International J.Math. Combin. Vol.4 (2009), 49-55

Minimal Retraction of Space-time and Their Foldings

A. E. El-Ahmady and H. Rafat

(Department of Mathematics, Faculty Science of Tanta University, Tanta, Egypt)

E-mail:

Abstract: A *Smarandache multi-spacetime* is such a union spacetime $\bigcup_{i=1}^{n} S_i$ of spacetimes $S_1, S_2, \cdots, S_n$ for an integer $n \geq 1$. In this article, we will be deduced the geodesics of space-time, i.e., a Smarandache multi-spacetime with $n = 1$ by using Lagrangian equations. The deformation retract of space-time onto itself and into a geodesics will be achieved. The concept of retraction and folding of zero dimension space-time will be obtained.The relation between limit of folding and retraction presented.

Key Words: Folding, deformation retract, space-time, Smarandache multi-spacetime.

AMS(2000): 53A35, 51H05, 58C05, 51F10, 58B34.

§1. Introduction

The folding of a manifold was, firstly introduced by Robertson in [1977] [14]. Since then many authers have studied the folding of manifolds such as in [4,6,12,13]. The deformation retracts of the manifolds defined and discussed in [5,7]. In this paper, we will discuss the folding restricted by a minimal retract and geodesic. We may also mention that folding has many important technical applications, for instance, in the engineering problems of buckling and post-buckling of elastic and elastoplastic shells [1]. More studies and applications are discussed in [4], [8], [9], [10], [13].

§2. Definitions

1. A subset A of a topological space X is called a retract of X, if there exists a continuous map $r : X \to A$ such that ([2]):

(i) X is open;
(ii)$r(a) = a, \ \forall a \in A$.

2. A subset A of atopological space X is said to be a deformation retract if there exists a retraction $r : X \to A$, and a homotopy $f : X \times I \to X$ such that([2]):

$f(x, 0) = x, \forall, x \in X;$

[1]Received Oct.15, 2009. Accepted Nov. 18, 2009.

$f(x,1) = r(x), \forall x \in X;$
$f(a,t) = a, \forall a \in A, t \in [0,1].$

3. Let M and N be two smooth manifolds of dimensions m and n respectively. A map $f : M \to N$ is said to be an isometric folding of M into N if and only if for every piecewise geodesic path $\gamma : J \to M$,the induced path $f \circ \gamma : J \to N$ is a piecewise geodesic and of the same length as γ ([14]). If $f$does not preserve the lengths, it is called topological folding.

4. Let M be an m-dimensional manifold. M is said to be minimal m-dimensional manifold if the mean curvature vanishes everywhere, i.e., $H(\sigma.p) = 0$ for all $p \in M$ ([3]).

5. A subset A of a minimal manifold M is a minimal retraction of M, if there exists a continuous map $r : M \to A$such that ([12]):

(i)M is open;
(ii)$r(M) = A$;
(iii)$r(a) = a, \forall a \in A$;
(iv)$r(M)$is minimal manifold.

§3. Main Results

Using the Neugebaure-Bcklund transformation, the space-time T take the form [11]

$$ds^2 = dt^2 - dp^2 - dz^2 - p^2 d\phi^2 \tag{1}$$

Using the relationship between the cylindrical and spherical coordinates, the metric becomes

$$\overline{ds}^2 = r^2(\sin^2\theta_2 - \cos^2\theta_2)\overline{d\theta}_2^2 - r^2\sin^2\theta_2\overline{d\theta}_1^2 + (\cos^2\theta_2 - \sin^2\theta_2)\overline{dr}^2$$
$$-r^2\sin^2\theta_1\sin^2\theta_2\overline{d\varphi}^2 - 4r\sin\theta_2\cos\theta_2 d\theta_2 dr.$$

The coordinates of space-time T are:

$$\left.\begin{aligned} y_1 &= \sqrt{c_1(r,\theta_2) - r^2\sin^2\theta_2\theta_1^2} \\ y_2 &= \sqrt{4r^2\cos 2\theta_2 + k_1} \\ y_3 &= \sqrt{r^2\cos 2\theta_2 + c_3(\theta_2)} \\ y_4 &= \sqrt{c_4(r,\theta_1,\theta_2) - r^2\sin^2\theta_1\sin^2\theta_2\phi^2} \end{aligned}\right\} \tag{2}$$

where c_1, k_1, c_3, c_4 are the constant of integrations. Applying the transformation

$x_1^2 = y_1^2 - c_1(r,\theta_2),$
$x_2^2 = y_2^2 - k_1,$
$x_3^2 = y_3^2 - c_3(\theta_2),$
$x_4^2 = y_4^2 - c_4(r,\theta_1,\theta_2)$

Then, the coordinates of space-time T becomes:

$$\left.\begin{aligned} x_1 &= ir\sin\theta_2\theta_1 \\ x_2 &= 2r\sqrt{\cos 2\theta_2} \\ x_3 &= r\sqrt{\cos 2\theta_2} \\ x_4 &= ir\ \sin\theta_1\sin\theta_2\phi. \end{aligned}\right\} \tag{3}$$

Now, we apply Lagrangian equations

$$\frac{d}{ds}\left(\frac{\partial T}{\partial G_i}\right)-\frac{\partial T}{\partial G_i}=0, i=1,2,3,4.$$

to find a geodesic which is a subset of the space-time T . Since

$$\begin{aligned} T &= \frac{1}{2}\{-r^2\cos 2\theta_2\theta_2'^2 - r^2\sin^2\theta_2\theta_1'^2 + \cos 2\theta_2 r'^2 - r^2\sin^2\theta_1\sin^2\theta_2\phi'^2 \\ &\quad -2r\sin 2\theta_2\theta_2' r'\} \end{aligned}$$

then, the Lagrangian equations for space-time T are:

$$\frac{d}{ds}(r^2\sin^2\theta_2\theta_1') + (r^2\sin\theta_1\cos\theta_1\sin^2\theta_2{\phi'}^2) = 0 \tag{4}$$

$$\begin{aligned} &\tfrac{d}{ds}(r^2\cos 2\theta_2\theta_2' + r\sin\theta_2 r') + (r^2\sin 2\theta_2\theta_2'^2 + r^2\sin\theta_2\cos\theta_2\theta_1'^1 \\ &+\sin 2\theta_2 r'^2 + r^2\sin^2\theta_1\sin\theta_2\cos\theta_2\phi'^2 + 2r\cos 2\theta_2\theta_2' r') = 0 \end{aligned} \tag{5}$$

$$\begin{aligned} &\tfrac{d}{ds}(\cos 2\theta_2 r' - r\sin 2\theta_2\theta_2') + (r\cos 2\theta_2\theta_2'^2 + r\sin^2\theta_2\theta_1'^2 + \\ &r\sin^2\theta_1\sin^2\theta_2\phi'^2 + \sin 2\theta_2\theta_2' r') = 0 \end{aligned} \tag{6}$$

$$\frac{d}{ds}(r^2\sin^2\theta_1\sin^2\theta_2\phi') = 0. \tag{7}$$

From equation (7) we obtain $r^2\sin^2\theta_1\sin^2\theta_2\phi^1 =$ constant μ. If $\mu = 0$, we obtain the following cases:

(i) If $r = 0$, hence we get the coordinates of space-time T_1, which are defined as

$$x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0,$$

which is a hypersphere T_1, $x_1^2 - x_2^2 - x_3^2 - x_4^2 = 0$ on the null cone since the distance between any two different points equal zero, it is a minimal retraction and geodesic.

(ii) If $sin^2\theta_1 = 0$, we get

$$x_1 = 0, x_2 = 2r\sqrt{\cos 2\theta_2}, x_3 = r\sqrt{\cos 2\theta_2}, x_4 = 0.$$

Thus, $x_1^2+x_2^2+x_3^2+x_4^2=5r^2\cos 2\theta_2$, which is a hypersphere S_1 in space-time T with $x_1 = x_4 = 0$. It is a geodesic and retraction.

(iii) If $\sin^2\theta_2 = 0$, then $\theta_2 = 0$ we obtain the following geodesic retraction

$$x_1 = 0, x_2 = 2r, x_3 = r, x_4 = 0, \quad x_1^2 + x_2^2 + x_3^2 - x_1^2 = 5r^2,$$

which is the hypersphere $S_2 \subset T$ with $x_1 = x_4 = 0$.

(iv) If $\phi' = 0$ this yields the coordinate of $T_2 \subset T$ given by

$$x_1 = ir\sin\theta_2\theta_1, x_2 = 2r\sqrt{\cos 2\theta_2}, x_3 = r\sqrt{\cos 2\theta_2}, x_4 = 0.$$

It is worth nothing that $x_4 = 0$ is a hypersurface $T_2 \subset T$. Hence, we can formulate the following theorem.

Theorem 1 *The retractions of space-time is null geodesic, geodesic hyperspher and hypersurface.*

Lemma 1 *In space-time the minimal retraction induces null-geodesic.*

Lemma 2 *A minimal geodesic in space-time is a necessary condition for minimal retration.*

The deformation retract of the space-time T is defined as

$$\rho : T \times I \to T$$

where T is the space-time and I is the closed interval [0,1]. The retraction of the space-time T is defined as

$$R : T \to T_1, T_2, S_1 \text{ and } S_2.$$

The deformation retract of space-time T into a geodesic $T_1 \subset T$ is defined by

$$\begin{aligned}\rho(m,t) &= (1-t)\{ir\,\sin\theta_2\,\theta_1, 2r\sqrt{\cos 2\theta_2}, r\sqrt{\cos 2\theta_2},\\ &\quad ir\sin\theta_1\sin\theta_2\phi\} + t\{0,0,0,0\}.\end{aligned}$$

where $\rho(m,0) = \{ir\sin\theta_2\theta_1, 2r\sqrt{\cos 2\theta_2}, r\sqrt{\cos 2\theta_2}, ir\sin\theta_1\sin\theta_2\phi\}$, $\rho(m,1) = \{0,0,0,0\}$.

The deformation retract of space-time T into a geodesic $T_2 \subset T$ is defined as

$$\begin{aligned}\rho(m,t) &= (1-t)\{ir\sin\theta_2\theta_1, 2r\sqrt{\cos 2\theta_2}, r\sqrt{\cos 2\theta_2}, ir\sin\theta_1\sin\theta_2\phi\}\\ &+ t\{ir\sin\theta_2\theta_1, 2r\sqrt{\cos 2\theta_2}, r\sqrt{\cos 2\theta_2}, 0\}.\end{aligned}$$

The deformation retract of space-time T into a geodesic $S_1 \subset T$ is defined by

$$\begin{aligned}\rho(m,t) &= (1-t)\{ir\sin\theta_2\theta_1, 2r\sqrt{\cos 2\theta_2}, r\sqrt{\cos 2\theta_2}, ir\sin\theta_1\sin\theta_2\phi\}\\ &+ t\{0, 2r\sqrt{\cos 2\theta_2}, r\sqrt{\cos 2\theta_2}, 0\}.\end{aligned}$$

The deformation retract of space-time T into a geodesic $S_2 \subset T$ is defined as

$$\rho(m,t) = (1-t)\{ir\sin\theta_2\theta_1, 2r\sqrt{\cos 2\theta_2}, r\sqrt{\cos 2\theta_2}, ir\sin\theta_1\sin\theta_2\phi\} + t\{0, 2r, r, 0\}.$$

Now we are going to discuss the folding $\Im$ of the space-time T. Let $\Im : T \to T$, where

$$\Im(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, |x_4|) \tag{8}$$

An isometric folding of the space-time T into itself may be defined as

$$\begin{aligned}\Im : \quad & \{ir\sin\theta_2\theta_1,\ 2r\sqrt{\cos 2\theta_2}, r\sqrt{\cos 2\theta_2}, ir\sin\theta_1\sin\theta_2\phi\} \\ \to \quad & \{ir\sin\theta_2\theta_1, 2r\sqrt{\cos 2\theta_2}, r\sqrt{\cos 2\theta_2}, |ir\sin\theta_1\sin\theta_2\phi|\}.\end{aligned}$$

The deformation retract of the folded space-time T into the folded geodesic T_1 is

$$\begin{aligned}\rho_\Im : \quad & \{ir\sin\theta_2\theta_1, 2r\sqrt{\cos 2\theta_2}, r\sqrt{\cos 2\theta_2}, |ir\sin\theta_1\sin\theta_2\phi|\} \times I \\ \to \quad & \{ir\sin\theta_2\theta_1, 2r\sqrt{\cos 2\theta_2}, r\sqrt{\cos 2\theta_2}, |ir\sin\theta_1\sin\theta_2\phi|\}\end{aligned}$$

with

$$\rho_\Im(m,t) = (1-t)\{ir\sin\theta_2\theta_1, 2r\sqrt{\cos 2\theta_2}, r\sqrt{\cos 2\theta_2}, |ir\sin\theta_1\sin\theta_2\phi|\} + t\{0, 0, 0, 0\}.$$

The deformation retract of the folded space-time T into the folded geodesicT_2 is

$$\begin{aligned}\rho_\Im(m,t) \quad = \quad & (1-t)\{ir\sin\theta_2\theta_1, 2r\sqrt{\cos 2\theta_2}, r\sqrt{\cos 2\theta_2}, |ir\sin\theta_1\sin\theta_2\phi|\} \\ + \quad & t\{ir\sin\theta_2\theta_1, 2r\sqrt{\cos 2\theta_2}, r\sqrt{\cos 2\theta_2}, 0\}.\end{aligned}$$

The deformation retract of the folded space-time T into the folded geodesic S_1 is

$$\begin{aligned}\rho_\Im(m,t) \quad = \quad & (1-t)\{ir\sin\theta_2\theta_1, 2r\sqrt{\cos 2\theta_2}, r\sqrt{\cos 2\theta_2}, |ir\sin\theta_1\sin\theta_2\phi|\} \\ + \quad & t\{0, 2r\sqrt{\cos 2\theta_2}, r\sqrt{\cos 2\theta_2}, 0\}.\end{aligned}$$

The deformation retract of the folded space-time T into the folded geodesic S_2 is

$$\begin{aligned}\rho_\Im(m,t) \quad = \quad & (1-t)\{ir\sin\theta_2\theta_1, 2r\sqrt{\cos 2\theta_2}, r\sqrt{\cos 2\theta_2}, |ir\sin\theta_1\sin\theta_2\phi|\} \\ + \quad & t\{0, 2r, r, 0\}\end{aligned}$$

Then, the following theorem has been proved.

Theorem 2 *Under the defined folding, the deformation retract of the folded space-time into the folded geodesics is the same as the deformation retract of space-time into the geodesics.*

Now, let the folding be defined as:

$$\Im^*(x_1, x_2, x_3, x_4) = (x, |x_2|, x_3, x_4). \tag{9}$$

The isometric folded space-time $\Im(T)$ is

$$\bar{R} = \{ir\sin\theta_2\theta_1, \left|2r\sqrt{\cos 2\theta_2}\right|, r\sqrt{\cos 2\theta_2}, ir\sin\theta_1\sin\theta_2\phi\}.$$

Hence, we can formulate the following theorem.

Theorem 3 *The deformation retract of the folded space-time ,i.e., $\rho\Im^*(T)$ is different from the deformation retract of space-time under condition (9).*

Now let $\Im_1 : T^n \to T^n$,

$\Im_2 : \Im_1(T^n) \to \Im_1(T^n)$,

$\Im_3 : \Im_2(\Im_1(T^n)) \to \Im_2(\Im_1(T^n)), \cdots$,

$\Im_n : \Im_{n-1}(\Im_{n-2}...(\Im_1(T^n))...)) \to \Im_{n-1}(\Im_{n-2}...(\Im_1(T^n))...))$,

$\lim\limits_{n\to\infty} \Im_{n-1}(\Im_{n-2}...(\Im_1(T^n))...)) = n-1$ dimensional space-time T^{n-1}.

Let $h_1 : T^{n-1} \to T^{n-1}$,

$h_2 : h_1(T^{n-1}) \to h_1(T^{n-1})$,

$h_3 : h_2(h_1(T^{n-1})) \to h_2(h_1(T^{n-1}), ...,$

$h_m : h_{m-1}(h_{m-2}...(h_1(T^{n-1}))...)) \to h_{m-1}(h_{m-2}...(h_1(T^{n-1}))...))$,

$\lim h_m(h_m : h_{m-1}(h_{m-2}...(h_1(T^{n-1}))...)) = n-2$ dimensional space-time T^{n-2}.

Consequently, $\lim\limits_{s\to\infty}\lim\limits_{m\to\infty}\lim\limits_{n\to\infty}...k_s(h_m(\Im_n(T^n))) =$ 0-dimensional space-time. Hence, we can formulate the following theorem.

Theorem 4 *The end of the limits of the folding of space-time T^n is a 0-dimensional geodesic, it is a minimal retraction.*

Now let f_1 be the foldings and r_i be the retractions. then we have

$T^n \xrightarrow{f_1^1} T_1^n \xrightarrow{f_2^1} T_2^n \longrightarrow \cdots T_{n-1}^n \xrightarrow{\lim f_i^1} T^{n-1}$,

$T^n \xrightarrow{r_1^1} T_1^n \xrightarrow{r_2^1} T_2^n \longrightarrow \cdots T_{n-1}^n \xrightarrow{\lim r_i^1} T^{n-1}$,

$T^n \xrightarrow{f_1^2} T_1^{n-1} \xrightarrow{f_2^2} T_2^{n-1} \longrightarrow \cdots T_{n-1}^n \xrightarrow{\lim f_i^2} T^{n-2}, \cdots$,

$T^{n-1} \xrightarrow{r_1^1} T_1^{n-1} \xrightarrow{r_2^2} T_2^{n-1} \longrightarrow \cdots\ T_{n-1}^n \xrightarrow{\lim r_i^2} T^{n-2}, \cdots$,

$T^1 \xrightarrow{f_1^n} T_1^1 \xrightarrow{f_2^n} T_2^1 \longrightarrow \cdots T_{n-1}^1 \xrightarrow{\lim f_i^n} T^0$,

$T^1 \xrightarrow{r_1^n} T_1^1 \xrightarrow{r_2^n} T_2^1 \longrightarrow \cdots T_{n-1}^1 \xrightarrow{\lim f_i^n} T^0$.

Then the end of the limits of foldings = the limit of retractions = 0-dimensional space-time. Whence, the following theorem has been proved.

Theorem 5 *In space-time the end of the limits of foldings of T^n into itself coincides with the minimal retraction.*

References

[1] M. J. Ablowitz and P. A. Clarkson: *Solutions, Nonlinear evolution equations and inverse scattering* Cambridge University press(1991)

[2] M. A. Armstrong: *Basic topology*, McGrow-Hill(1979).

[3] M. P. Docarmo:*Riemannian geometry*, Boston,Birkhauser(1992).

[4] A. E. El-Ahmady: Fuzzy folding of fuzzy horocycle, Circolo Matematico, di Palermo, Serie II, Tomo LIII, 443-450, 2004.

[5] A. E. El-Ahmady: The deformation retract and topological folding of buchdahi space, *Periodica Mathematica Hungarica*, 28(1),1994, 19-30.

[6] A. E. El-Ahmady: Fuzzy Lobachevskian space and its folding, *The Joural of Fuzzy Mathematics*, 12(2),2004, 255-260.

[7] A. E. El-Ahmady and H. M. Shamara: Fuzzy deformation retract of fuzzy horospheres, *Indian J.Pure Appel.Math.*, 32(10), 2001 1501-1506.

[8] A. E. El-Ahmady, The deformation retract of a manifold adimting a simple transitive group of motions and its topological folding, *Bull. Cal. Math. Soc.*, 96 (4), 2004, 279-284.

[9] A. E. El-Ahmady, Limits of fuzzy retractions of fuzzy hyperspheres and their foldings, *Tamkang Journal of Mathematics* (accepted).

[10] A. E. El-Ahmady and H. Rafat, A calculation of geodesics in chaotic flat space and its folding, *Chaos, Solitions and Fractals* (accepted).

[11] C. Rogers and W. F. Shadwick: *Backland transformations and their applications*, Academic Press, 1982.

[12] M. El-Ghoul and K. Khalifa, The folding of minimal manifolds and its deformation, *Chaos, Solitons and Fractals U.K.*, 13(2002), 1031-1035.

[13] M. El-Ghoul, A. E. El-Ahmady and H. Rafat, Folding-retraction of chaotic dynamical manifold and the VAK of vacuum fluction, *Chaos, Solitions and Fractal, UK*, 20(2004) 209-217.

[14] S. A. Robertson, Isometric folding of Riemannian manifolds, *Proc. Roy. Soc. Edinburgh*, 77(1977), 275-284.