1}^{\infty} 2^{-k}\|f_m(k)\| \\ &< \varepsilon/3\left(\sum_{k=0}^{\infty} 2^{-k}\right) + \varepsilon/3 = \varepsilon, \end{aligned}$$

for all $m \geqslant n_0$.

In order to show the uniqueness of the limit of the sequence $(f_n)_{n\in\mathbb{N}}$ assume that there exists $g \in \mathcal{C}_X^*$ such that $(f_n)_{n\in\mathbb{N}}$ is $\tau(d_{\|\cdot\|_{\mathcal{C}_X^*}})$ convergent to g. Consequently, given $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $f \leqslant_{\mathcal{C}_M^*} f_n$ and $g \leqslant_{\mathcal{C}_M^*} f_n$ with $d_{\|\cdot\|_{\mathcal{C}_X^*}}(f, f_n) < \varepsilon$ and $d_{\|\cdot\|_{\mathcal{C}_X^*}}(g, f_n) < \varepsilon$ for all $n \geqslant n_0$. Whence there exist $h_n, t_n \in \mathcal{C}_M^*$ such that $f_n = f + h_n$, $f_n = g + t_n$ with $\max\{\|h_n\|_{\mathcal{C}_X^*}, \|t_n\|_{\mathcal{C}_X^*}\} < \varepsilon$ for all $n \geqslant n_0$. It follows that $g \leqslant_{\mathcal{C}_M^*} f_n$ for all $n \in \mathbb{N}$. Hence $g \leqslant_{\mathcal{C}_M^*} f$. So there exists $h \in \mathcal{C}_X^*$ such that $f = g + h$. The preceding equalities imply that $g + t_n = g + h + h_n$ for all $n \geqslant n_0$. By the cancellativity of $(X, +)$ we deduce that $t_n = h + h_n$ for all $n \geqslant n_0$. By Proposition 10 we have that the norm $\|\cdot\|_{\mathcal{C}_X^*}$ is convex, and as a consequence we obtain that $\|t_n\|_{\mathcal{C}_X^*} = \|h\|_{\mathcal{C}_X^*} + \|h_n\|_{\mathcal{C}_X^*}$ for all $n \geqslant n_0$. Of course the preceding equality implies that $\|h\|_{\mathcal{C}_X^*} = 0$. Therefore, we conclude that $f = g$. The proof is complete. □

Corollary 15. *Let $(X, M, \leqslant_M, \|\cdot\|)$ be a meet complete convex ordered normed monoid. Then every right K-Cauchy sequence bounded below has a unique limit point in $(\mathcal{C}_X^*, d_{\|\cdot\|_{\mathcal{C}_X^*}})$.*

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a right K-Cauchy sequence bounded below. Then given $\varepsilon > 0$ we can find $n_0 \in \mathbb{N}$ such that $d_{\|\cdot\|_{\mathcal{C}_X^*}}(f_m, f_n) < \varepsilon$ for all $m \geqslant n \geqslant n_0$. Consequently, it is clear that $(f_n)_{n\in\mathbb{N}}$ is decreasing eventually so the result follows from Theorem 14. □

In the case that the strict monoid M coincides with the whole monoid X then we obtain from the preceding result the right K-sequential completeness of the extended dual complexity space $(\mathcal{C}_X^*, d_{\|\cdot\|_{\mathcal{C}_X^*}})$, since $0 \leqslant_X x$ for all $x \in X$.

Corollary 16. *Let $(X, \leqslant_X, \|\cdot\|)$ be a meet complete convex ordered normed monoid. Then the extended quasi-metric space $(\mathcal{C}_X^*, d_{\|\cdot\|_{\mathcal{C}_X^*}})$ is right K-sequentially complete.*

2.3. A connection with Interval Analysis

In 1959, R. Moore introduced the Interval Analysis as a new branch of applied mathematics [16]. In this new approach the intervals of real numbers are considered as a new kind of numbers in such a way that each interval can be seen as an approximation of the real numbers that it contains. Thus if one considers a fixed real number and an interval containing it, then the left and the right endpoints of the interval can be considered as a lower and upper bound of the real number, respectively. In the last decades it has been possible to develop efficient automatic control techniques of the errors in numerical computation that arise, from uncertain initial data or as a consequence of rounding (or truncation) operations, during the computation from the interval approach proposed by Moore (see [16] and the references in there for a detailed discussion). Since Moore's theory was introduced, the interval mathematics has become very important as a mathematical foundation of processes that arise in a natural way in several fields of computation as Denotational Semantics [23,7,8], Measurement Domain Theory [15,14] and Exact Computation [2–6,8].

In numerical analysis the methods for finding upper and lower bounds on solutions to operators equations play a central role. The Interval Analysis approach has provided many tests for the existence of solutions and the convergence of iterative methods for systems of equations, as for instance differential equations and integral equations (see [16]). All these tests are based on two simple but useful results which can be enunciated as follows:

Proposition 17. *Every decreasing sequence $(X_n)_{n\in\mathbb{N}}$ in $(\mathrm{I}(\mathbb{R}), \leqslant_{\mathrm{I}(\mathbb{R})_0})$ with lower bound $X \in \mathrm{I}(\mathbb{R})$ has a unique limit point $\bigcap_{n\in\mathbb{N}} X_n$ in $(\mathrm{I}(\mathbb{R}), D_{\mathrm{I}(\mathbb{R})})$ with $X \subseteq \bigcap_{n\in\mathbb{N}} X_n$.*

Proposition 18. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence in $(\mathrm{I}(\mathbb{R}), \leqslant_{\mathrm{I}(\mathbb{R})_0})$ such that there exists $X \in \mathrm{I}(\mathbb{R})$ satisfying that $X \subseteq X_n$ for all $n \in \mathbb{N}$. Then the sequence $(Y_n)_{n\in\mathbb{N}}$ with $Y_1 = X_1$ and $Y_{n+1} = X_{n+1} \cap Y_n$ for all $n > 1$ is a decreasing sequence with limit point $\bigcap_{n\in\mathbb{N}} X_n$ and $X \subseteq Y_n$ for all $n \in \mathbb{N}$.*

Propositions 17 and 18 are key to obtain an appropriate interval version of the Picard iteration method and the Newton method, respectively. In both methods the interval X represents an approximation of the solution of the equations system in such a way that the endpoints of X are an upper and lower bound sufficiently near to the solution of the problem. Note that this is the typical case of those problems in which we only can know the solution under a certain level of uncertainty because all the coefficients of the equations are given by empirical measures. Moreover, each interval X_n represents a successive approximation of the solution, which is provided by the iteration n of the method and gives an upper and lower bound of the solution with more accuracy than the approximations associated to the preceding steps of the process.

Next we show that the framework $(\mathrm{I}(\mathbb{R}), d_{\|\cdot\|_{\mathrm{I}(\mathbb{R})}})$ is a suitable tool to model iterative processes in the same way that $(\mathrm{I}(\mathbb{R}), D_{\mathrm{I}(\mathbb{R})})$. In particular we prove an asymmetric version of Propositions 17 and 18 via interval valued complexity functions.

Proposition 19. *Every decreasing sequence $(X_n)_{n\in\mathbb{N}}$ in $(\mathrm{I}(\mathbb{R}), \leqslant_{\mathrm{I}(\mathbb{R})_0})$ with lower bound $X \in \mathrm{I}(\mathbb{R})$ has a unique limit point in $(\mathrm{I}(\mathbb{R}), d_{\|\cdot\|_{\mathrm{I}(\mathbb{R})}})$ given by $\bigcap_{n\in\mathbb{N}} X_n$.*

Moreover, if $\lim_{n\to+\infty} \|X_n\|_{\mathrm{I}(\mathbb{R})} = 0$, then $\bigcap_{n\in\mathbb{N}} X_n \in \mathbb{R}$.

Proof. For each $n \in \mathbb{N}$, let a_n and b_n be the left and right endpoints of X_n, respectively. Define $f_n : \omega \to \mathrm{I}(\mathbb{R})$ by $f_n(k) = [a_n, b_n]$ for all $k \in \omega$ and $n \in \mathbb{N}$. Then $f_n \in \mathcal{C}^*_{\mathrm{I}(\mathbb{R})}$ for all $n \in \mathbb{N}$ because of $\sum_{k=0}^{\infty} 2^{-k}\|[a_n, b_n]\| = 2(b_n - a_n) < +\infty$. Since $X_{n+1} \subseteq X_n$ for all $n \in \mathbb{N}$ we have that $f_{n+1} \leqslant_{\mathcal{C}^*_{\mathrm{I}(\mathbb{R})}} f_n$ for all $n \in \mathbb{N}$. Thus the sequence $(f_n)_{n\in\mathbb{N}}$ is decreasing in $(\mathcal{C}^*_{\mathrm{I}(\mathbb{R})}, \leqslant_{\mathcal{C}^*_{\mathrm{I}(\mathbb{R})}})$.

Now define the function $g : \omega \to \mathrm{I}(\mathbb{R})$ given by $g(k) = [a, b]$, where we denote by a, b the left and right endpoints of X, respectively. Then g is a lower bound of $(f_n)_{n\in\mathbb{N}}$, since $X \leqslant_{\mathrm{I}(\mathbb{R})_0} X_n$ for all $n \in \mathbb{N}$. Thus $g \leqslant_{\mathcal{C}^*_{\mathrm{I}(\mathbb{R})}} f_n$ for all $n \in \mathbb{N}$. By Theorem 14 we have that the sequence $(f_n)_{n\in\mathbb{N}}$ converges to $f \in \mathcal{C}^*_{\mathrm{I}(\mathbb{R})}$ in $(\mathrm{I}(\mathbb{R}), d_{\|\cdot\|_{\mathcal{C}^*_{\mathrm{I}(\mathbb{R})}}})$, where $f(k) = \inf_{n\in\mathbb{N}} f_n(k)$ for all $k \in \omega$. Whence we deduce that $f(0) \leqslant_{\mathrm{I}(\mathbb{R})_0} f_n(0)$ for all $n \in \mathbb{N}$ so $f(0) \subseteq \bigcap_{n\in\mathbb{N}} [a_n, b_n]$. Denote by $a_{f(0)}, b_{f(0)}$ the left and right endpoint of the interval $f(0)$.

On the other hand, the fact that $\lim_{n\to+\infty} d_{\|\cdot\|_{\mathcal{C}^*_{\mathrm{I}(\mathbb{R})}}}(f, f_n) = 0$ implies that $\lim_{n\to+\infty} d_{\|\cdot\|_{\mathrm{I}(\mathbb{R})}}(f(0), f_n(0)) = 0$. So $\lim_{n\to+\infty} a_n = a_{f(0)}$ and $\lim_{n\to+\infty} b_n = b_{f(0)}$. Consequently $a_{f(0)} = \sup_{n\in\mathbb{N}} a_n$ and $b_{f(0)} = \inf_{n\in\mathbb{N}} b_n$. Therefore we conclude that $[a_{f(0)}, b_{f(0)}] = \bigcap_{n\in\mathbb{N}} X_n$. Theorem 14 guarantees that $\bigcap_{n\in\mathbb{N}} X_n$ is the unique limit point of $(X_n)_{n\in\mathbb{N}}$ in $(\mathrm{I}(\mathbb{R}), d_{\|\cdot\|_{\mathrm{I}(\mathbb{R})}})$.

Finally, if $\lim_{n\to+\infty} \|X_n\|_{\mathrm{I}(\mathbb{R})} = 0$ then we have that $\lim_{n\to+\infty} b_n - a_n = 0$. Whence we obtain that $a_{f(0)} = b_{f(0)}$, and, thus $a_{f(0)} = \bigcap_{n\in\mathbb{N}} X_n$, i.e. $\bigcap_{n\in\mathbb{N}} X_n \in \mathbb{R}$. □

Corollary 20. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence in $(\mathrm{I}(\mathbb{R}), \leqslant_{\mathrm{I}(\mathbb{R})_0})$ such that there exists $X \in \mathrm{I}(\mathbb{R})$ satisfying that $X \subseteq X_n$ for all $n \in \mathbb{N}$. Then the sequence $(Y_n)_{n\in\mathbb{N}}$ with $Y_1 = X_1$ and $Y_{n+1} = X_{n+1} \cap Y_n$ for all $n > 1$ is a decreasing sequence with limit point $\bigcap_{n\in\mathbb{N}} Y_n$ in $(\mathrm{I}(\mathbb{R}), d_{\|\cdot\|_{\mathrm{I}(\mathbb{R})}})$ and $X \subseteq Y_n$ for all $n \in \mathbb{N}$.*

Proof. Since $X \subseteq X_n$ for all $n \in \mathbb{N}$ we have that $Y_n \neq \varnothing$ for all $n \in \mathbb{N}$. Furthermore, by construction the sequence $(Y_n)_{n\in\mathbb{N}}$ is decreasing in $(\mathrm{I}(\mathbb{R}), \leqslant_{\mathrm{I}(\mathbb{R})_0})$ and it has as a lower bound the interval X. Applying Proposition 19 we obtain the desired conclusion. □

Note that in the preceding result the lower bound X represents an approximation of the solution of the problem, and that the solution given by the iterative process contains it.

Of course Proposition 19 and Corollary 20 establish the basis for a whole range of applications of quantitative asymmetric "metric" tools to Interval Analysis.

Observe that, contrary to the case of $(\mathrm{I}(\mathbb{R}), d_{\|\cdot\|_{\mathrm{I}(\mathbb{R})}})$, the quasi-metric space $(\mathrm{I}(\mathbb{R}), d_{\mathrm{I}(\mathbb{R})})$ is not a suitable framework, from a quantitative viewpoint, to model iterative processes in Interval Analysis as it has been shown in Example 4. Now let us consider the decreasing sequence $([0, 1+\frac{1}{n}])_{n\in\mathbb{N}}$ in $(\mathrm{I}(\mathbb{R}), \leqslant_{\mathrm{I}(\mathbb{R})_0})$ provided by an iterative process. We have that $\lim_{n\to+\infty} d_{\mathrm{I}(\mathbb{R})}([0,1],[0,1+\frac{1}{n}]) = 0$ and that $\lim_{n\to+\infty} d_{\mathrm{I}(\mathbb{R})}([0,0],[0,1+\frac{1}{n}]) = 0$. So we cannot decide which of both limits is the appropriate approximation of the solution of the iterative process. In addition if the solution of the original problem is in $]0,1]$ we make an error considering as a good approximation of it the interval $[0,0]$. However, in our context this situation is not possible because Theorem 14 guarantees the uniqueness of the limit of decreasing sequences. Moreover the above fact shows that $\tau(d_{\mathrm{I}(\mathbb{R})})$ is not Hausdorff. Nevertheless we have that $D_{\mathrm{I}(\mathbb{R})}([a,b],[c,d]) \leqslant d_{\|\cdot\|_{\mathrm{I}(\mathbb{R})}}([a,b],[c,d])$ for all $[a,b],[c,d] \in \mathrm{I}(\mathbb{R})$ and, thus, $\tau(d_{\|\cdot\|_{\mathrm{I}(\mathbb{R})}})$ is Hausdorff.

Recently in [15], K. Martin has established the correctness of the bisection method via techniques based on asymmetric topology (measurements, fixed point theory and interval numbers). We end the paper showing the correctness of the mentioned method using Proposition 19.

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose that $[a,b]$ is an element of $\mathrm{I}(\mathbb{R})$ containing only one root of f and such that $f(a)\cdot f(b) < 0$. Define the interval operator $R : \mathrm{I}(\mathbb{R}) \to \mathrm{I}(\mathbb{R})$ by

$$R\big([c,d]\big) = \begin{cases} \mathrm{left}[c,d] & \text{if } f(c)\cdot f(\frac{c+d}{2}) < 0, \\ \mathrm{right}[c,d] & \text{if } f(c)\cdot f(\frac{c+d}{2}) > 0, \\ [\frac{c+d}{2}, \frac{c+d}{2}] & \text{if } f(\frac{c+d}{2}) = 0, \end{cases}$$

where $\mathrm{left}[c,d] = [c, \frac{c+d}{2}]$ and $\mathrm{right}[c,d] = [\frac{c+d}{2}, d]$. Then it is clear that the sequence $(R^n([a,b]))_{n\in\mathbb{N}}$ is decreasing in $(\mathrm{I}(\mathbb{R}), \leqslant_{\mathrm{I}(\mathbb{R})_0})$ with lower bound $[r,r]$. By Proposition 19 we have that the sequence $(R^n([a,b]))_{n\in\mathbb{N}}$ has a unique limit point $\bigcap_{n\in\mathbb{N}} R^n([a,b])$ and $[r,r] \subseteq \bigcap_{n\in\mathbb{N}} R^n([a,b])$. Since $\|R^n([a,b])\|_{\mathrm{I}(\mathbb{R})} = \frac{b-a}{2^n}$ for all $n \in \mathbb{N}$, $\lim_{n\to+\infty} \|R^n([a,b])\|_{\mathrm{I}(\mathbb{R})} = 0$. Therefore $[r,r] = \bigcap_{n\in\mathbb{N}} R^n([a,b])$.

Acknowledgement

The authors are grateful to the referee for the useful suggestions that led to the improvement of the original version of the paper.

References

[1] B.M. Acióly, R.C. Bedregal, A quasi-metric topology compatible with inclusion monotonicity on interval space, Reliab. Comput. 3 (1997) 305–313.
[2] A. Edalat, A. Lieutier, D. Pattinson, A computational model for multi-variable differential calculus, in: Proc. 8th International Conference on Foundations of Software Science and Computation Structures, 2006, Edinburgh, UK, in: Lecture Notes in Comput. Sci., vol. 3441, Springer, 2005, pp. 505–519.
[3] A. Edalat, D. Pattinson, A domain theoretic account of Euler's method for solving initial value problems, in: Proc. Workshop on State-of-the-Art in Scientific Computing, 2004, Copenhagen, Denmark, in: Lecture Notes in Comput. Sci., vol. 3732, Springer, 2006, pp. 112–121.
[4] A. Edalat, D. Pattinson, A domain theoretic account of Picard's theorem, in: Proc. 31th International Colloquium on Automata, Languages and Programming, 2004, Turku, Finland, in: Lecture Notes in Comput. Sci., vol. 3142, Springer, 2004, pp. 494–505.
[5] A. Edalat, M. Krznaric, A. Lieutier, Domain-theoretic solution of differential equations (Scalar Fields), Electron. Notes Theor. Comput. Sci. 83 (2004), 15 pp.
[6] A. Edalat, A. Lieutier, Domain theory and differential calculus (Functions of one variable), Math. Structures Comput. Sci. 14 (2004) 771–802.
[7] M.H. Escardó, PCF extended with real numbers, Theoret. Comput. Sci. 162 (1996) 79–115.
[8] M.H. Escardó, T. Streicher, Induction and recursion on the partial real line with applications to Real PCF, Theoret. Comput. Sci. 210 (1999) 121–157.
[9] P. Fletcher, W.F. Lindgren, Quasi-Uniform Spaces, Lect. Notes Pure Appl. Math., vol. 77, Marcel Dekker, 1982.
[10] H.P.A. Künzi, Nonsymmetric topology, in: Proc. Colloquium on Topology, 1993, Szekszárd, Hungary, in: Colloq. Math. Soc. János Bolyai Math. Stud., vol. 4, 1995, pp. 303–338.
[11] H.P.A. Künzi, S. Romaguera, Spaces of continuous functions and quasi-uniform convergence, Acta Math. Hungar. 75 (1997) 287–298.
[12] H.P.A. Künzi, C. Ryser, The Bourbaki quasi-uniformity, Topology Proc. 20 (1995) 161–183.
[13] B. Fuchssteiner, W. Lusky, Convex Cones, North-Holland, 1981.
[14] K. Martin, Powerdomains and zero finding, Electron. Notes Theor. Comput. Sci. 59 (2002) 173–184.
[15] K. Martin, The measurement process in domain theory, in: Proc. 27th International Colloquium on Automata, Languages and Programming, 2000, Geneva, Switzerland, in: Lecture Notes in Comput. Sci., vol. 1853, Springer, 2000, pp. 116–126.
[16] R.E. Moore, Methods and Applications of Interval Analysis, SIAM Stud. Appl. Math., Philadelphia, 1979.
[17] S. Romaguera, E.A. Sánchez-Pérez, O. Valero, Computing complexity distances between algorithms, Kybernetika 39 (2003) 569–582.
[18] S. Romaguera, E.A. Sánchez-Pérez, O. Valero, Quasi-normed monoids and quasi-metrics, Publ. Math. Debrecen 62 (2003) 53–69.
[19] S. Romaguera, M.P. Schellekens, Quasi-metric properties of complexity spaces, Topology Appl. 98 (1999) 311–322.
[20] S. Romaguera, M.P. Schellekens, Duality and quasi-normability for complexity spaces, Appl. Gen. Topol. 3 (2002) 91–112.

[21] M.P. Schellekens, Extendible spaces, Appl. Gen. Topol. 3 (2002) 169–184.
[22] M.P. Schellekens, The Smyth completion: A common foundation for the denotational semantics and complexity analysis, in: Proc. MFPS 11, Electron. Notes Theor. Comput. Sci. 1 (1995) 211–232.
[23] D. Scott, Lattice theory, data types and semantics, in: Proc. Courant Computer Science Symposium on Formal Semantic of Programming Languages, 1972, Englewood Cliffs, USA, Prentice Hall, 1972, pp. 66–106.
[24] W.F. Vasantha Kandasamy, Bialgebraic Structures and Smarandache Bialgebraic Structures, American Research Press, Rehoboth, 2003.