MEAN VALUE OF A NEW ARITHMETIC FUNCTION

Liu Yanni
Department of Mathematics, Northwest University
Xi'an, Shaanxi, P.R.China

Gao Peng
School of Economics and Management,Northwest University
Xi'an, Shaanxi, P.R.China

Abstract The main purpose of this paper is using elementary method to study a new arithmetic function, and give an interesting asymptotic formula for it.

Keywords: Arithmetic function; Mean value; Asymptotic formula

§1. Introduction

For any positive integer n, we have $n = u^k v$, where v is a k-power free number. Let $b_k(n)$ be the k-power free part of n. Let p be any fixed prime, n be any positive integer, $e_p(n)$ denotes the largest exponent of power p. That is, $e_p(n) = m$,if $p^m|n$ and $p^{m+1} \dagger n$. In [1], Professor F.Smarandache asked us to study the properties of these two arithmetic functions. It seems that no one knows the relationship between these two arithmetic functions before. The main purpose of this paper is to study the mean value properties of $e_p(b_k(n))$, and obtain an interesting mean value formula for it. That is, we shall prove the following conclusion:

Theorem. *Let p be a prime, k be any fixed positive integer. Then for any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{n \leq x} e_p(b_k(n)) = \left(\frac{p^k - p}{(p^k - 1)(p - 1)} - \frac{k - 1}{p^k - 1} \right) x + O\left(x^{\frac{1}{2} + \epsilon} \right),$$

where ϵ denotes any fixed positive number.

Taking $k = 2$ in the theorem, we may immediately obtain the following

Corollary. *For any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{n \leq x} e_p(b_k(n)) = \frac{1}{p+1}x + O\left(x^{\frac{1}{2}+\epsilon}\right).$$

§2. Proof of the theorem

In this section, we shall use analytic method to complete the proof of the theorem. In fact we know that $e_p(n)$ is not a multiplicative function, but we can use the properties of the Riemann zeta-function to obtain a generating function. For any complex s, if Re$(s) > 1$, we define the Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} \frac{e_p(b_k(n))}{n^s}.$$

Let positive integer $n = p^\alpha n_1$, where $(n_1, p) = 1$, then from the definition of $e_p(n)$ and $b_k(n)$, we have:

$$e_p(b_k(n)) = e_p(b_k(p^\alpha n_1)) = e_p(b_k(p^\alpha)).$$

From the above formula and the Euler product formula (See Theorem 11.6 of [3]) we can get

$$\begin{aligned} f(s) &= \sum_{n=1}^{\infty} \frac{e_p(b_k(n))}{n^s} \\ &= \sum_{\alpha=0}^{\infty} \sum_{\substack{n_1=1 \\ (n_1,p)=1}}^{\infty} \frac{e_p(b_k(p^\alpha))}{p^{\alpha s} n_1^s} \\ &= \zeta(s)(1 - \frac{1}{p^s}) \sum_{\alpha=1}^{\infty} \frac{e_p(b_k(p^\alpha)}{p^{\alpha s}}. \end{aligned}$$

Let

$$\begin{aligned} A &= \sum_{\alpha=1}^{\infty} \frac{e_p(b_k(p^\alpha)}{p^{\alpha s}} \\ &= \frac{1}{p^s} + \frac{2}{p^{2s}} + \cdots + \frac{k-1}{p^{(k-1)s}} + \frac{1}{p^{(k+1)s}} + \frac{2}{p^{(k+2)s}} + \cdots + \frac{k-1}{p^{(2k-1)s}} \\ &\quad + \cdots + \frac{1}{p^{(uk+1)s}} + \frac{2}{p^{(uk+2)s}} + \cdots + \frac{k-1}{p^{(uk+k-1)s}} \\ &= \sum_{u=0}^{\infty} \frac{1}{p^{uks}} \sum_{r=1}^{k-1} \frac{r}{p^{rs}} \\ &= \frac{1}{1-\frac{1}{p^{ks}}} \frac{1}{1-\frac{1}{p^s}} \left(\frac{1 - \frac{1}{p^{(k-1)s}}}{p^s - 1} - \frac{k-1}{p^{ks}} \right). \end{aligned}$$

So we have

$$f(s) = \sum_{n=1}^{\infty} \frac{e_p(b_k(n))}{n^s} = \left(\frac{p^{ks} - p^s}{(p^{ks} - 1)(p^s - 1)} - \frac{k-1}{p^{ks} - 1} \right) \zeta(s).$$

Because the Riemann zeta-function $\zeta(s)$ have a simple pole point at $s = 1$ with the residue 1, we know $f(s)\frac{x^s}{s}$ also have a simple pole point at $s = 1$ with the residue $\left(\frac{p^k - p}{(p^k-1)(p-1)} - \frac{k-1}{p^k-1} \right) x$. By Perron formula (See [2]), taking $s_0 = 0$, $b = \frac{3}{2}$, $T > 1$, then we have

$$\sum_{n \le x} e_p(b_k(n)) = \frac{1}{2\pi i} \int_{\frac{3}{2}-iT}^{\frac{3}{2}+iT} f(s) \frac{x^s}{s} ds + O\left(\frac{x^{\frac{3}{2}}}{T} \right),$$

we move the integral line to Re $s = \frac{1}{2} + \epsilon$, then taking $T = x$, we can get

$$\begin{aligned}
\sum_{n \le x} e_p(b_k(n)) &= \left(\frac{p^k - p}{(p^k - 1)(p - 1)} - \frac{k-1}{p^k - 1} \right) x \\
&\quad + \frac{1}{2\pi i} \int_{\frac{1}{2}+\epsilon-iT}^{\frac{1}{2}+\epsilon+iT} f(s) \frac{x^s}{s} ds + O\left(x^{\frac{1}{2}+\epsilon} \right) \\
&= \left(\frac{p^k - p}{(p^k - 1)(p - 1)} - \frac{k-1}{p^k - 1} \right) x \\
&\quad + O\left(\int_{-T}^{T} \left| f(\frac{1}{2} + \epsilon + it) \right| \frac{x^{\frac{1}{2}+\epsilon}}{1 + |t|} dt \right) + O\left(x^{\frac{1}{2}+\epsilon} \right) \\
&= \left(\frac{p^k - p}{(p^k - 1)(p - 1)} - \frac{k-1}{p^k - 1} \right) x + O\left(x^{\frac{1}{2}+\epsilon} \right).
\end{aligned}$$

This completes the proof of Theorem.

References

[1] F.Smarandache, Only Problems,Not Solutions, Chicago, Xiquan Publishing House, 1993.

[2] Pan Chengdong and Pan Chengbiao, Foundation of Analytic Number Theory, Beijing, Science Press, 1991.

[3] Tom M. Apstol, Introduction to Analytic Number Theory, New York, Springer-Verlag, 1976.