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ON FIVE SMARANDACHE'S PROBLEMS

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The eight problem from [1] (see also 16-th problem from [2]) is the following:

Smarandache mobile periodicals (I):

$$\begin{array}{ccccccccccccccc}
\ldots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & \ldots \\
\ldots & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & \ldots \\
\ldots & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & \ldots \\
 & & & \vdots & & & & \vdots & & & & \vdots & & &
\end{array}$$

This sequence has the form

$$\underbrace{1, 111, 11011, 111, 1}_{5}, \underbrace{1, 111, 11011, 1100011, 11011, 111, 1}_{7},$$

$$\underbrace{1, 111, 11011, 1100011, 110000011, , 1100011, 11011, 111, 1}_{9}, \ldots$$

All digits from the above table generate an infinite matrix A. We shall describe the elements of A.

Let us take a Cartesian coordinate system C with origin in the point containing element "1" in the topmost (i.e., the first) row of A. We assume that this row belongs to the ordinate axis of C (see Fig. 1) and that the points to the right of the origin have positive ordinates.

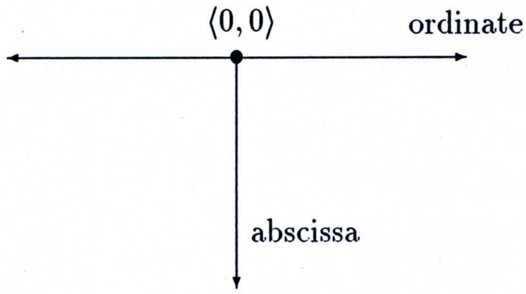

Fig. 1.

The above digits generate an infinite sequence of squares, located in the half-plane (determined by C) where the abscisses of the points are nonnegative. Their diameters have the form

"110...011".

Exactly one of the diameters of each of considered squares lies on the abscissa of C. It can be seen (and proved, e.g., by induction) that the s-th square, denoted by G_s $(s = 0, 1, 2, ...)$ has a diameter with lenght $2s+4$ and the same square has a highest vertex with coordinates $\langle s^2 + 3s, 0\rangle$ in C and a lowest vertex with coordinates $\langle s^2 + 5s + 4, 0\rangle$ in C.

Let us denote by $a_{k,i}$ an element of A with coordinates $\langle k, i\rangle$ in C.

First, we determine the minimal nonnegative s for which the inequality

$$s^2 + 5s + 4 \geq k$$

holds. We denote it by $s(k)$. Directly it is seen the following

Lemma. The number $s(k)$ admits the explicit representation:

$$s(k) = \begin{cases} 0, & \text{if } 0 \leq k \leq 4 \\ [\frac{\sqrt{4k+9}-5}{2}], & \text{if } k \geq 5 \text{ and } 4k+9 \text{ is a square of an integer} \\ [\frac{\sqrt{4k+9}-5}{2}] + 1, & \text{if } k \geq 5 \text{ and } 4k+9 \text{ is not a square of an integer} \end{cases} \quad (1)$$

and the inequalities

$$(s(k))^2 + 3s(k) \leq k \leq (s(k))^2 + 5s(k) + 4 \tag{2}$$

hold.

Second, we introduce the integeres $\delta(k)$ and $\varepsilon(k)$ by

$$\delta(k) \equiv k - (s(k))^2 - 3s(k), \tag{3}$$

$$\varepsilon(k) \equiv (s(k))^2 + 5s(k) + 4 - k. \tag{4}$$

From (2) we have $\delta(k) \geq 0$ and $\varepsilon(k) \geq 0$. Let P_k be the infinite strip orthogonal to the abscissa of C and lying between the straight lines passing through those vertices of the square $G_{s(k)}$ lying on the abscissa of C. Then $\delta(k)$ and $\varepsilon(k)$ characterize the location of point with coordinates $\langle k, i\rangle$ in C in strip P_k. Namely, the following assertion is true.

Proposition 1. The elements $a_{k,i}$ of the infinite matrix A are described as follows: if $k \leq (s(k))^2 + 4s(k) + 2$, then

$$a_{k,i} = \begin{cases} 0, & \text{if } \delta(k) < |i| \text{ or } \delta(k) \geq |i| + 2, \\ 1, & \text{if } |i| \leq \delta(k) \leq |i| + 1 \end{cases} ; \tag{5}$$

if $k > (s(k))^2 + 4s(k) + 2$, then

$$a_{k,i} = \begin{cases} 0, & \text{if } \varepsilon(k) < |i| \text{ or } \varepsilon(k) \geq |i| + 2, \\ 1, & \text{if } |i| \leq \varepsilon(k) \leq |i| + 1 \end{cases} , \tag{6}$$

where here and below $s(k)$ is given by (1), $\delta(k)$ and $\varepsilon(k)$ are given by (3) and (4), respectively.

Omitting the obvious proof (it can be done, e.g., by induction), we note that (5) gives a description of $a_{k,i}$ for the case when $\langle k, i\rangle$ belongs to the strip that is orthogonal to the abscissa of C and lying between the straight lines through the points in C with coordinates $\langle (s(k))^2+3s(k), 0\rangle$ and $\langle (s(k))^2+4s(k)+2, 0\rangle$ (involving these straight lines), while (6) gives a description of $a_{k,i}$ for the case when $\langle k, i\rangle$ belongs to the strip that is also orthogonal to the abscissa of C, but lying between the straight lines through the points in C with coordinates $\langle (s(k))^2 + 4s(k) + 2, 0\rangle$ and $\langle (s(k))^2 + 5s(k) + 4, 0\rangle$ (without involving the straight line passing through the point in C with coordinates $\langle (s(k))^2 + 4s(k) + 2, 0\rangle$).

Below, we propose another description of elements of A, which can be proved (e.g., by induction) using the same considerations.

$$a_{k,i} = \begin{cases} 1, & \text{if } \langle k,i\rangle \in \\ & \{\langle (s(k))^2+3s(k),0\rangle, \langle (s(k))^2+5s(k)+4,0\rangle\} \\ & \cup\{\langle (s(k))^2+3s(k)+j,-j\rangle, \\ & \langle (s(k))^2+3s(k)+j,-j+1\rangle, \\ & \langle (s(k))^2+3s(k)+j,j-1\rangle, \\ & \langle (s(k))^2+3s(k)+j,j\rangle : 1 \le j \le s(k)+2\} \\ & \langle (s(k))^2+5s(k)+4-j,-j\rangle, \\ & \langle (s(k))^2+5s(k)+4-j,-j+1\rangle, \\ & \langle (s(k))^2+5s(k)+4-j,j-1\rangle, \\ & \langle (s(k))^2+5s(k)+4-j,j\rangle : \\ & 1 \le j \le s(k)+1\} \\ 0, & \text{otherwise} \end{cases} \tag{7}$$

Similar representations are possible for all of the next problems.

Let us denote by $\overline{u_1 u_2 ... u_s}$ an s-digit number.

For Smarandache's sequence from Problem 8

$$1, 111, 11011, 111, 1, 111, 11011, 1100011, 11011, 111, 1, ...,$$

that is given above, if we denote it by $\{b_k\}_{k=0}^{\infty}$, then we obtain the representation

$$b_k = \begin{cases} \overline{a_{k,\delta(k)}a_{k,\delta(k)-1}...a_{k,0}a_{k,1}...a_{k,\delta(k)-1}a_{k,\delta(k)}}, \\ \qquad \text{if } k \le (s(k))^2+4s(k)+2 \\ \overline{a_{k,\varepsilon(k)}a_{k,\varepsilon(k)-1}...a_{k,0}a_{k,1}...a_{k,\varepsilon(k)-1}a_{k,\varepsilon(k)}}, \\ \qquad \text{if } k > (s(k))^2+4s(k)+2 \end{cases}$$

where $a_{k,i}$ are given in an explicit form by (5).

The 10-th Smarandache problem is dual to the above one:

Smarandache infinite numbers (I):

$$
\begin{array}{ccccccccccccccc}
\dots & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & \dots \\
\dots & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & \dots \\
\dots & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & \dots \\
\dots & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & \dots \\
\dots & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & \dots \\
\dots & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & \dots \\
\dots & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & \dots \\
\dots & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & \dots \\
\dots & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & \dots \\
\dots & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & \dots \\
\dots & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & \dots \\
\dots & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & \dots \\
\dots & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & \dots \\
\dots & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & \dots \\
\dots & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & \dots \\
\dots & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & \dots \\
\dots & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & \dots \\
\dots & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & \dots \\
\dots & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & \dots \\
\dots & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & \dots \\
\dots & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & \dots \\
\dots & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & \dots \\
\dots & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & \dots \\
\dots & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \dots \\
\dots & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & \dots \\
 & & & \vdots & & & & \vdots & & & & \vdots & & &
\end{array}
$$

Further, we will keep the notations: A (for the matrix) and $a_{k,i}$ (for its elements) from the 8-th Smarandache's problem, for each one of the next problems in this section.

Proposition 2. The elements $a_{k,i}$ of the infinite matrix A are described as follows if $k \leq (s(k))^2 + 4s(k) + 2$, then

$$a_{k,i} = \begin{cases} 0, & \text{if } |i| \leq \delta(k) \leq |i| + 1 \\ 1, & \text{if } \delta(k) < |i| \text{ or } \delta(k) \geq |i| + 2 \end{cases};$$

if $k > (s(k))^2 + 4s(k) + 2$, then

$$a_{k,i} = \begin{cases} 0, & \text{if } |i| \leq \varepsilon(k) \leq |i| + 1 \\ 1, & \text{if } \varepsilon(k) < |i| \text{ or } \varepsilon(k) \geq |i| + 2 \end{cases}.$$

The 9-th Smarandache problem is a modification and extension of the 8-th problem:

...	0	0	0	0	0	0	1	0	0	0	0	0	0	...
...	0	0	0	0	0	1	1	1	0	0	0	0	0	...
...	0	0	0	0	1	1	2	1	1	0	0	0	0	...
...	0	0	0	0	0	1	1	1	0	0	0	0	0	...
...	0	0	0	0	0	0	1	0	0	0	0	0	0	...
...	0	0	0	0	0	1	1	1	0	0	0	0	0	...
...	0	0	0	0	1	1	2	1	1	0	0	0	0	...
...	0	0	0	1	1	2	3	2	1	1	0	0	0	...
...	0	0	0	0	1	1	2	1	1	0	0	0	0	...
...	0	0	0	0	0	1	1	1	0	0	0	0	0	...
...	0	0	0	0	0	0	1	0	0	0	0	0	0	...
...	0	0	0	0	0	1	1	1	0	0	0	0	0	...
...	0	0	0	0	1	1	2	1	1	0	0	0	0	...
...	0	0	0	1	1	2	3	2	1	1	0	0	0	...
...	0	0	1	1	2	3	4	3	2	1	1	0	0	...
...	0	0	0	1	1	2	3	2	1	1	0	0	0	...
...	0	0	0	0	1	1	2	1	1	0	0	0	0	...
...	0	0	0	0	0	1	1	1	0	0	0	0	0	...
...	0	0	0	0	0	0	1	0	0	0	0	0	0	...
...	0	0	0	0	0	1	1	1	0	0	0	0	0	...
...	0	0	0	0	1	1	2	1	1	0	0	0	0	...
...	0	0	0	1	1	2	3	2	1	1	0	0	0	...
...	0	0	1	1	2	3	4	3	2	1	1	0	0	...
...	0	1	1	2	3	4	5	4	3	2	1	1	0	...
...	0	0	1	1	2	3	4	3	2	1	1	0	0	...
			⋮				⋮				⋮			

This sequence has the form

$$\underbrace{1,111,11211,111,1}_{5},\underbrace{1,111,11211,1123211,11211,111,1}_{7},$$

$$\underbrace{1,111,11211,1123211,112343211,,1123211,11211,111,1}_{9},\ldots$$

Proposition 3. The elements $a_{k,i}$ of infinite matrix A are described as follow: if $k \le (s(k))^2 + 4s(k) + 2$, then

$$a_{k,i} = \begin{cases} 0, & \text{if } \delta(k) < |i| \\ 1, & \text{if } \delta(k) = |i| \\ \delta(k) - |i|, & \text{if } \delta(k) > |i| \end{cases}; \tag{8}$$

if $k > (s(k))^2 + 4s(k) + 2$, then

$$a_{k,i} = \begin{cases} 0, & \text{if } \varepsilon(k) < |i| \\ 1, & \text{if } \varepsilon(k) = |i| \\ \varepsilon(k) - |i|, & \text{if } \varepsilon(k) > |i| \end{cases}, \tag{9}$$

For the above sequence

$$1, 111, 11211, 111, 1, 111, 11211, 1123211, 11211, 111, 1, \ldots$$

if we denote it by $\{c_k\}_{k=0}^{\infty}$, then we obtain the representation

$$c_k = \begin{cases} \overline{\overline{\overline{a_{k,\delta(k)}a_{k,\delta(k)-1}}\ldots\overline{a_{k,0}a_{k,1}}\ldots\overline{a_{k,\delta(k)-1}a_{k,\delta(k)}}}}, \\ \qquad \text{if } k \leq (s(k))^2 + 4s(k) + 2 \\ \overline{\overline{\overline{a_{k,\varepsilon(k)}a_{k,\varepsilon(k)-1}}\ldots\overline{a_{k,0}a_{k,1}}\ldots\overline{a_{k,\varepsilon(k)-1}a_{k,\varepsilon(k)}}}}, \\ \qquad \text{if } k > (s(k))^2 + 4s(k) + 2 \end{cases}$$

where $a_{k,i}$ are given in an explicit form by (8).

The 11-th Smarandache problem is a modification of the 9-th problem:

Smarandache infinite numbers (II):

$$
\begin{array}{ccccccccccccccc}
\dots & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & \dots \\
\dots & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & \dots \\
\dots & 1 & 1 & 1 & 1 & 2 & 2 & 3 & 2 & 2 & 1 & 1 & 1 & 1 & \dots \\
\dots & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & \dots \\
\dots & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & \dots \\
\dots & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & \dots \\
\dots & 1 & 1 & 1 & 1 & 2 & 2 & 3 & 2 & 2 & 1 & 1 & 1 & 1 & \dots \\
\dots & 1 & 1 & 1 & 2 & 2 & 3 & 4 & 3 & 2 & 2 & 1 & 1 & 1 & \dots \\
\dots & 1 & 1 & 1 & 1 & 2 & 2 & 3 & 2 & 2 & 1 & 1 & 1 & 1 & \dots \\
\dots & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & \dots \\
\dots & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & \dots \\
\dots & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & \dots \\
\dots & 1 & 1 & 1 & 1 & 2 & 2 & 3 & 2 & 2 & 1 & 1 & 1 & 1 & \dots \\
\dots & 1 & 1 & 1 & 2 & 2 & 3 & 4 & 3 & 2 & 2 & 1 & 1 & 1 & \dots \\
\dots & 1 & 1 & 2 & 2 & 3 & 4 & 5 & 4 & 3 & 2 & 2 & 1 & 1 & \dots \\
\dots & 1 & 1 & 1 & 2 & 2 & 3 & 4 & 3 & 2 & 2 & 1 & 1 & 1 & \dots \\
\dots & 1 & 1 & 1 & 1 & 2 & 2 & 3 & 2 & 2 & 1 & 1 & 1 & 1 & \dots \\
\dots & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & \dots \\
\dots & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & \dots \\
\dots & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & \dots \\
\dots & 1 & 1 & 1 & 1 & 2 & 2 & 3 & 2 & 2 & 1 & 1 & 1 & 1 & \dots \\
\dots & 1 & 1 & 1 & 2 & 2 & 3 & 4 & 3 & 2 & 2 & 1 & 1 & 1 & \dots \\
\dots & 1 & 1 & 2 & 2 & 3 & 4 & 5 & 4 & 3 & 2 & 2 & 1 & 1 & \dots \\
\dots & 1 & 2 & 2 & 3 & 4 & 5 & 6 & 5 & 4 & 3 & 2 & 2 & 1 & \dots \\
\dots & 1 & 1 & 2 & 2 & 3 & 4 & 5 & 4 & 3 & 2 & 2 & 1 & 1 & \dots \\
 & & & \cdot & & & & \cdot & & & & \cdot & & & \\
 & & & \cdot & & & & \cdot & & & & \cdot & & &
\end{array}
$$

Proposition 4. The elements $a_{k,i}$ of the infinite matrix A are described as follows: if $k \leq (s(k))^2 + 4s(k) + 2$, then

$$
a_{k,i} = \begin{cases} 1, & \text{if } \delta(k) < |i| \\ 2, & \text{if } |i| \leq \delta(k) \leq |i| + 1 \\ \delta(k) - |i| + 1, & \text{if } \delta(k) \geq |i| + 2 \end{cases} ;
$$

if $k > (s(k))^2 + 4s(k) + 2$, then

$$
a_{k,i} = \begin{cases} 1, & \text{if } \varepsilon(k) < |i| \\ 2, & \text{if } |i| \leq \varepsilon(k) = |i| + 1 \\ \varepsilon(k) - |i| + 1, & \text{if } \varepsilon(k) \geq |i| + 2 \end{cases} ,
$$

Now, we introduce modifications of the above problems, giving formulae for their (k, i)-th members $a_{k,i}$.

We modify the first of the above problems, now – with a simple countours of the squares in the matrix:

$$
\begin{array}{ccccccccccccccc}
\ldots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \ldots \\
\ldots & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \ldots \\
\ldots & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \ldots \\
 & & & \vdots & & & & \vdots & & & & \vdots & & &
\end{array}
$$

Proposition 5. The elements $a_{k,i}$ of the infinite matrix A are described as follows:

if $k \le (s(k))^2 + 4s(k) + 2$, then

$$a_{k,i} = \begin{cases} 0, & \text{if } \delta(k) < |i| \text{ or } \delta(k) > |i| \\ 1, & \text{if } \delta(k) = |i| \end{cases}; \tag{10}$$

if $k > (s(k))^2 + 4s(k) + 2$, then

$$a_{k,i} = \begin{cases} 0, & \text{if } \varepsilon(k) < |i| \text{ or } \varepsilon(k) > |i| \\ 1, & \text{if } \varepsilon(k) = |i| \end{cases}, \tag{11}$$

Next, we will modify the third of the above problems, again with a simple countour of the squares:

$$
\begin{array}{ccccccccccccccc}
\dots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 2 & 1 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 2 & 1 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 3 & 2 & 1 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 2 & 1 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 2 & 1 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 3 & 2 & 1 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 1 & 2 & 3 & 4 & 5 & 4 & 3 & 2 & 1 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 3 & 2 & 1 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 2 & 1 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 2 & 1 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 3 & 2 & 1 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 1 & 2 & 3 & 4 & 5 & 4 & 3 & 2 & 1 & 0 & 0 & \dots \\
\dots & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 5 & 4 & 3 & 2 & 1 & 0 & \dots \\
\dots & 0 & 0 & 1 & 2 & 3 & 4 & 5 & 4 & 3 & 2 & 1 & 0 & 0 & \dots \\
 & & & \cdot & & & & \cdot & & & & \cdot & & & \\
 & & & \cdot & & & & \cdot & & & & \cdot & & &
\end{array}
$$

Proposition 6. The elements $a_{k,i}$ of the infinite matrix A are described as follow:

if $k \leq (s(k))^2 + 4s(k) + 2$, then

$$
a_{k,i} = \begin{cases} 0, & \text{if } \delta(k) < |i| \\ \delta(k) - |i| + 1, & \text{if } \delta(k) \geq |i| \end{cases}; \tag{12}
$$

if $k > (s(k))^2 + 4s(k) + 2$, then

$$
a_{k,i} = \begin{cases} 0, & \text{if } \varepsilon(k) < |i| \\ \varepsilon(k) - |i| + 1, & \text{if } \varepsilon(k) \geq |i| \end{cases}, \tag{13}
$$

Third, we will fill the interior of the squares with Fibonacci numbers

$$
\begin{array}{ccccccccccccccc}
\ldots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 1 & 1 & 2 & 3 & 2 & 1 & 1 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 1 & 1 & 2 & 3 & 2 & 1 & 1 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 1 & 1 & 2 & 3 & 5 & 3 & 2 & 1 & 1 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 1 & 1 & 2 & 3 & 2 & 1 & 1 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 1 & 1 & 2 & 3 & 2 & 1 & 1 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 1 & 1 & 2 & 3 & 5 & 3 & 2 & 1 & 1 & 0 & 0 & \ldots \\
\ldots & 0 & 1 & 1 & 2 & 3 & 5 & 8 & 5 & 3 & 2 & 1 & 1 & 0 & \ldots \\
\ldots & 0 & 0 & 1 & 1 & 2 & 3 & 5 & 3 & 2 & 1 & 1 & 0 & 0 & \ldots \\
 & & & \vdots & & & & \vdots & & & & \vdots & & &
\end{array}
$$

Proposition 7. The elements $a_{k,i}$ of the infinite matrix A are described as follows: if $k \leq (s(k))^2 + 4s(k) + 2$, then

$$
a_{k,i} = \begin{cases} 0, & \text{if } \delta(k) < |i| \\ F_{\delta(k)-|i|}, & \text{if } \delta(k) \geq |i| \end{cases}; \tag{14}
$$

if $k > (s(k))^2 + 4s(k) + 2$, then

$$
a_{k,i} = \begin{cases} 0, & \text{if } \varepsilon(k) < |i| \\ F_{\varepsilon(k)-|i|}, & \text{if } \varepsilon(k) \geq |i| \end{cases}, \tag{15}
$$

where F_m $(m = 0, 1, 2, ...)$ is the m-th Fibonacci number.

Fourth, we will fill the interior of the squares with powers of 2:

...	0	0	0	0	0	0	1	0	0	0	0	0	0	...
...	0	0	0	0	0	1	2	1	0	0	0	0	0	...
...	0	0	0	0	1	2	4	2	1	0	0	0	0	...
...	0	0	0	0	0	1	2	1	0	0	0	0	0	...
...	0	0	0	0	0	0	1	0	0	0	0	0	0	...
...	0	0	0	0	0	1	2	1	0	0	0	0	0	...
...	0	0	0	0	1	2	4	2	1	0	0	0	0	...
...	0	0	0	1	2	4	8	4	2	1	0	0	0	...
...	0	0	0	0	1	2	4	2	1	0	0	0	0	...
...	0	0	0	0	0	1	2	1	0	0	0	0	0	...
...	0	0	0	0	0	0	1	0	0	0	0	0	0	...
...	0	0	0	0	0	1	2	1	0	0	0	0	0	...
...	0	0	0	0	1	2	4	2	1	0	0	0	0	...
...	0	0	0	1	2	4	8	4	2	1	0	0	0	...
...	0	0	1	2	4	8	16	8	4	2	1	0	0	...
...	0	0	0	1	2	4	8	4	2	1	0	0	0	...
...	0	0	0	0	1	2	4	2	1	0	0	0	0	...
...	0	0	0	0	0	1	2	1	0	0	0	0	0	...
...	0	0	0	0	0	0	1	0	0	0	0	0	0	...
...	0	0	0	0	0	1	2	1	0	0	0	0	0	...
...	0	0	0	0	1	2	4	2	1	0	0	0	0	...
...	0	0	0	1	2	4	8	4	2	1	0	0	0	...
...	0	0	1	2	4	8	16	8	4	2	1	0	0	...
...	0	1	2	4	8	16	32	16	8	4	2	1	0	...
...	0	0	1	2	4	8	16	8	4	2	1	0	0	...
			.				.				.			
			.				.				.			

Proposition 8. The elements $a_{k,i}$ of the infinite matrix A are described as follows: if $k \leq (s(k))^2 + 4s(k) + 2$, then

$$a_{k,i} = \begin{cases} 0, & \text{if } \delta(k) < |i| \\ 2^{\delta(k)-|i|}, & \text{if } \delta(k) \geq |i| \end{cases} ; \tag{16}$$

if $k > (s(k))^2 + 4s(k) + 2$, then

$$a_{k,i} = \begin{cases} 0, & \text{if } \varepsilon(k) < |i| \\ 2^{\varepsilon(k)-|i|}, & \text{if } \varepsilon(k) \geq |i| \end{cases} , \tag{17}$$

Fifth, we will fill the interior of the squares with values 1 and -1 as in the next table:

...	0	0	0	0	0	0	*1*	0	0	0	0	0	...
...	0	0	0	0	0	*1*	*-1*	*1*	0	0	0	0	...
...	0	0	0	0	*1*	*-1*	*1*	*-1*	*1*	0	0	0	...
...	0	0	0	0	0	*1*	*-1*	*1*	0	0	0	0	...
...	0	0	0	0	0	0	*1*	0	0	0	0	0	...
...	0	0	0	0	0	*1*	*-1*	*1*	0	0	0	0	...
...	0	0	0	0	*1*	*-1*	*1*	*-1*	*1*	0	0	0	...
...	0	0	0	*1*	*-1*	*1*	*-1*	*1*	*-1*	*1*	0	0	...
...	0	0	0	0	*1*	*-1*	*1*	*-1*	*1*	0	0	0	...
...	0	0	0	0	0	*1*	*-1*	*1*	0	0	0	0	...
...	0	0	0	0	0	0	*1*	0	0	0	0	0	...
...	0	0	0	0	0	*1*	*-1*	*1*	0	0	0	0	...
...	0	0	0	0	*1*	*-1*	*1*	*-1*	*1*	0	0	0	...
...	0	0	0	*1*	*-1*	*1*	*-1*	*1*	*-1*	*1*	0	0	...
...	0	0	*1*	*-1*	*1*	*-1*	*1*	*-1*	*1*	*-1*	*1*	0	...
...	0	0	0	*1*	*-1*	*1*	*-1*	*1*	*-1*	*1*	0	0	...
...	0	0	0	0	*1*	*-1*	*1*	*-1*	*1*	0	0	0	...
...	0	0	0	0	0	*1*	*-1*	*1*	0	0	0	0	...
...	0	0	0	0	0	0	*1*	0	0	0	0	0	...
...	0	0	0	0	0	*1*	*-1*	*1*	0	0	0	0	...
...	0	0	0	0	*1*	*-1*	*1*	*-1*	*1*	0	0	0	...
...	0	0	0	*1*	*-1*	*1*	*-1*	*1*	*-1*	*1*	0	0	...
...	0	0	*1*	*-1*	*1*	*-1*	*1*	*-1*	*1*	*-1*	*1*	0	...
...	0	*1*	*-1*	*1*	*-1*	*1*	*-1*	*1*	*-1*	*1*	*-1*	*1*	...
...	0	0	*1*	*-1*	*1*	*-1*	*1*	*-1*	*1*	*-1*	*1*	0	...
...	0	0	0	*1*	*-1*	*1*	*-1*	*1*	*-1*	*1*	0	0	...
			.				.				.		
			.				.				.		

Proposition 9. The elements $a_{k,i}$ of the infinite matrix A are described as follows: if $k \leq (s(k))^2 + 4s(k) + 2$, then

$$a_{k,i} = \begin{cases} 0, & \text{if } \delta(k) < |i| \\ (-1)^{\delta(k)-|i|}, & \text{if } \delta(k) \geq |i| \end{cases}; \tag{18}$$

if $k > (s(k))^2 + 4s(k) + 2$, then

$$a_{k,i} = \begin{cases} 0, & \text{if } \varepsilon(k) < |i| \\ (-1)^{\varepsilon(k)-|i|}, & \text{if } \varepsilon(k) \geq |i| \end{cases}, \tag{19}$$

The following infinite matrix A is a generalization of all previous schemes:

$$
\begin{array}{ccccccccccc}
\dots & 0 & 0 & 0 & 0 & F(0) & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & F(0) & F(1) & F(0) & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & F(0) & F(1) & F(2) & F(1) & F(0) & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & F(0) & F(1) & F(0) & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & F(0) & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & F(0) & F(1) & F(0) & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & F(0) & F(1) & F(2) & F(1) & F(0) & 0 & 0 & \dots \\
\dots & 0 & F(0) & F(1) & F(2) & F(3) & F(2) & F(1) & F(0) & 0 & \dots \\
\dots & 0 & 0 & F(0) & F(1) & F(2) & F(1) & F(0) & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & F(0) & F(1) & F(0) & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & F(0) & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & F(0) & F(1) & F(0) & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & F(0) & F(1) & F(2) & F(1) & F(0) & 0 & 0 & \dots \\
\dots & 0 & F(0) & F(1) & F(2) & F(3) & F(2) & F(1) & F(0) & 0 & \dots \\
\dots & F(0) & F(1) & F(2) & F(3) & F(4) & F(3) & F(2) & F(1) & F(0) & \dots \\
\dots & 0 & F(0) & F(1) & F(2) & F(3) & F(2) & F(1) & F(0) & 0 & \dots \\
\dots & 0 & 0 & F(0) & F(1) & F(2) & F(1) & F(0) & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & F(0) & F(1) & F(0) & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & F(0) & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & F(0) & F(1) & F(0) & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & F(0) & F(1) & F(2) & F(1) & F(0) & 0 & 0 & \dots \\
\dots & 0 & F(0) & F(1) & F(2) & F(3) & F(2) & F(1) & F(0) & 0 & \dots \\
\dots & F(0) & F(1) & F(2) & F(3) & F(4) & F(3) & F(2) & F(1) & F(0) & \dots \\
\dots & F(1) & F(2) & F(3) & F(4) & F(5) & F(4) & F(3) & F(2) & F(1) & \dots \\
\dots & F(0) & F(1) & F(2) & F(3) & F(4) & F(3) & F(2) & F(1) & F(0) & \dots \\
\dots & 0 & F(0) & F(1) & F(2) & F(3) & F(2) & F(1) & F(0) & 0 & \dots \\
 & & \cdot & & & \cdot & & & \cdot & & \\
 & & \cdot & & & \cdot & & & \cdot & &
\end{array}
$$

where F is an arbitrary arithmetical function such that the number $F(0)$ is defined.
Proposition 10. The elements $a_{k,i}$ of the infinite matrix A are described as follows:

if $k \leq (s(k))^2 + 4s(k) + 2$, then

$$
a_{k,i} = \begin{cases} 0, & \text{if } \delta(k) < |i| \\ F(\delta(k) - |i|), & \text{if } \delta(k) \geq |i| \end{cases}; \tag{20}
$$

if $k > (s(k))^2 + 4s(k) + 2$, then

$$
a_{k,i} = \begin{cases} 0, & \text{if } \varepsilon(k) < |i| \\ F(\varepsilon(k) - |i|), & \text{if } \varepsilon(k) \geq |i| \end{cases}, \tag{21}
$$

Let we put

$$
F(n) = G(H(n)), \ n = 0, 1, 2, ... \tag{22}
$$

where $H : \mathcal{N} \cup \{0\} \to E$ and $G : E \to \mathcal{N} \cup \{0\}$, are arbitrary functions and E is a fixed set, for example, $E = \mathcal{N} \cup \{0\}$. Then many applications are possible. For example, if

$$G(n) = \psi(n),$$

where function ψ is described in **A6** and $H(n) = 2^n$, we obtain the infinite matrix as given below

$$
\begin{array}{ccccccccccc}
\ldots & 0 & 0 & 0 & 0 & \psi(1) & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & \psi(1) & \psi(2) & \psi(1) & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & \psi(1) & \psi(2) & \psi(4) & \psi(2) & \psi(1) & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & \psi(1) & \psi(2) & \psi(1) & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & \psi(1) & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & \psi(1) & \psi(2) & \psi(1) & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & \psi(1) & \psi(2) & \psi(4) & \psi(2) & \psi(1) & 0 & 0 & \ldots \\
\ldots & 0 & \psi(1) & \psi(2) & \psi(4) & \psi(8) & \psi(4) & \psi(2) & \psi(1) & 0 & \ldots \\
\ldots & 0 & 0 & \psi(1) & \psi(2) & \psi(4) & \psi(2) & \psi(1) & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & \psi(1) & \psi(2) & \psi(1) & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & \psi(1) & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & \psi(1) & \psi(2) & \psi(1) & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & \psi(1) & \psi(2) & \psi(4) & \psi(2) & \psi(1) & 0 & 0 & \ldots \\
\ldots & 0 & \psi(1) & \psi(2) & \psi(4) & \psi(8) & \psi(4) & \psi(2) & \psi(1) & 0 & \ldots \\
\ldots & \psi(1) & \psi(2) & \psi(4) & \psi(8) & \psi(16) & \psi(8) & \psi(4) & \psi(2) & \psi(1) & \ldots \\
\ldots & 0 & \psi(1) & \psi(2) & \psi(4) & \psi(8) & \psi(4) & \psi(2) & \psi(1) & 0 & \ldots \\
\ldots & 0 & 0 & \psi(1) & \psi(2) & \psi(4) & \psi(2) & \psi(1) & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & \psi(1) & \psi(2) & \psi(1) & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & \psi(1) & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & \psi(1) & \psi(2) & \psi(1) & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & \psi(1) & \psi(2) & \psi(4) & \psi(2) & \psi(1) & 0 & 0 & \ldots \\
\ldots & 0 & \psi(1) & \psi(2) & \psi(4) & \psi(8) & \psi(4) & \psi(2) & \psi(1) & 0 & \ldots \\
\ldots & \psi(1) & \psi(2) & \psi(4) & \psi(8) & \psi(16) & \psi(8) & \psi(4) & \psi(2) & \psi(1) & \ldots \\
\ldots & \psi(2) & \psi(4) & \psi(8) & \psi(16) & \psi(32) & \psi(16) & \psi(8) & \psi(4) & \psi(2) & \ldots \\
\ldots & \psi(1) & \psi(2) & \psi(4) & \psi(8) & \psi(16) & \psi(8) & \psi(4) & \psi(2) & \psi(1) & \ldots \\
\ldots & 0 & \psi(1) & \psi(2) & \psi(4) & \psi(8) & \psi(4) & \psi(2) & \psi(1) & 0 & \ldots \\
 & \vdots & & & \vdots & & & \vdots & & &
\end{array}
$$

or in calculation form:

...	0	0	0	0	0	0	1	0	0	0	0	0	0	...
...	0	0	0	0	0	1	2	1	0	0	0	0	0	...
...	0	0	0	0	1	2	4	2	1	0	0	0	0	...
...	0	0	0	0	0	1	2	1	0	0	0	0	0	...
...	0	0	0	0	0	0	1	0	0	0	0	0	0	...
...	0	0	0	0	0	1	2	1	0	0	0	0	0	...
...	0	0	0	0	1	2	4	2	1	0	0	0	0	...
...	0	0	0	1	2	4	8	4	2	1	0	0	0	...
...	0	0	0	0	1	2	4	2	1	0	0	0	0	...
...	0	0	0	0	0	1	2	1	0	0	0	0	0	...
...	0	0	0	0	0	0	1	0	0	0	0	0	0	...
...	0	0	0	0	0	1	2	1	0	0	0	0	0	...
...	0	0	0	0	1	2	4	2	1	0	0	0	0	...
...	0	0	0	1	2	4	8	4	2	1	0	0	0	...
...	0	0	1	2	4	8	7	8	4	2	1	0	0	...
...	0	0	0	1	2	4	8	4	2	1	0	0	0	...
...	0	0	0	0	1	2	4	2	1	0	0	0	0	...
...	0	0	0	0	0	1	2	1	0	0	0	0	0	...
...	0	0	0	0	0	0	1	0	0	0	0	0	0	...
...	0	0	0	0	0	1	2	1	0	0	0	0	0	...
...	0	0	0	0	1	2	4	2	1	0	0	0	0	...
...	0	0	0	1	2	4	8	4	2	1	0	0	0	...
...	0	0	1	2	4	8	7	8	4	2	1	0	0	...
...	0	1	2	4	8	7	5	7	8	4	2	1	0	...
...	0	0	1	2	4	8	7	8	4	2	1	0	0	...
...	0	0	0	1	2	4	8	4	2	1	0	0	0	...
			.				.				.			
			.				.				.			

The elements of this matrix are described by (20) and (21), if we take

$$F(n) = \psi(2^n), n = 0, 1, 2, ...$$

The elements of the following matrix A has alphabetical form:

$$\begin{array}{ccccccccccccccc}
\dots & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & a & b & a & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & a & b & c & b & a & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & a & b & a & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & a & b & a & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & a & b & c & b & a & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & a & b & c & d & c & b & a & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & a & b & c & b & a & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & a & b & a & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & a & b & a & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & a & b & c & b & a & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & a & b & c & d & c & b & a & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & a & b & c & d & e & d & c & b & a & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & a & b & c & d & c & b & a & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & a & b & c & b & a & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & a & b & a & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & a & b & a & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & a & b & c & b & a & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & a & b & c & d & c & b & a & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & a & b & c & d & e & d & c & b & a & 0 & 0 & \dots \\
\dots & 0 & a & b & c & d & e & f & e & d & c & b & a & 0 & \dots \\
\dots & 0 & 0 & a & b & c & d & e & d & c & b & a & 0 & 0 & \dots \\
 & & & \vdots & & & & \vdots & & & & \vdots & & &
\end{array}$$

and they are described using (20) and (21), because we may put:

$$F(0) = 1, F(1) = b, F(2) = c, F(3) = d, F(4) = e$$

etc.

Of course, by analogy, we can construct different schemes, e.g., the schemes of Problems 12, 13 and 14 from [1], but the benefit of these schemes is not clear.

Essentially more interesting is Problem 103 from [1]:

Smarandache numerical carpet:

has the general form

$$
\begin{array}{c}
\cdot \\
\cdot \\
\cdot \\
1 \\
1 \quad a \quad 1 \\
1 \quad a \quad b \quad a \quad 1 \\
1 \quad a \quad b \quad c \quad b \quad a \quad 1 \\
1 \quad a \quad b \quad c \quad d \quad c \quad b \quad a \quad 1 \\
1 \quad a \quad b \quad c \quad d \quad e \quad d \quad c \quad b \quad a \quad 1 \\
1 \quad a \quad b \quad c \quad d \quad e \quad f \quad e \quad d \quad c \quad b \quad a \quad 1 \\
1 \quad a \quad b \quad c \quad d \quad e \quad f \quad g \quad f \quad e \quad d \quad c \quad b \quad a \quad 1 \\
1 \quad a \quad b \quad c \quad d \quad e \quad f \quad e \quad d \quad c \quad b \quad a \quad 1 \\
1 \quad a \quad b \quad c \quad d \quad e \quad d \quad c \quad b \quad a \quad 1 \\
1 \quad a \quad b \quad c \quad d \quad c \quad b \quad a \quad 1 \\
1 \quad a \quad b \quad c \quad b \quad a \quad 1 \\
1 \quad a \quad b \quad a \quad 1 \\
1 \quad a \quad 1 \\
1 \\
\cdot \\
\cdot \\
\cdot
\end{array}
$$

On the border of level 0, the elements are equal to "1";
they form a rhomb.
Next, on the border of level 1, the elements are equal to "a";
where "a" is the sum of all elements of the previous border;
the "a"s form a rhomb too inside the previous one.
Next again, on the border of level 2, the elements are equal to "b";
where "b" is the sum of all elements of the previous border;
the "b"s form a rhomb too inside the previous one.
And so on ...

The above square, that Smarandache named *"rhomb"*, corresponds to the square from our construction for the case of $s = 6$, if we begin to count from $s = 1$, instead of $s = 0$. In [1] a particular solution of the Problem 103 is given, but there a complete solution is not introduced. We will give a solution below firstly for the case of Problem 103 and then for a more general case.

It can be easily seen that the number of the elements of the s-th square side is $s + 2$ and therefore the number of the elements from the countour of this square is just equal to $4s + 4$.

The s-th square can be represented as a set of sub-squares, each one included in the next. Let us number them inwards, so that the outmost (boundary) square is the first one. As it is written in Problem 103, all of its elements are equal to 1. Hence, the values of the elements of the subsequent (second) square will be (using also the notation from Problem

103):

$$a_1 = a = (s+2) + (s+1) + (s+1) + s = 4(s+1);$$

the values of the elements of the third square will be

$$a_2 = b = a(4(s-1) + 4 + 1) = 4(s+1)(4s+1);$$

the values of the elements of the fourth square will be

$$a_3 = c = b(4(s-2) + 4 + 1) = 4(s+1)(4s+1)(4s-3);$$

the values of the elements of the fifth square will be

$$a_4 = d = c(4(s-3) + 4 + 1) = 4(s+1)(4s+1)(4s-3)(4s-7);$$

etc., where the square, corresponding to the initial square *(rhomb),* from Problem 103 has the form

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & . & . & . & & & & \\
 & & 1 & a_1 & . & . & . & a_1 & 1 & & \\
 & 1 & a_1 & a_2 & . & . & . & a_2 & a_1 & 1 & \\
1 & a_1 & a_2 & a_3 & . & . & . & a_3 & a_2 & a_1 & 1 \\
 & 1 & a_1 & a_2 & . & . & . & a_2 & a_1 & 1 & \\
 & & 1 & a_1 & . & . & . & a_1 & 1 & & \\
 & & & & . & . & . & & & & \\
 & & & & & 1 & & & & &
\end{array}$$

It can be proved by induction that the elements of this square that stay on t-th place are given by the formula

$$a_t = 4(s+1)\prod_{i=0}^{t-2}(4s+1-4i).$$

If we would like to generalize the above problem, we can construct, e.g., the following extension:

$$\begin{array}{ccccccccccc}
 & & & & & x & & & & & \\
 & & & & . & . & . & & & & \\
 & & x & a_1 & . & . & . & a_1 & x & & \\
 & x & a_1 & a_2 & . & . & . & a_2 & a_1 & x & \\
x & a_1 & a_2 & a_3 & . & . & . & a_3 & a_2 & a_1 & x \\
 & x & a_1 & a_2 & . & . & . & a_2 & a_1 & x & \\
 & & x & a_1 & . & . & . & a_1 & x & & \\
 & & & & . & . & . & & & & \\
 & & & & & x & & & & &
\end{array}$$

where x is a given number. Then we obtain

$$a_1 = 4(s+1)x;$$

$$a_2 = 4(s+1)(4s+1)x;$$

$$a_3 = 4(s+1)(4s+1)(4s-3)x;$$

$$a_4 = 4(s+1)(4s+1)(4s-3)(4s-7)x;$$

etc. and for $t \geq 1$

$$a_t = 4(s+1)(\prod_{i=0}^{t-2}(4s+1-4i))x,$$

where it is assumed that

$$\prod_{i=0}^{-1} \bullet = 1.$$

References

[1] C. Dumitrescu, V. Seleacu, Some Sotions and Questions in Number Theory, Erhus Univ. Press, Glendale, 1994.

[2] F. Smarandache, Only Problems, Not Solutions!. Xiquan Publ. House, Chicago, 1993.