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On the mean value of the Pseudo-Smarandache-Squarefree function

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Abstract For any positive integer n, the Pseudo Smarandache Squarefree function $Z_w(n)$ is defined as $Z_w(n) = \min\{m : n|m^n, m \in N\}$, and the function $Z(n)$ is defined as $Z(n) = \min\left\{m : n \leq \frac{m(m+1)}{2},\ m \in N\right\}$. The main purpose of this paper is using the elementary methods to study the mean value properties of the function $Z_w(Z(n))$, and give a sharper mean value formula for it.

Keywords Pseudo-Smarandache-Squarefree function $Z_w(n)$, function $Z(n)$, mean value, asymptotic formula.

§1. Introduction and result

For any positive integer n, the Pseudo-Smarandache-Squarefree function $Z_w(n)$ is defined as the smallest positive integer m such that $n \mid m^n$. That is,

$$Z_w(n) = \min\{m : n|m^n, m \in N\}.$$

For example $Z_w(1) = 1$, $Z_w(2) = 2$, $Z_w(3) = 3$, $Z_w(4) = 2$, $Z_w(5) = 5$, $Z_w(6) = 6$, $Z_w(7) = 7$, $Z_w(8) = 2$, $Z_w(9) = 3$, $Z_w(10) = 10$, $\cdots$. About the elementary properties of $Z_w(n)$, some authors had studied it, and obtained some interesting results. For example, Felice Russo [1] obtained some elementary properties of $Z_w(n)$ as follows:

Property 1. The function $Z_w(n)$ is multiplicative. That is, if $GCD(m, n) = 1$, then $Z_w(m \cdot n) = Z_w(m) \cdot Z_w(n)$.

Property 2. $Z_w(n) = n$ if and only if n is a squarefree number.

The main purpose of this paper is using the elementary method to study the mean value properties of $Z_w(Z(n))$, and give a sharper asymptotic formula for it, where $Z(n)$ is defined as $Z(n) = \min\left\{m : n \leq \frac{m(m+1)}{2}, m \in N\right\}$. That is, we shall prove the following conclusion:

Theorem. For any real number $x \geq 2$, we have the asymptotic formula

$$\sum_{n\leq x} Z_w(Z(n)) = \left(1 + \prod_p \left(1 + \frac{1}{p(p^2-1)}\right)\right) \cdot \frac{4\sqrt{2}}{\pi^2} \cdot x^{\frac{3}{2}} + O\left(x^{\frac{5}{4}}\right),$$

where $\prod_p$ denotes the product over all primes.

§2. Some lemmas

To complete the proof of the theorem, we need the following several lemmas.

Lemma 1. For any real number $x \geq 2$, we have the asymptotic formula

$$\sum_{m \leq x} \mu^2(m) = \frac{6}{\pi^2}x + O(\sqrt{x}). \tag{1}$$

Proof. See reference [2].

Lemma 2. For any real number $x \geq 2$, we have the asymptotic formula

$$\sum_{\substack{m \leq x \\ m \in A}} m^2 = \frac{2}{\pi^2}x^3 + O\left(x^{\frac{5}{2}}\right),$$

where A denotes the set of all square-free integers.

Proof. By the Abel's summation formula (See Theorem 4.2 of [3]) and Lemma 1, we have

$$\begin{aligned} \sum_{\substack{m \leq x \\ m \in A}} m^2 &= \sum_{m \leq x} m^2\mu^2(m) = x^2 \cdot \left(\frac{6}{\pi^2}x + O(\sqrt{x})\right) - 2\int_1^x t\left(\frac{6}{\pi^2}t + O(\sqrt{t})\right)\mathrm{d}t \\ &= \frac{6}{\pi^2}x^3 + O\left(x^{\frac{5}{2}}\right) - \frac{4}{\pi^2}x^3 = \frac{2}{\pi^2}x^3 + O\left(x^{\frac{5}{2}}\right). \end{aligned}$$

This proves Lemma 2.

Lemma 3. For any real number $x \geq 2$ and $s > 1$, we have the inequality

$$\sum_{\substack{m \leq x \\ m \in B}} \frac{Z_w(m)}{m^s} < \prod_p \left(1 + \frac{1}{p^{s-1}(p^s - 1)}\right).$$

Specially, if $s > \frac{3}{2}$, then we have the asymptotic formula

$$\sum_{\substack{m \leq x \\ m \in B}} \frac{Z_w(m)}{m^s} = \prod_p \left(1 + \frac{1}{p^{s-1}(p^s - 1)}\right) + O\left(x^{\frac{3}{2}-s}\right),$$

where B denotes the set of all square-full integers.

Proof. First we define the arithmetical function $a(m)$ as follows:

$$a(m) = \begin{cases} 1 & \text{if } m \in B \ ; \\ 0 & \text{otherwise.} \end{cases}$$

From Property 1 and the definition of $a(m)$ we know that the function $Z_w(m)$ and $a(m)$ are multiplicative. If $s > 1$, then by the Euler product formula (See Theorem 11.7 of [3]) we have

$$\begin{aligned} \sum_{\substack{m \leq x \\ m \in B}} \frac{Z_w(m)}{m^s} < \sum_{\substack{m=1 \\ m \in B}}^{\infty} \frac{Z_w(m)}{m^s} &= \sum_{m=1}^{\infty} \frac{Z_w(m)}{m^s}a(m) \\ &= \prod_p \left(1 + \frac{p}{p^{2s}} + \frac{p}{p^{3s}} + \cdots\right) \\ &= \prod_p \left(1 + \frac{1}{p^{s-1}(p^s - 1)}\right). \end{aligned}$$

Note that if $m \in B$, then $Z_w(m) \leq \sqrt{m}$. Hence, if $s > \frac{3}{2}$, then we have

$$\begin{aligned}
\sum_{\substack{m \leq x \\ m \in B}} \frac{Z_w(m)}{m^s} &= \sum_{\substack{m=1 \\ m \in B}}^{\infty} \frac{Z_w(m)}{m^s} - \sum_{\substack{m > x \\ m \in B}} \frac{Z_w(m)}{m^s} \\
&= \sum_{\substack{m=1 \\ m \in B}}^{\infty} \frac{Z_w(m)}{m^s} + O\Big(\sum_{m > x} \frac{1}{m^{s-\frac{1}{2}}}\Big) \\
&= \prod_p \Big(1 + \frac{1}{p^{s-1}(p^s - 1)}\Big) + O\Big(x^{\frac{3}{2}-s}\Big).
\end{aligned}$$

This proves Lemma 3.

§3. Proof of the theorem

In this section, we shall use the elementary method to complete the proof of the theorem. Note that if $\frac{(m-1)m}{2} + 1 \leq n \leq \frac{m(m+1)}{2}$, then $Z(n) = m$. That is, the equation $Z(n) = m$ has m solutions as follows:

$$n = \frac{(m-1)m}{2} + 1, \frac{(m-1)m}{2} + 2, \cdots, \frac{m(m+1)}{2}$$

Since $n \leq x$, from the definition of $Z(n)$ we know that if $Z(n) = m$, then $1 \leq m \leq \frac{\sqrt{8x+1}-1}{2}$.

Note that $Z_w(n) \leq n$, we have

$$\begin{aligned}
\sum_{n \leq x} Z_w(Z(n)) &= \sum_{\substack{n \leq x \\ Z(n) = m}} Z_w(m) = \sum_{m \leq \frac{\sqrt{8x+1}-1}{2}} m \cdot Z_w(m) + O(x) \\
&= \sum_{m \leq \sqrt{2x}} m \cdot Z_w(m) + O(x). \qquad (2)
\end{aligned}$$

We separate all integer m in the interval $[1, \sqrt{2x}]$ into three subsets A, B, and C as follows: A: the set of all square-free integers; B: the set of all square-full integers; C: the set of all positive integer m such that $m \in [1, \sqrt{2x}]/A \bigcup B$.

Note that (2), we have

$$\sum_{n \leq x} Z_w(Z(n)) = \sum_{\substack{m \leq \sqrt{2x} \\ m \in A}} m \cdot Z_w(m) + \sum_{\substack{m \leq \sqrt{2x} \\ m \in B}} m \cdot Z_w(m) + \sum_{\substack{m \leq \sqrt{2x} \\ m \in C}} m \cdot Z_w(m) + O(x). \qquad (3)$$

From Property 2 and Lemma 2 we know that if $m \in A$, then we have

$$\sum_{\substack{m \leq \sqrt{2x} \\ m \in A}} m \cdot Z_w(m) = \sum_{\substack{m \leq \sqrt{2x} \\ m \in A}} m^2 = \frac{4\sqrt{2}}{\pi^2} x^{\frac{3}{2}} + O\Big(x^{\frac{5}{4}}\Big). \qquad (4)$$

It is clear that if $m \in B$, then $Z_w(m) \leq \sqrt{m}$. Hence

$$\sum_{\substack{m \leq \sqrt{2x} \\ m \in B}} m \cdot Z_w(m) \ll \sum_{\substack{m \leq \sqrt{2x} \\ m \in B}} m^{\frac{3}{2}} \ll x^{\frac{5}{4}}. \qquad (5)$$

If $m \in C$, then we write m as $m = q \cdot n$, where q is a square-free integer and n is a square-full integer. From Property 1, Property 2, Lemma 2 and Lemma 3 we have

$$\begin{aligned}
\sum_{\substack{m\leq\sqrt{2x}\\ m\in C}} m\cdot Z_w(m) &= \sum_{n\leq\sqrt{2x}} nZ_w(n)a(n)\sum_{q\leq\frac{\sqrt{2x}}{n}} q^2\mu^2(q)\\
&= \sum_{n\leq\sqrt{2x}} nZ_w(n)a(n)\left(\frac{4\sqrt{2}}{\pi^2}\cdot\frac{x^{\frac{3}{2}}}{n^3}+O\left(\frac{x^{\frac{5}{4}}}{n^{\frac{5}{2}}}\right)\right)\\
&= \frac{4\sqrt{2}}{\pi^2}x^{\frac{3}{2}}\sum_{n\leq\sqrt{2x}}\frac{Z_w(n)a(n)}{n^2}+O\left(x^{\frac{5}{4}}\sum_{n\leq\sqrt{2x}}\frac{Z_w(n)a(n)}{n^{\frac{3}{2}}}\right)\\
&= \frac{4\sqrt{2}}{\pi^2}\prod_p\left(1+\frac{1}{p(p^2-1)}\right)x^{\frac{3}{2}}+O\left(x^{\frac{5}{4}}\right). \qquad (6)
\end{aligned}$$

Combining (3), (4), (5) and (6), we may immediately deduce the asymptotic formula

$$\sum_{n\leq x} Z_w(Z(n)) = \left(1+\prod_p(1+\frac{1}{p(p^2-1)}\right)\cdot\frac{4\sqrt{2}}{\pi^2}\cdot x^{\frac{3}{2}}+O\left(x^{\frac{5}{4}}\right).$$

This completes the proof of Theorem.

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