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On the mean value of the Smarandache ceil function[1]

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Abstract For any fixed positive integer n, the Smarandache ceil function of order k is denoted by $N^* \to N$ and has the following definition:

$$S_k(n) = \min\{x \in N : n \mid x^k\}, \qquad \forall n \in N^*.$$

In this paper, we study the mean value properties of the Smarandache ceil function, and give a sharp asymptotic formula for it.

Keywords Smarandache ceil function; Mean value; Asymptotic formula.

§1. Introduction

For any fixed positive integer n, the Smarandache ceil function of order k is denoted by $N^* \to N$ and has the following definition:

$$S_k(n) = \min\{x \in N : n \mid x^k\}, \qquad \forall n \in N^*.$$

For example, $S_2(1) = 1$, $S_2(2) = 2$, $S_2(3) = 3$, $S_2(4) = 2$, $S_2(5) = 5$, $S_2(6) = 6$, $S_2(7) = 7$, $S_2(8) = 4$, $S_2(9) = 3$, $\cdots$. This was introduced by Smarandache who proposed many problems in [1]. There are many papers on the Smarandache ceil function. For example, Ibstedt [2] [3] studied this function both theoretically and computationally, and got the following conclusions:

$$(a, b) = 1 \Rightarrow S_k(ab) = S_k(a)S_k(b), \qquad a, b \in N^*.$$

$$S_k(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}) = S_k\left(p_1^{\alpha_1}\right) \cdots S_k\left(p_r^{\alpha_r}\right).$$

In this paper, we study the mean value properties of the Smarandache ceil function, and give a sharp asymptotic formula for it. That is, we shall prove the following:

Theorem. For any real number $x \geq 2$, we have the asymptotic formula

$$\sum_{n \leq x} \frac{1}{S_2(n)} = \frac{3}{2\pi^2} \ln^2 x + A_1 \ln x + A_2 + O(x^{-\frac{1}{4}+\epsilon}),$$

where A_1 and A_2 are two computable constants, ϵ is any fixed positive integer.

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§2. Proof of the theorem

To complete the proof of the theorem, we need the following Lemma, which is called the Perron's formula (See reference [4]):

Lemma. Suppose that the Dirichlet series $f(s) = \sum_{n=1}^{\infty} a(n)n^{-s}$, $s = \sigma + it$, convergent absolutely for $\sigma > \sigma_a$, and that there exist a positive increasing function $H(u)$ and a function $B(u)$ such that

$$a(n) \leq H(n), \qquad n = 1, 2, \cdots,$$

and

$$\sum_{n=1}^{\infty} |a(n)|\, n^{-\sigma} \leq B(\sigma), \qquad \sigma > \sigma_a.$$

Then for any $s_0 = \sigma_0 + it_0$, $b_0 > \sigma_a$, $b_0 \geq b > 0$, $b_0 \geq \sigma_0 + b > \sigma_a$, $T \geq 1$ and $x \geq 1$, x not to be an integer, we have

$$\begin{aligned}\sum_{n\leq x} a(n)n^{-s_0} &= \frac{1}{2\pi i}\int_{b-iT}^{b+iT} f(s_0+s)\frac{x^s}{s}\mathrm{d}s + O\left(\frac{x^b B(b+\sigma_0)}{T}\right)\\ &+ O\left(x^{1-\sigma_0}H(2x)\min\left(1, \frac{\log x}{T}\right)\right) + O\left(x^{-\sigma_0}H(N)\min\left(1, \frac{x}{T\,||\,x\,||}\right)\right),\end{aligned}$$

where N is the nearest integer to x, $||\,x\,|| = |N - x|$.

Now we complete the proof of the theorem. Let $s = \sigma + it$ be a complex number and

$$f(s) = \sum_{n=1}^{\infty} \frac{1}{S_2(n)n^s}.$$

Note that $|\frac{1}{S_2(n)}| \leq \frac{1}{\sqrt{n}}$, so it is clear that $f(s)$ is a Dirichlet series absolutely convergent for $\mathrm{Re}(s) > \frac{1}{2}$, by Euler product formula [5] and the definition of $S_2(n)$ we have

$$\begin{aligned} f(s) &= \prod_p \left(1 + \frac{1}{S_2(p)p^s} + \frac{1}{S_2(p^2)p^{2s}} + \frac{1}{S_2(p^3)p^{3s}}\right.\\ &\quad \left. + \frac{1}{S_2(p^4)p^{4s}} + \cdots + \frac{1}{S_2(p^{2k})p^{2ks}} + \frac{1}{S_2(p^{2k+1})p^{(2k+1)s}} + \cdots\right)\\ &= \prod_p \left(1 + \frac{1}{p^{s+1}} + \frac{1}{p^{2s+1}} + \frac{1}{p^{3s+2}} + \frac{1}{p^{4s+2}} + \cdots + \frac{1}{p^{2ks+k}} + \frac{1}{p^{(2k+1)s+k+1}}\right.\\ &\quad \left. + \frac{1}{p^{2(k+1)s+k+1}} + \frac{1}{p^{(2(k+2)+1)s+k+2}} + \cdots\right)\\ &= \prod_p \frac{1}{1 - \frac{1}{p^{2s+1}}}\left(1 + \frac{1}{p^{s+1}}\right)\\ &= \frac{\zeta(2s+1)\zeta(s+1)}{\zeta(2s+2)}, \end{aligned}$$

where $\zeta(s)$ is the Riemann zeta-function and $\prod_p$ denotes the product over all primes.

Taking

$$H(x) = 1; \qquad B(\sigma) = \frac{2}{2\sigma - 1}, \qquad \sigma > \frac{1}{2};$$

$s_0 = 0$; $b = 1$; $T = x^{\frac{5}{4}}$ in the above Lemma we may get

$$\sum_{n \le x} \frac{1}{S_2(n)} = \frac{1}{2i\pi} \int_{1-ix^{\frac{5}{4}}}^{1+ix^{\frac{5}{4}}} f(s) \frac{x^s}{s} ds + O(x^{-\frac{1}{4}+\varepsilon}).$$

To estimate the main term, we move the integral line in the above formula from $s = 1 \pm ix^{\frac{5}{4}}$ to $s = -\frac{1}{4} \pm ix^{\frac{5}{4}}$. This time, the function $f(s)\frac{x^s}{s}$ have a third order pole point at $s = 0$ with residue

$$\frac{3}{2\pi^2} \ln^2 x + A_1 \ln x + A_2,$$

where A_1 and A_2 are two computable constants.

Hence, we have

$$\frac{1}{2\pi i} \left(\int_{1-ix^{\frac{5}{4}}}^{1+ix^{\frac{5}{4}}} + \int_{1+ix^{\frac{5}{4}}}^{-\frac{1}{4}+ix^{\frac{5}{4}}} + \int_{-\frac{1}{4}+ix^{\frac{5}{4}}}^{-\frac{1}{4}-ix^{\frac{5}{4}}} + \int_{-\frac{1}{4}-ix^{\frac{5}{4}}}^{1-ix^{\frac{5}{4}}} \right) \frac{\zeta(2s+1)\zeta(s+1)x^s}{\zeta(2s+2)s} ds$$
$$= \frac{3}{2\pi^2} \ln^2 x + A_1 \ln x + A_2.$$

We can easily get the estimate

$$\left| \frac{1}{2\pi i} \left(\int_{1+ix^{\frac{5}{4}}}^{-\frac{1}{4}+ix^{\frac{5}{4}}} + \int_{-\frac{1}{4}+i\frac{5}{4}}^{-\frac{1}{4}-i\frac{5}{4}} + \int_{-\frac{1}{4}-ix^{\frac{5}{4}}}^{1-ix^{\frac{5}{4}}} \right) \frac{\zeta(2s+1)\zeta(s+1)x^s}{\zeta(2s+2)s} ds \right| \ll x^{-\frac{1}{4}+\epsilon}.$$

From above we may immediately get the asymptotic formula:

$$\sum_{n \le x} \frac{1}{S_2(n)} = \frac{3}{2\pi^2} \ln^2 x + A_1 \ln x + A_2 + O(x^{-\frac{1}{4}+\epsilon}).$$

This completes the proof of the theorem.

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