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ON SOME CHARACTERIZATION OF SMARANDACHE - BOOLEAN NEAR - RING WITH SUB-DIRECT SUM STRUCTURE

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Abstract In this paper, we introduced Samarandache-2-algebraic structure of Boolean-near-ring namely Smarandache-Boolean-near-ring. A Samarandache-2-algebraic structure on a set N means a weak algebraic structure S_1 on N such that there exist a proper subset M of N, which is embedded with a stronger algebraic structure S_2, stronger algebraic structure means satisfying more axioms, that is $S_1 << S_2$, by proper subset one understands a subset different from the empty set, from the unit element if any, from the whole set [3]. We define Smarandache-Boolean-near-ring and obtain the some of its characterization through Boolean-ring with sub-direct sum structure. For basic concept of near-ring we refer to G.Pilz [11].

Keywords Boolean-ring, Boolean-near-ring, Smarandache-Boolean-near-ring, Compatibility, Maximal set, Idempotent and uni-element.

§1. Preliminaries

Definition 1.1. A (Left) near ring A is a system with two Binary operations, addition and multiplication, such that
(i) the elements of A form a group (A, +) under addition,
(ii) the elements of A form a multiplicative semi-group,
(iii) x(y+z) = xy+xz, for all x, y and z∈A.
In particular, if A contains a multiplicative semi-group S whose elements generates (A, +) and satisfy,
(iv) (x+y)s = xs+ys, for all x,y∈A and s∈S, then we say that A is a distributively generated near-ring.

Definition 1.2. A near-ring (B, +,.) is Boolean-near-ring if there exists a Boolean-ring (A,+, ∧,1) with identity such that . is defined interms of +, ∧ and 1, and for any b∈B, b.b=b.

Definition 1.3. A near-ring (B, +,.) is said to be idempotent if x^2=x, for all x∈B.

(ie) If (B, +,.) is an idempotent ring, then for all a,b∈B, a+a= 0 and a.b=b.a

Definition 1.4. Compatibility a∈b (ie) "a is compatibility to b") if ab^2=a^2b.

Definition 1.5. Let A = (..., a,b,c,...) be a set of pairwise compatible elements of an associate ring R. Let A be maximal in the sense that each element of A is compatible with every other element of A and no other such elements may be found in R. Then A is said to be a maximal compatible set or a maximal set.

Definition 1.6. If a sub-direct sum R of domains has an identity, and if R has the property that with each element a, it contains also the associated idempotent a^0 of a, then R is called an associate subdirect sum or an associate ring.

Definition 1.7. If the maximal set A contains an element u which has the property that $a < u$, for all $a \in A$, then u is called the uni-element of A.

Definition 1.8. Left zero divisors are right zero divisors, if ab=0 implies ba=0.

Now we have introduced a new definition by[3]

Definition 1.9. A Boolean- near- ring B is said to be Samarandache- Boolean- near -ring whose proper subset A is a Boolean- ring with respect to same induced operation of B.

Theorem 1.1. A Boolean-near-ring $(B, \vee, \wedge)$ is having the proper subset A , is a maximal set with uni-element in an associate ring R, with identity under suitable definitions for (B,+,.) with corresponding lattices $(A, \leq)$ $(A, <)$ and

$a \vee b = a + b - 2a^0 b = (a \cup b) - (a \cap b)$

$a \wedge b = a \cap b = a^0 b = ab^0$.

Then B is a Smarandache-Boolean-near-ring.

Proof. Given $(B, \vee, \wedge)$ is a Boolean-near- ring whose proper subset $(A, \vee, \wedge)$ is a maximal set with uni-element in an associate Ring R, and if the maximal set A is also a subset of B.

Now to prove that B is Smarandache-Boolean-near-ring. It is enough to prove that the proper subset A of B is a Boolean-ring. Let a and b be two constants of A : If a is compatible to b, we define $a \wedge b$ as follows :

If $a_i = b_i$ in the i-component, let $(a \wedge b)_i = 0_i$

If $a_i \neq b_i$, then since $a \sim b$ precisely one of these is zero.

If $a_i = 0$, let $(a \wedge b)_i = b_i \neq 0$;

If $b_i = 0$, let $(a \wedge b)_i = a_i \neq 0$

It is seen that if $a \wedge b$ belongs to the associate ring R, then $a \wedge b < u$, where u is the uni-element of A, and therefore,$a \wedge b \in A$

Consider $a \wedge b = a + b - 2a^0 b$

If in the i-component, $0 \neq a_i - b_i$, then since $(a^0)_i = 1_i = (b^o)_i$,

we have $(a + b - 2a^0 b)_i = 0_i$ and,

If $0_i = a_i = b_i$, then $(a^0)_i = 0$ and $(b^0)_i = 1$, whence,

$(a + b - 2a^0 b)_i = b_i$

If $a_i \neq 0$ and $b_i = 0$ then $(a + b - 2a^0 b) = 0_i$

Therefore $a \wedge b \in A$, the maximal set.

Similarly, the element $a \wedge b = a \cap b = a^0 b = ab^0 = glb(a,b)$ has defined and shown to belong to A as the glb (a,b) Now let us show that $(A, \vee, \wedge)$ is a Boolean - ring: Firstly, $a \vee a = 0$, since $a_i = a_i$ in every i-component, whence $(a \vee a)_i$ vanishes, by our definition of '$\vee$'. Secondly $a \wedge a =$

a∩a = a^0a=a, and so a is idempotent under ∧. We shown that A is closed under ∧ is ∨. And associativity is a direct verification, and each element is itself inverse under ∧.

To prove associativity under ∧ :

For a∧ (b∧ c) = a^0(b∧ c)
= a^0(b^0c)
= a^0(bc^0)
= (a^0b) c^0
= (a∧ b)^{0}c= (a∧ b) ∧c
⇒a∧(b∧c) = (a∧b)∧c, for all a, b, c∈R
For distributivity of ∧ over ∧,
Let c be an arbitrary in A
Now c∧(a∨b) = c^0 (a∨b)
= c^0(a∪b)-c^0(a∩b)
= (c^0a∪c^0b)-$c^0$$a^0$b
= c^0a+c^0b - $c^0$$a^0$b-$c^0$$a^0$b
= c^0a+c^0b - 2$c^0$$a^0$b
= (c∧a) ∨ (c∧b)
⇒c∧(a∨b) = (c∧a)∨(c∧b)
Hence (A ∨, ∧) is a Boolean-ring.

∴It follows that the proper subset A, a maximal set of B forms a Boolean ring.

∴B is a Boolean-near-ring , whose proper subset is a Boolean-ring, Then by definition, B is a Smarandache-Boolean-near-ring.

Theorem 1.2. A Boolean-near-ring (B, ∨, ∧) is having the proper subset (A,+,∧,1) is an associate ring in which the relation of compatibility is transitive for non-zero elements with identity under suitable definitions for (B,+,.) with corresponding lattices (A,≤) (A, <) and

a∨b = a+b - 2a^0b = (a∪b) - (a∩b)

a∧b = a∩b = a^0b = ab^0.

Then B is a Smarandache-Boolean-near-ring.

Proof.

Assume that (B, +,.) be Boolean- near-ring having a proper subset A is an associate ring in which the relation of compatibility is transitive for non-zero elements.

Now to prove that B is a Smarandache-Boolean-near-ring.

(ie) to prove that if the proper subset of B is a Boolean-ring, then by definition B is Smarandache-Boolean-near-ring. we have 0 is compatible with all elements, whence all elements are compatible with A and therefore, are idempotent.

Then assume that transitivity holds for compatibility of non-zero elements. It follows that non-zero elements from two maximal sets cannot be compatible (much less equal), and hence, except for the element 0, the maximal sets are disjoint.

Let a be a arbitrary, non-zero element of R. If a is a zero-divisor of R, then the idempotent element A-a^0 ≠0.

Further A-a^0 belongs to the maximal set generated by the non-zero divisor a′=a+A-a^0,

since it is $(A\text{-}a^0)a' = (A\text{-}a^0)\ (a+A\text{-}a^0)$
$= (A\text{-}a^0) = (A\text{-}a^0)^2$

(ie) $A\text{-}a^0 < a'$. Since also $a < a'$ and $a \sim A - a^0$ Therefore, a is idempotent.

(ie) All the zero-divisors of R are idempotent which is a maximal set then by theorem 1 and by definition A is a Boolean-ring. Then by definition, Therefore B is Smarandache-Boolean-near-ring.

Theorem 1.3.

A Boolean-near-ring $(B, \vee, \wedge)$ is having the proper subset A ,the set A of idempotent elements of a ring R, with suitable definitions for $\vee$ and $\wedge$,

$a \vee b = a + b - 2a^0 b = (a \cup b) - (a \cap b)$

$a \wedge b = a \cap b = a^0 b = ab^0$.

Then B is a Smarandache-Boolean-near-ring.

Proof.

Assume that the set A of idempotent elements of a ring R, which is also a subset of B. Now to prove that B is a Smarandache-Boolean-near-ring. It is sufficient to prove that the set A of idempotent elements of a ring R with identity forms a maximal set in R with uni-element.

By the definition of compatible, then we have every element of R is compatible with every other idempotent element.

If $a \in R$ is not idempotent then,

$a^2.1 \neq a.1^2$, since the definition of compatible. Hence no non-idempotent can belong to this maximal set. Thus the set A is idempotent element of R with identity forms a maximal set in R whose uni-element is the identity of R, by theorem 1 and by definition. A, a maximal set of B forms a Boolean ring

Then by definition

It conclude that B is Smarandache-Boolean-near-ring.

Theorem 1.4.

A Boolean-near-ring $(B, \vee, \wedge)$ is having the proper subset ,having a non-zero divisor of A, as an associate ring. with suitable definitions for $\vee$ and $\wedge$,

$a \vee b = a + b - 2a^0 b = (a \cup b) - (a \cap b)$

$a \wedge b = a \cap b = a^0 b = ab^0$.

Then B is a Smarandache-Boolean-near-ring.

Proof.

Let B is Boolean-near-ring whose proper subset having a non-zero divisor of associate ring A.

Now to prove that B Smarandache-Boolean-near-ring.

It is enough to prove that every non-divisor of A determines uniquely a maximal set of A with uni-element.

Let a be the uni-element of a maximal set A then we have $b < a$, for $b \in A$

Consider all the elements of A in whose sub-direct display one or more component ai duplicate the corresponding component ui of u, the other components of a being zeros.

(ie) all the element a such that $a < u$, becomes u is uni-element.

Clearly, these elements are compatible with each other and together with u form a maximal set

in A, for which u is the uni-element.
Hence A is a maximal set with uni-element and by theorem 1 and definition A, a maximal set of B forms a Boolean ring .

Then by definition Therefore B is Smarandache-Boolean-near-ring.

Theorem 1.5.

A Boolean-near-ring (B, $\vee, \wedge$) is having the proper subset A , associate ring is of the form A = u_J, where u is a non-zero of A and J is the set of idempotent elements of A, with suitable definitions for $\vee$ and $\wedge$,

$a\vee b = a+b - 2a^0b = (a\cup b) - (a\cap b)$

$a\wedge b = a\cap b = a^0b = ab^0$.

Then B is a Smarandache-Boolean-near-ring.

Proof.

Assume that the proper subset A of a Boolean-near-ring B is of the form A = u_J, where u is non-zero divisor of A and J is the set of idempotent elements of A.
Now to prove B is Smarandache-Boolean-near-ring.
It is enough to prove that A is a maximal set with uni-element.

(i) It is sufficient to show that the set uJ is a maximal set having u as its uni-element and

(ii) If b belongs to the maximal set determined by u, then b has the required form b = eu, for some e∈J

Proof of (i) It is seen that ue∼uf (ie) ue is compatible to uf with uni-element u, for all e, f∈J, since idempotent belogs to the center of A. Also, ue<u, since $ue.u=u^2e=(ue)^2$

Proof of (ii) We know that A is an associate ring, the associated idempotent a^0 of a has the property:
if a∼b then $a0b = ab^0 = b^0a = ba^0$
If $a\in A_u$ then since $a < u$ and u^0=1,

we have $A=u^0a=au^0=a^0u$, for all $a^0\in J$

Hence A is a maximal set with uni-element of of B by suitable definition and by theorem 1 then we have A is a Boolean-ring.
Then by definition,

Hence B is Smarandache-Boolean-near-ring.

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