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Smarandache Triple Tripotents in Z_n and in Group Ring Z_2G

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Abstract

In this paper, we study tripotent elements and Smarandache triple tripotents (S-T. tripotents) in Z_n, the ring of integers modulo n, and in group ring Z_2G where G is a cyclic group of order 2n (n is an odd number).

Keywords: Tripotent, Smarandache triple tripotent

Introduction

The concept of m-idempotent was introduced by H. Chaoling and G. Yonghua at 2010,[2]. Smarandache concepts introduced by Florentin Smarandache [7]. Smarandache idempotent element in rings defined by Vasantha Kandasamy [8]. This paper has two sections. In section one we introduce the concept of Smarandache triple tripotent in rings (S-T. tripotent). We find the number of tripotents and S-T. tripotents and their forms in Z_n,the ring of integers modulo n. In section two, we study tripotents and S-T. tripotents in the group ring Z_2G, where G is a cyclic group of order 2n (n is an odd number), in particular, when n is a Mersenne prime that is a prime of the form $2^k - 1$ for some prime k, and we obtain their numbers.

§1. Tripotents and S-T. tripotents in the ring Z_n.

In this section the concept of S-T. tripotent introduced. We study tripotents and S - T. tripotents in Z_n, for $n = 2^k$, pq, pqr, for distinct primes p, q and r, we find the number of tripotents and S-T. tripotents and their forms.

Definition1.1.[2]. An element α of a ring R is called tripotent (3–idempotent), if $\alpha^3 = \alpha$. A tripotent element is said to be anon trivial tripotent if $\alpha^2 \neq \alpha$.

Now, we introduce the concept of S-T. tripotent.

Definition 1.2. Three distinct non trivial tripotents x, y, z in a commutative ring R called Smarandache triple tripotent (S-T. tripotent) if xy=z, xz=y and yz=x.

The proof of the following result is easy.

Proposition 1.3. In Z_n, n > 2, the element [n-1] (equivalence class of n-1) is a non trivial tripotent (we write n-1 instead of [n-1]).

The following useful Lemma is needed.

Lemma 1.4 . If x is a non trivial idempotent of Z_n and $x-1 \neq \frac{n}{2}$, then x-1 and 2x-1 are non trivial tripotents.

Proof : Let x be a non trivial idempotent of Z_n with $x-1 \neq \frac{n}{2}$. Then $x^2 \equiv x \pmod{n}$. Now, $(x-1)^3 \equiv x-1 \pmod{n}$, hence x-1 is a tripotent. We have to show that (x-1) is not an idempotent. If $(x-1)^2 \equiv x-1 \pmod{n}$, then $1-x \equiv x-1 \pmod{n}$. Hence $2(x-1) \equiv 0 \pmod{n}$. This means that n|2(x-1). If n is an odd number, then n | (x-1), hence $x \equiv 1 \pmod{n}$ which is a trivial idempotent. If n is an even number, then $x-1 \equiv 0 \pmod{\frac{n}{2}}$, so $x-1 \equiv \frac{n}{2} \pmod{n}$ which is a contradiction with the assumption. Therefore x-1 is a non trivial tripotent. Similarly we can show (2x-1) is a non trivial tripotent. ■

The converse of Lemma 1.4 is not true in general (if y and 2y+1 are non trivial tripotents, then it is not necessary that y+1 is an idempotent and $y \neq \frac{n}{2}$).

Example 1.5. In Z_{60}, the ring of integers modulo 60, take y=4, then 2y+1=9. Clearly y and 2y+1 are non trivial tripotents, but y+1=5 is not an idempotent.

In the following result, a condition under which the converse of Lemma1.4 is true is given.

Proposition 1.6. Let y and 2y+1 be non trivial tripotents in Z_n, such that (n, 12)=1. Then y+1 is a nontrivial idempotent and $y \neq \frac{n}{2} \pmod{n}$.

Proof: From the assumption we have $y^3 \equiv y \pmod{n}$ and $(2y+1)^3 \equiv 2y+1 \pmod{n}$. This implies that $n|12(y^2+y)$. But (n,12)=1, so $y^2+y \equiv 0 \pmod{n}$. Consequently $(y+1)^2 \equiv y+1 \pmod{n}$. Hence (y+1) is a non trivial idempotent, and clearly $y \neq \frac{n}{2} \pmod{n}$. ■

Proposition 1.7. The ring Z_{2^n}, $n > 2$ has exactly three non trivial tripotents, furthermore they forms a S -T. tripotent.

Proof: By Proposition 1.3, the element (2^n-1) is a non trivial tripotent, and easily one show that $2^{n-1}-1, 2^{n-1}+1$ are non trivial tripotents, and that the triple 2^n-1, $2^{n-1}-1$, $2^{n-1}+1$, forms a S-T. tripotent. Now, suppose that x is any other non trivial tripotent. Then $x(x^2-1) \equiv 0 \pmod{2^n}$, so $2^n \mid x(x^2-1)$. This means that, either $2^n \mid x$ or $2^n \mid x^2-1$. If $2^n \mid x$, then $x \equiv 0 \pmod{2^n}$ which is a contradiction with $x \not\equiv 0 \pmod{2^n}$. Thus $2^n \mid x^2-1$, hence $x^2 \equiv 1 \pmod{2^n}$.This congruence has four solutions they are 1, $2^{n-1}-1$, $2^{n-1}+1$ and 2^n-1, [6]. The solution 1 is trivial and the others are the same as above. Hence Z_{2^n} has exactly three non trivial tripotents, and it is easy to show that the triple $(2^{n-1}-1)$, $(2^{n-1}+1)$, (2^n-1) is a S -T. tripotent. ■

Proposition 1.8. Let p be an odd prime. Then Z_{p^n}, for $n \geq 1$ has only one non trivial tripotent.

Proof: By Proposition 1.3, the element p^n-1 is a non trivial tripotent. Suppose x is any other non trivial tripotent in Z_{p^n}. Then $x(x^2-1) \equiv 0 \pmod{p^n}$, this means that $p^n \mid x(x^2-1)$. If n=1, $p \mid x(x^2-1)$, then $p|x$ or $p|x^2-1$, but $p \nmid x$ because otherwise $x\equiv 0 \pmod p$, hence $p|x^2-1$, so $x^2\equiv 1 \pmod p$. The solutions of the congruence $x^2\equiv 1 \pmod p$ are 1, p-1 [1], but 1 is a trivial idempotent and p-1 is the same idempotent obtained by Proposition 1.3. Therefore there is exactly one nontrivial tripotent. Now, suppose n>1 and that $x \neq p^n-1$ is any non trivial tripotent. Then $x(x^2-1)\equiv 0 \pmod{p^n}$. Since p is a prime, either $p^n|x$ or $p^n|x^2-1$. If $p^n|x$, then $x\equiv 0 \pmod{p^n}$ contradiction with $x\not\equiv 0 \pmod{p^n}$, therefore $p^n|x^2-1$, that means $x^2\equiv 1 \pmod{p^n}$, but this congruence has exactly two incongruent solutions [6], either $x\equiv 1 \pmod{p^n}$ which is a trivial idempotent or $x \equiv p^n-1 \pmod{p^n}$ which his the tripotent obtained from Proposition 1.3. Hence Z_{p^n} has exactly one non trivial tripotent. ■

Recall that if a, b are positive integers with (a, b) =d, then the Diophantine equation xa+yb=c has infinite solutions if d| c and has no solution if $d \nmid c$, we give the following result.

Theorem 1.9. Let n=pq, where p and q are distinct odd primes. Then Z_n has exactly five non trivial tripotents, and one S-T. tripotent.

Proof: By Proposition 1.3, the element pq-1 is a non trivial tripotent of Z_n. By Diophantine equation, there exist t, s $\in Z$, $t > 0$ such that tq-sp=1 and there exist t_1, $s_1 \in Z$, $t > 0$ such that $t_1p-s_1q=1$. It is shown in [4], that tq and t_1p are non trivial idempotents (In fact $t_1p=n+1 - tq \pmod{pq}$) of Z_{pq}. Then by Lemma 1.4 the elements tq-1, 2tq-1, n-tq and 1-2tq are non trivial tripotents. So we get five non trivial tripotents. Suppose that x be any other non trivial tripotent, thus

$x^3 \equiv x$ (mod pq), so $x(x^2-1) \equiv 0$ (mod pq), which means $pq \mid x(x^2-1)$. There are three cases:
(1) $p|x$ and $q|x^2-1$.
(2) $p|x^2-1$ and $q|x$, and
(3) $pq \mid x^2-1$.
In case(1), $x \equiv 0$ (mod p), hence $x \equiv kp$ (mod pq), for some k, $0 \le k \le q-1$, and $q|x^2-1$, then $x^2 \equiv 1$ (modq), by [1], $x \equiv 1$ (mod q) or $x \equiv q-1$(mod q). If $x \equiv 1$ (mod q), hence $x \equiv 1+rq$ (mod pq) for some r, $0 \le r \le p-1$, then kp-rq=1 which means x=kp is an idempotent (it is a trivial tripotent). When $x \equiv q-1$ (mod q), we get $x \equiv s_3q-1$ (mod pq) for some s_3, $0 \le s_3 \le p-1$. Therefore $s_3q-kp=1$, this means x=kp is a non trivial tripotent which is obtained before.
Case (2) is similar.
Case (3) $pq|x^2-1$, then $x^2 \equiv 1$ (mod pq). This congruence has four solutions 1, 1-2tq, 2tq-1 and pq-1, [6]. The solution 1 is trivial, the others was obtained before. Therefore Z_{pq} has exactly five non trivial tripotents, and a simple calculation shows that the triple (n-1), (1-2tq), (2tq-1) is a S-T. tripotent.■

Proposition 1.10. Let p be an odd prime. Then Z_{2p} has exactly two non trivial tripotents.

Proof: It is shown in [4], that Z_{2p} has only two non trivial idempotents namely p and p+1. Then by Lemma1.4 the elements p-1 and 2p-1 are non trivial tripotents in Z_{2p}. But 2p-1 is the same tripotent obtained by Proposision1.3. Hence Z_{2p} has two non trivial tripotents. Suppose that x is any other non trivial tripotent, then $x(x^2-1) \equiv 0$ (mod 2p), there are two cases:
(1) $2|x$ and $p|x^2-1$.
(2) $2|x^2-1$ and $p|x$.
In case (1), $x \equiv 0$ (mod 2), hence $x \equiv 2t_1$ (mod 2p), for some $0 \le t_1 \le p-1$, and $p|x^2-1$, then $x^2 \equiv 1$(mod p). The congruent $x^2 \equiv 1$ (mod p) has exactly two solutions 1, p-1, [1]. If $x \equiv 1$ (mod p), then $x \equiv 1+kp$ (mod 2p) for some $0 \le k \le 2p-1$, hence $2t_1-kp=1$ which means $x=2t_1$ is an idempotent. When $x \equiv p-1$ (mod p), hence $x \equiv s_1p-1$ (mod 2p) for some $0 \le s_1 \le 2p-1$. Therefore $s_1p-2t_1=1$ this means $x=2t_1$ is a non trivial tripotent which is obtained before.
Case (2), is similar.
Hence Z_{2p} has exactly two non trivial tripotents.■

Theorem 1.11. Let $n=p^nq$, where p, q are distinct odd primes. Then Z_n has exactly five non trivial tripotents, and one S-T. tripotent.

Proof: By Diophantine equation, there exist t, s $\in Z$, $t > 0$ such that $tq-sp^n=1$. By similar method used in the proof of Theorem 1.9 one can show that p^nq-1, tq-1, 2tq-1, n-tq and 1-2tq are non trivial tripotents in Z_{p^nq}, and it is easy to show that the triple n-1, 1-2tq and 2tq-1, is a S-T. tripotent. ■

Theorem 1.12. Let n=2pq where p and q are distinct odd primes. Then Z_n has exactly ten non trivial tripotents and two S-T. tripotents.

***Proof*:** By Proposition 1.3, the element 2pq-1 is a non trivial tripotent. Suppose that $p < q$. Then by Diophantine equation, there exist t, s $\in$ Z, t > 0 such that tq-sp=1 as (p, q)=1. As it is shown in [4], Z_n has exactly 6 non trivial idempotents they are pq, pq+1, tq, 2pq+1-tq, pq+tq and 1-tq+pq. By Lemma 1.4 the elements pq-1, 2pq-1, tq-1, 2tq-1, 2pq-tq, 1-2tq and pq+tq-1 are non trivial tripotents. The element 2pq-1 is the same non trivial tripotent obtained by Proposition 1.3, so we obtain seven non trivial tripotents and it is not difficult to show that 1-2tq+pq, 2tq-1+pq and 2pq-tq+pq are also non trivial tripotents. Hence Z_{2pq} has ten non trivial tripotents. Suppose that x is any other non trivial tripotent for Z_{2pq}. Then $x(x^2-1) \equiv 0$ (mod 2pq), this means that $pq| x(x^2-1)$. There are three cases:
(1) $pq|x$ or $pq|x^2-1$
(2) $q| x$ and $p| x^2-1$
(3) $q | x^2-1$ and $p| x$.
In case(1), x≡pq (mod 2pq), but pq is an idempotent, so it is a trivial tripotent. If $pq|x^2-1$, then $x^2\equiv 1$(mod pq), this congruence has the following four solutions, 1which is trivial, 2pq-1,1-2tq and 2tq-1 are obtained before.
In case (2), x≡0 (mod q), hence $x\equiv t_1q$ (mod 2pq) for some $t_1, 0 \le t_1 \le 2p-1$, and $p|x^2-1$, then $x^2\equiv 1$ (mod p), by [1] x≡1 (mod p), or x≡p-1 (mod p), are solutions of the congruence $x^2\equiv 1$ (mod p). If x≡1 (mod p), hence x≡1+r p (mod 2pq), for some $1 \le r \le 2q-1$, then t_1q- rp=1, this means $x=t_1q$ is an idempotent. When x≡p-1 (modp), hence $x\equiv s_1p-1$(mod 2pq) for some $1 \le s_1 \le 2q-1$. Therefore $s_1p-t_1q=1$, which means $x=t_1q$ is a non trivial tripotent which is obtained before. Case (3) is similar.
Hence Z_{2pq} has exactly ten non trivial tripotents.
Now, we show that the triple (2pq-1), (2tq-1), (1-2tq) is a S-T. tripotent.
(2pq-1) (2tq-1) ≡4tqpq – 2pq – 2tq+1 (mod 2pq)
≡ 1-2tq (mod 2pq),
(2pq-1) (1-2tq) ≡ 2tq-1 (mod 2pq), and
(2tq- 1) (1-2tq) ≡ 2pq-1 (mod2pq)
Therefore (2pq-1), (2tq-1) and (1-2tq) is a S-T. tripotent. Similarly (pq-1), (1-2tq +pq), (2tq-1+pq) forms a S-T. tripotent. Hence Z_{2pq} has two S-T. tripotents. ■

The following example illustrates the above results.

Example 1.13.
1) The non trivial tripotents of Z_8 are 3, 5 and7. The triple 3, 5, 7 is a S-T. tripotent, (proposition 1.7).
2) Z_{243} has only one non trivial tripotent, namely 242, (proposition 1.8).
3) Consider Z_n, n=3.7=21. Now, 1(7) -2(3) =1 by Theorem 1.10, the tripotents are 6, 14, 20, 8, 13, and the triple 20, 8, 13 is a S-T. tripotent,(Theorem 1.9).

4) Z_{10} has exactly two non trivial tripotents they are 4 and 9, (proposition 1.10)
5) In Z_{135}, 11(5)-2(27) =1. By Theorem 1.11, Z_{135} has five non trivial tripotents 54, 80, 26, 109, 134, and the triple 26, 109, 134 is a S-T. tripotent.
6) In Z_{154}, 2(11)-3(7)=1. By Theorem 1.12 the elements 76, 21, 132, 98, 153, 43, 55, 120, 111 and 34 are non trivial tripotents, the triples 153, 43, 111 and 76, 120, 34 are S- T. tripotents.

Theorem 1.14. Let n=pqr for distinct odd primes p, q, r. Then Z_n has exactly 19 non trivial tripotents, and at least three S-T. tripotents.

Proof: By Proposition1.3, the element pqr-1 is a non trivial tripotent. Since (pq, r)=1 there are t, s $\in$ Z with t > 0 such that spq-tr=1, also there exist $s_1, t_1 \in$ Z such that $s_1qr-t_1r=1$, and there are s_2, t_2 such that $s_2pr-t_2q=1$. It is shown in [4], Z_{pqr} has 6 non trivial idempotents, they are spq, s_1qr, s_2pr, pqr+1-spq, pqr+1-s_1qr and pqr+1-s_2pr. By Lemma 1.4 the elements spq-1, 2spq-1, s_1qr-1, $2s_1qr$-1, s_2pr-1, $2s_2pr$-1, 1-2spq, 1-$2s_1qr$, 1-$2s_2pr$, pqr-spq, pqr-s_1qr and pqr-s_2pr are non trivial tripotents. We can also show that the following six elements spq-s_1qr, s_1qr-spq, spq-s_2pr, s_2pr-spq, s_1qr-s_2pr and s_2pr-s_1qr are also non trivial tripotents in Z_n. Suppose that x is any other non trivial tripotent of Z_n, then $x(x^2-1)\equiv 0$ (mod pqr). This means that pqr | $x(x^2-1)$. If pqr | x then x≡0 (mod pqr), contradiction with x≢0 (mod pqr). So we have the cases:

(1) pqr|x^2-1. (2) p|x and qr|x^2-1. (3) p| x^2-1 and qr|x
(4) pq|x and r|x^2-1. (5) pq|x^2-1 andr| x. (6) pr|x and q| x^2-1.
(7) pr|x^2-1 and q| x.

In case (1), pqr| x^2-1, then $x^2\equiv 1$ (mod pqr). This congruence has 8 solutions they are 1, 2spq-1, $2s_1qr$-1, $2s_2pr$-1, 1-2spq, 1-$2s_1qr$, 1-$2s_2pr$ and pqr-1, [6]. But all of them were obtained before.

In case (2), x ≡ 0 (mod p), then x≡t_3p (mod pqr), for some t_3, $0 \le t_3 \le$ qr-1, and $x^2\equiv 1$ (mod qr), hence, $x^2 \equiv 1+s_3qr$ (mod pqr), for some s_3, $0 \le s_3 \le$ p-1, thus $(t_3p)^2-s_3qr$ =1, hence $x=(t_3p)^2$ is an idempotent.

Case (3) $x^2\equiv 1$ (mod p). Then x ≡ 1 (mod p) or x≡p-1(mod p). If x≡1 (mod p), hence x≡1+ kp (mod pqr), for some k, $0 \le k \le$ qr-1. When x ≡ p-1(mod p), then x≡k_1p-1 (mod pqr), for some k_1, $1 \le k_1 \le$ qr-1, and x ≡ qr (mod pqr), hence x ≡ jqr (mod pqr), for some j, $1 \le j \le$ p-1, therefore jqr-kp =1 which means x=jqr is an idempotent, also k_1p-jqr=1, Leeds to a contradiction, since x=jqr is a non trivial tripotent.

Cases(4) and (6) are Similar to case (3).
Cases (5) and (7) are Similar to case (2).

Hence Z_{pqr} has exactly 19 non trivial tripotents. One can show that the triples 2spq-1, 1-2spq, pqr-1**;** 1-$2s_1qr$, $2s_1qr$-1, pqr-1 and $2s_2pr$, 1-$2s_2pr$, pqr-1 are a S-T. tripotents. ■

We have to mention here that in general there are more than three S-T. tripotents but we could not find their forms.

Example 1.15. The non trivial tripotents of Z_{105}, are 20, 104, 90, 35, 84, 71, 99, 29, 34, 64, 49, 76, 56, 6, 50, 55, 41, 14, 69 and the triples : 29, 41, 34**;** 69, 99, 6**;** 4, 49, 56**;** 20, 55, 50**;** 104, 41, 64**;** 71, 76, 41**;** 64, 71, 29**;** 64, 76, 34**;** 104,71,34**;** 104, 76, 24 are S-T. tripotents.

§2. Smarandache triple tripotents in the group ring Z_2G

In this section we study tripotents and S-T. tripotents in the group ring Z_2G, where G is a cyclic group of order 2n (n is an odd number) generated by g, specially, when n is a Mersenne prime, and we obtain their numbers. For definition of group ring see[3]. We start by the following definition.

Definition 2.1.[8]. Let R be a ring. An element $0 \neq x \in R$ is a Smarandache idempotent (S-idempotent) of R if

1) $x^2=x$.
2) There exists $a \in R \setminus \{0, 1, x\}$
i) $a^2 = x$ and
ii) xa = a (ax =a) or ax=x (xa =x).
a called the Smarandache co-idempotent (S-co-idempotent).

The following lemma is needed.

Lemma 2.2. Let α be a S-idempotent of the group ring Z_2G, where G is a cyclic group of order 2n (n is an odd number) generated by g and β be a S-co-idempotent of α with αβ=β. Then β, $\alpha+\beta+g^n$ and $\alpha+\beta+1$ are non trivial tripotents.

Proof**:** Since β is a S-co-idempotent of α , we get $\beta^2 = \alpha \neq \beta$, consequently $\beta^3 = \beta$, hence β is a non trivial tripotent. Then $(\alpha + \beta + g^n)^3 = \alpha + \beta + g^n$, hence $\alpha + \beta + g^n$ is a non trivialtripotent.Similarly $\alpha + \beta + 1$ is a non trivial tripotent. ■

Theorem 2.3. In the group ring Z_2G, where G is a cyclic group of order 2n (n is an odd number) generated by g, for any k distinct integers $t_1<t_2<\cdots<t_k$, 0 <k, $t_i \leq$ n-1 for each i, $g^{t_1}+g^{t_2}+\cdots+g^{t_k}+g^n+g^{n+t_1}+g^{n+t_2}+\cdots+g^{n+t_k}$ and $1+g^{t_1}+g^{t_2}+\cdots+g^{t_k}+g^{n+t_1}+g^{n+t_2}+\cdots+g^{n+t_k}$ are non trivial tripotents. Moreover the number of non trivial tripotents is equal to $\sum_0^{n-1}\binom{n-1}{s} + \sum_1^{n-1}\binom{n-1}{s}$.

Proof**:** Let $t_1, t_2,\cdots,t_k$ be any k distinct integers with $0 < t_1<t_2<\cdots<t_k$, t_i, k ≤ n-1. Let $d_k=g^{t_1}+g^{t_2}+\cdots+g^{t_k}+g^n+g^{n+t_1}+g^{n+t_2}+\cdots+g^{n+t_k}$. Then $d_k^2=g^{2t_1}+g^{2t_2}+\cdots+g^{2t_k}+g^{2n}+g^{2n+2t_1}+g^{2n+2t_2}+\cdots+g^{2n+2t_k}=1\neq d_k$.
Hence $d_k^3= d_k$, so d_k is a nontrivial tripotent. Using some known facts from probability theory, the number of such tripotents is $\sum_1^{n-1}\binom{n-1}{s}$.Clearly g^n is also a non trivial tripotent we denote $d_0=g^n$.
Let $f_k = 1+g^{t_1}+g^{t_2}+\cdots+g^{t_k}+g^{n+t_1}+g^{n+t_2}+\cdots+g^{n+t_k}$. Then
$f_k^2 = 1+g^{t_1}+g^{t_2}+\cdots+g^{t_k}+g^{n+t_1}+g^{n+t_2}+\cdots+g^{n+t_k}=1\neq f_k$.

Hence $f_k^3 = f_k$, so f_k is a non trivial tripotent. Also using some known facts from probability theory, we get the number of such tripotents is $\sum_1^{n-1}\binom{n-1}{s}$. Hence the number of non trivial tripotents we obtain is $\sum_0^{n-1}\binom{n-1}{s} + \sum_1^{n-1}\binom{n-1}{s}$.■

Remark 2.4. If $\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_m$ are m non trivial tripotents of Z_2G, where G is a cyclic group of order 2n (n is an odd number) generated by g, and $\alpha_i \neq \beta$ for all i, where β is a S-co- idempotent of the S–idempotent $\alpha = g^2 + g^4 + \cdots + g^{2n-2}$,[5] where $\alpha \neq \alpha_i$, then $\alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_m$ is a tripotent if m is an odd number, and $1 + \alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_m$ is a tripotent if m is an even number.

Proposition 2.5. In the group ring Z_2G, where G is a cyclic group of order 2n (n is an odd number) generated by g, if α_1, α_2 are any two non trivial tripotents in Z_2G, then the triple α_1, α_2, $1 + \alpha_1 + \alpha_2$ is a S-T. tripotent.

Proof: Since in the group ring Z_2G the tripotents obtained are of the form d_k or f_k given in Theorem 2.3, then we have the following cases:

Case 1: α_1, α_2 are of the type d_k. Let
$\alpha_1 = d_e = g^{\ell_1} + g^{\ell_2} + \ldots + g^{\ell_e} + g^n + g^{n+\ell_1} + g^{n+\ell_2} + \ldots + g^{n+\ell_e}$, and
$\alpha_2 = d_h = g^{s_1} + g^{s_2} + \cdots + g^{s_h} + g^n + g^{n+s_1} + g^{n+s_2} + \cdots + g^{n+s_h}$,
where $\ell_1, \ell_2, \ldots, \ell_e$ and $s_1, s_2, \ldots, s_h$ are e and h distinct integers respectively, $\ell_j \leq n-1$, $s_i \leq n-1$ for each i, j. By Remark 2.4, $1 + \alpha_1 + \alpha_2$ is also a non trivial tripotent. We claim that the triple α_1, α_2, $1 + \alpha_1 + \alpha_2$ isa S - T. tripotent. For this purpose we describe the multiplication $\alpha_1\alpha_2$ in the following array say $\mathcal{A}$:

$$\begin{pmatrix}
g^{l_1+s_1} & g^{l_1+s_2} & \cdots & g^{l_1+s_h} & \boxed{g^{n+l_1}} & g^{n+l_1+s_1} & g^{n+l_1+s_2} & \cdots & g^{n+l_1+s_h} \\
g^{l_2+s_1} & g^{l_2+s_2} & \cdots & g^{l_2+s_h} & \boxed{g^{n+l_2}} & g^{n+l_2+s_1} & g^{n+l_2+s_2} & \cdots & g^{n+l_2+s_h} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
g^{l_e+s_1} & g^{l_e+s_2} & \cdots & g^{l_e+s_h} & \boxed{g^{n+l_e}} & g^{n+l_e+s_1} & g^{n+l_e+s_2} & \cdots & g^{n+l_e+s_h} \\
\boxed{g^{n+s_1}} & \boxed{g^{n+s_2}} & \cdots & \boxed{g^{n+s_h}} & \boxed{g^{2n}=1} & \boxed{g^{2n+s_1}} & \boxed{g^{2n+s_2}} & \cdots & \boxed{g^{2n+s_h}} \\
g^{n+l_1+s_1} & g^{n+l_1+s_2} & \cdots & g^{n+l_1+s_h} & \boxed{g^{2n+l_1}} & g^{2n+l_1+s_1} & g^{2n+l_1+s_2} & \cdots & g^{2n+l_1+s_h} \\
g^{n+l_2+s_1} & g^{n+l_2+s_2} & \cdots & g^{n+l_2+s_h} & \boxed{g^{2n+l_2}} & g^{2n+l_2+s_1} & g^{2n+l_2+s_2} & \cdots & g^{2n+l_2+s_h} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
g^{n+l_e+s_1} & g^{n+l_e+s_2} & \cdots & g^{n+l_e+s_h} & \boxed{g^{2n+l_e}} & g^{2n+l_e+s_1} & g^{2n+l_e+s_2} & \cdots & g^{2n+l_e+s_h}
\end{pmatrix}$$

$\mathcal{A}=[a_{ij}]_{(2e+1)\times(2h+1)}$, where a_{ij} is the summand of $\alpha_1\alpha_2$ which is equal to the product of the ith summand of α_1 with jth summand of α_2. Considering the first and the (e+2) th rows of this array we see that if g^i occurs in one of them it occurs in both of them for each i except (i= n+ℓ_1, ℓ_1), (as $g^{2n+\ell_1} = g^{\ell_1}$). By adding the terms of these two rows it remains only $g^{\ell_1} + g^{n+\ell_1}$ (observing that the coefficient of each g^i , i=1, ⋯, 2n-1 is in Z_2G). Again by adding the second and the (e+3) th rows in this array, according to the same argument it remains only $g^{\ell_2} + g^{n+\ell_2}$. Proceeding in this manner we will get the (e) th and the (2e+1) th rows, adding there terms it remains only $g^{\ell_e} + g^{n+\ell_e}$. So by adding all terms of this array we get, $1+ g^{\ell_1}+g^{\ell_2}+\dots+g^{\ell_e}+g^{n+\ell_1}+g^{n+\ell_2}+\dots+g^{n+\ell_e} + g^{s_1} + g^{s_2} + \cdots + g^{s_h} + g^{n+s_1} + g^{n+s_2} + \cdots + g^{n+s_h} = 1 + \alpha_1 + \alpha_2$, which is clearly a non trivial tripotent in second type. By the same way we get $\alpha_1(1 + \alpha_1 + \alpha_2) = \alpha_2$ and $\alpha_2(1 + \alpha_1 + \alpha_2) = \alpha_1$. Therefore the triple $\alpha_1, \alpha_2, 1 + \alpha_1 + \alpha_2$ is a S-T. tripotent.

Case 2: α_1, α_2 are of the type f_k.
Let $\alpha_1 = f_i = 1 + g^{r_1}+ g^{r_2}+\dots+g^{r_i} +g^{n+r_1}+g^{n+r_2}+\dots+g^{n+r_i}$, and
$\alpha_2 = f_j = 1 + g^{m_1}+g^{m_2} +\dots+g^{m_j}+g^{n+m_1}+g^{n+m_2}+\dots+g^{n+m_j}$,
such that $r_1, r_2, \dots, r_i$ and $m_1, m_2, \dots, m_j$ are i and j distinct integers respectively, $r_k \le$ n-1, $m_t \le$ n-1 for each k, t, so by Remark 2.4, $1 + \alpha_1 + \alpha_2$ is also a non trivial tripotent. By using same method as in case1, we get that the triple $\alpha_1, \alpha_2, 1 + \alpha_1 + \alpha_2$ is a S-T. tripotent.

Case 3: α_1 of the type d_k and α_2 of the type f_k , where
$\alpha_1 = d_h = g^{s_1} + g^{s_2} + \cdots + g^{s_h} + g^n + g^{n+s_1} + g^{n+s_2} + \cdots + g^{n+s_h}$,
$\alpha_2 = f_e = 1 + g^{\ell_1}+g^{\ell_2}+\dots+g^{\ell_e}+g^{n+\ell_1}+g^{n+\ell_2}+\dots+g^{n+\ell_e}$ and
Such that $s_1, s_2, \dots, s_h$ and $\ell_1, \ell_2, \dots, \ell_e$ are h and e distinct integers respectively, $s_i \le$ n-1, $\ell_j \le$ n-1 for each i, j, then by Remark 2.4 the element $1 + \alpha_1 + \alpha_2$ is also a non trivial tripotent. If $1+\alpha_1+\alpha_2$ belongs to first type, then we get case 1 if it belongs to second type, then we get case 2. Hence the triple $\alpha_1, \alpha_2, 1 + \alpha_1 + \alpha_2$ is a S-T. tripotent. , ■

Theorem 2.6. The group ring Z_2G, where G is a cyclic group of order 2n (n is an odd number) has at least 2^n non trivial tripotens and $\binom{2^{n-1}}{2} + \frac{1}{3}\binom{2^{n-1}-1}{2}$ S-T. tripotents.

***Proof*:** By Theorem 2.3, the group ring Z_2G, has $2^{n-1} + 2^{n-1}-1=2^n-1$ non trivial tripotents. It is shown in [5], that if *G* is generated by g, then $\alpha = g^2 + g^4 + \cdots + g^{n-1} + g^{n+1} + \cdots + g^{2n-2}$ is a S-idempotent and $\beta = g+g^3 + \cdots + g^{n-2} + g^{n+2} + \cdots + g^{2n-1}$ is a S-co-idempotent. By Lemma 2.2, β is also a non trivial tripotent. Then Z_2G has at least 2^n non trivial tripotents. By Proposition 2.5, for any two non trivial tripotents α_1, α_2 in Z_2G, the triple $\alpha_1, \alpha_2, 1 + \alpha_1 + \alpha_2$ is a

S-T. tripotent. Using some probability theory we get that, the number of such S-T. tripotents is $\binom{2^{n-1}}{2} + \frac{1}{3}\binom{2^{n-1}-1}{2}$. ■

Example 2.7. Consider the group ring Z_2G, where $G = \langle g \mid g^{10}=1 \rangle$ is a cyclic group of order 10, generated by g. Then by Theorem 2.6, the group ring Z_2G has 32 non trivial tripotents and the number of S-T. tripotents is 155. We list some of non trivial tripotents and S-T. tripotents
g^5, $g+g^5+g^6$, $g+g^2+g^5+g^6+g^7$, $g+g^2+g^3+g^5+g^6+g^7+g^8$, $g+g^2+g^3+g^4+g^5+g^6+g^7+g^8+g^9$
$1+g+g^6$, $1+g+g^2+g^6+g^7$, $1+g^2+g^4+g^7+g^9$, $1+g+g^2+g^3+g^6+g^7+g^8$, $1+g+g^2+g^3+g^4+g^6+g^7+g^8+g^9$, $g+g^3+g^7+g^9$ are non trivial tripotents. The triples g^5, $g+g^5+g^6$, $1+g+g^6$ and $1+g+g^6$, $1+g^4+g^9$, $1+g+g^4+g^6+g^9$ are S-T. tripotents

Theorem 2.8. The group ring Z_2G, where G is a cyclic group of order 2p (p is Mersenne prime) has at least $2^p + (2^m - 2)$ non trivial tripotents and $\binom{2^{p-1}}{2} + \frac{1}{3}\binom{2^{p-1}-1}{2}$ S-T. tripotents, where $m = \frac{p-1}{k}$.

***Proof*:** By Theorem 2.6, the group ring Z_2G has at least 2^p non trivial tripotents. It is shown in [5] that if G is generated by g, then every element of the form $\alpha_\ell = g^{2\ell} + g^{2^2\ell} + g^{2^3\ell} + \cdots + g^{2^k\ell}$ is a S-idempotent of the group ring Z_2G, where ℓ is an odd number less than p, and $\beta_\ell = g^\ell + g^{t_2} + g^{t_3} + \cdots + g^{t_{k-1}} + g^{t_k}$, is a S-co-idempotent of α_ℓ with $\alpha_\ell\beta_\ell = \beta_\ell$, where t_i is defined *by*

$$t_i = \begin{cases} \frac{1}{2}x_i & \text{if } \frac{1}{2}x_i \text{ is odd} \quad (2 \le i \le k) \\ \frac{1}{2}x_i + p & \text{if } \frac{1}{2}x_i \text{ is even } (2 \le i \le k). \end{cases}$$

and x_i, $i \ge 2$ is the smallest positive integer such that $x_i < 2p$.
Thus $x_i \equiv 2^i\ell \pmod{2p}$, this means $x_i = 2^i\ell - 2pr$, for some $r \in Z^+$. S-idempotents of the form α_ℓ called basic S-idempotents. Moreover it is shown that the sum of any number S-idempotents is also a S-idempotents, also it is proved that if α is any such S-idempotent and β is a S-co-idempotent of α, then αβ=β. By Lemma 2.2, S-co-idempotent are non trivial tripotents. Since the number of such S-co-idempotents is 2^m-1, each of which is a non trivial tripotent. But one of these 2^m-1 S-co-idempotents namely $\beta = g+g^3 + \cdots + g^{n-2} + g^{n+2} + \cdots + g^{2n-1}$ is one of the 2^n non trivial tripotents obtained from Theorem 2.6, and no three of them form S-T. tripotent. Therefore the number of non trivial tripotents we obtained is $2^p + (2^m - 2)$ and the number of S-T. tripotents obtained is $\binom{2^{p-1}}{2} + \frac{1}{3}\binom{2^{p-1}-1}{2}$. ■

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