A1 and A3, A3 lies between A2 and A4, etc. and the segments AA1, A1A2, A2A3, A3A4, ... are congruent to one another.

Then, among this series of points, not always there exists a certain point An such that B lies between A and An.

For example:

let A be a point in delta1-f1, and B a point on f1, B different from P;

on the line AB consider the points A1, A2, A3, A4, ...

in between A and B, such that AA1, A1A2, A2A3, A3A4, etc. are congruent to one another;

then we find that there is no point behind B (considering the direction from A to B), because B is a limit point (the line AB ends in B).

The Bolzano's (intermediate value) theorem may not hold in the Critical Zone of the Model.

Can you readers find a better model for this anti-geometry?

References:

[1] Charles Ashbacher, "Smarandache Geometries", <Smarandache Notions Journal>, Vol. 8, No. 1-2-3, Fall 1997, pp. 212-215.

[2] Mike Mudge, "A Paradoxist Mathematician, His Function, Paradoxist Geometry, and Class of Paradoxes", <Smarandache Notions Journal>, Vol. 7, No. 1-2-3, August 1996, pp. 127-129.
reviewed by David E. Zitarelli, <Historia Mathematica>, Vol. 24, No. 1, p. 114, #24.1.119, 1997.

[3] Marian Popescu, "A Model for the Smarandache Paradoxist Geometry", <Abstracts of Papers Presented to the American Mathematical Society Meetings>, Vol. 17, No. 1, Issue 103, 1996, p. 265.

[4] Florentin Smarandache, "Collected Papers" (Vol. II), University of Kishinev Press, Kishinev, pp. 5-28, 1997.

[5] Florentin Smarandache, "Paradoxist Mathematics" (lecture), Bloomsburg University, Mathematics Department, PA, USA, November 1985.

On certain new inequalities and limits for the Smarandache function

József Sándor

Department of Mathematics, Babes-Bolyai University,
3400 Cluj - Napoca, Romania

I. Inequalities

1) If n $\geq$ 4 is an even number, then $S(n) \leq \frac{n}{2}$.

—Indeed, $\frac{n}{2}$ is integer, $\frac{n}{2} > 2$, so in $(\frac{n}{2})! = 1 \cdot 2 \cdot 3 \cdots \frac{n}{2}$ we can simplify with 2, so $n \mid (\frac{n}{2})!$.

This simplies clearly that $S(n) \leq \frac{n}{2}$.

2) If $n > 4$ is an even number, then $S(n^2) \leq n$

—By $n! = 1 \cdot 2 \cdot 3 \cdots \frac{n}{2} \cdots n$, since we can simplify with 2, for $n > 4$ we get that $n^2 | n!$. This clearly implies the above stated inequality. For factonials, the above inequality can be much improved, namely one has:

3) $S\left((m!)^2\right) \leq 2m$ and more generally, $S\left((m!)^n\right) \leq n \cdot m$ for all positive integers m and n.

—First remark that $\frac{(mn)!}{(m!)^n} = \frac{(mn)!}{m!(mn-m)!} \cdot \frac{(mn-m)!}{m!(mn-2m)!} \cdots \frac{(2m)!}{m! \cdot m!} =$

$= C_{2m}^m \cdot C_{3m}^m \cdots C_{nm}^m$, where $C_n^k = \binom{n}{k}$ denotes a binomial coefficient. Thus $(m!)^n$ divides $(m\,n)!$, implying the stated inequality. For $n = 2$ one obtains the first part.

4) Let $n > 1$. Then $S\left((n!)^{(n-1)!}\right) \leq n!$

—We will use the well-known result that the product of n consecutive integers is divisible by $n!$. By $(n!)! = 1 \cdot 2 \cdot 3 \cdots n \cdot \left((n+1)(n+2)\cdots 2n\right) \cdots \left((n-1)!-1\right) \cdots (n-1)!\, n$

each group is divisible by $n!$, and there are $(n-1)!$ groups, so $(n!)^{(n-1)!}$ divides $(n!)!$. This gives the stated inequality.

5) For all m and n one has $[S(m), S(n)] \leq S(m \cdot S(n) \leq [m, n]$. where $[a, b]$ denotes the

$\ell \cdot c \cdot m$ of a and b.

—If $m = \prod P_i^{a_i}$, $n = \prod q_j^{b_j}$ are the canonical representations of m, resp. n, then it is well-known that $S(m) = S(P_i^{a_i})$ and $S(n) = S(q_j^{b_j})$, where $S(P_i^{a_i}) = \max\{S(P_i^{a_i}) : i = 1, \cdots, r\}$; $S(q_j^{b_j}) = \max\{S(q_j^{b_j}) : j = 1, \cdots, h\}$, with r and h the number of prime divisors of m, resp. n. Then clearly $[S(m), S(n)] \leq S(m) \cdot S(n) \leq P_i^{a_i} \cdot q_j^{b_j} \leq [m, n]$

6) $(S(m), S(n)) \geq \frac{S(m) \cdot S(n)}{mn} \cdot (m, n)$ for all m and n

—Since $(S(m), S(m)) = \frac{S(m) \cdot S(n)}{[S(m), S(n)]} \geq \frac{S(m) \cdot S(n)}{[m, n]} = \frac{S(m) \cdot S(n)}{nm} \cdot (m, n)$

by 5) and the known formula $[m,n] = \frac{mn}{(m,n)}$.

7) $\frac{(S(m), S(n))}{(m, n)} \geq \left(\frac{S(mn)}{mn}\right)^2$ for all m and n

—Since $S(mn) \leq m\, S(n)$ and $S(mn) \leq n\, S(m)$ (See [1]), we have $\left(\frac{S(mn)}{mn}\right)^2 \leq \frac{S(m)\, S(n)}{mn}$,

and the result follows by 6).

8) We have $\left(\frac{S(mn)}{mn}\right)^2 \leq \frac{S(m)\, S(n)}{mn} \leq \frac{1}{(mn)}$

—This follows by 7) and the stronger inequality from 6), namely $S(m)\, S(n) \leq [m\, n] = \frac{mn}{(m, n)}$

Corollary $S(m\, n) \leq \frac{mn}{\sqrt{mn}}$

9) Max $\{S(m), S(n)\} \geq \frac{S(mn)}{(mn)}$ for all m, n; where (m,n) denotes the $g \cdot c \cdot d$ of m and n.

—We apply the known result: max $\{S(m), S(n)\} = S([m, n])$ On the other hand, since $[m, n] \mid m \cdot n$, by Corollary 1 from our paper [1] we get $\frac{S(mn)}{mn} \leq \frac{S([m, n])}{[m, n]}$.

Since $[m, n] = \frac{mn}{(m, n)}$,

The result follows:

Remark. Inequality g) compliments Theorem 3 from [1],

namely that max $\{S(m), S(n)\} \leq S(m\, n)$.

10) Let $d(n)$ be the number of divisors of n. Then $\frac{S(n!)}{n!} \leq \frac{S\left(n^{d(n)/2}\right)}{n^{d(n)/2}}$

—We will use the known relation $\prod_{k|n} k = n^{d(n)/2}$, where the product is extended over all divisors k of n. Since this product divides $\prod_{k\leq n} k = n!$, by Corollary 1 from [1] we can write

$\frac{S(n!)}{n!} \leq \frac{S(\prod_{k/n} k)}{\prod_{k/n} k}$, which gives the desired result.

Remark If n is of the form m^2, then $d(n)$ is odd, but otherwise $d(n)$ is even. So, in each case $n^{d(n)/2}$ is a positive integer.

11) For infinitely many n we have $S(n+1) < S(n)$, but for infinitely many m one has $S(m+1) > S(m)$.

—This is a simple application of 1). Indeed, let $n = p - 1$, where $p \geq 5$ is a prime. Then, by 1) we have $S(n) = S(p-1) \leq \frac{P-1}{2} < p$. Since $p = S(p)$, we have $S(p-1) < S(p)$. Let in the same manner $n = p + 1$. Then, as above, $S(p+1) \leq \frac{P+1}{2} < p = S(p)$.

12) Let p be a prime. Then $S(p!+1) > S(p!)$ and $S(p!-1) > S(p!)$

—Clearly, $S(p!) = p$. Let $p!+1 = \prod q_j^{\partial j}$ be the prime factorization of $p!+1$. Here each $q_j > p$, thus $S(p!+1) = S(q_j^{\partial j})$ (for certain j) $\geq S(p^{\partial j}) \geq S(p) = p$. The same proof applies to the case $p! - 1$.

Remark: This offers a new proof for M).

13) Let P_k be the kth prime number. Then $S(p_1p_2...P_k + 1) > S(p_1p_2\cdots P_k)$ and

-3-

$S(p_1p_2...P_k - 1) > S(p_1p_2\cdots P_k)$

—Almost the same proof as in 12) is valid, by remarking that $S(p_1p_2\cdots P_k) = P_k$ (since $p_1 < p_2 < \cdots < p_k$).

14) For infinitely many n one has $\left(S(n)^2\right) < S(n-1) \cdot S(n+1)$ and for infinitely many m, $\left(S(m)\right)^2 > S(m-1) \cdot S(m+1)$.

—By $S(p+1) < p$ and $S(p-1) < p$ (See the proof in 11) we have

$\frac{S(p+1)}{S(p)} < \frac{S(p)}{S(p)} < \frac{S(p)}{S(p-1)}$. Thus $\left(S(p)\right)^2 > S(p-1) \cdot S(p+1)$.

On the other hand, by putting $x_n = \frac{S(n+1)}{S(n)}$, we shall see in part II,

that $\limsup_{n\to\infty} x_n = +\infty$. Thus $x_{n-1} < x_n$ for infinitely many n, giving

$\left(S(n)\right)^2 < S(n-1) \cdot S(n+1)$.

II. Limits:

1) $\liminf_{n\to\infty} \frac{S(n)}{n} = 0$ and $\limsup_{n\to\infty} \frac{S(n)}{n} = 1$

—Clearly, $\frac{S(n)}{n} > 0$. Let $n = 2^m$. Then, since $S(2^m) \leq 2m$, and $\lim_{m\to\infty} \frac{2m}{2m} = 0$, we have $\lim_{m\to\infty} \frac{S(2^m)}{2^m} = 0$, proving the first part. On the other hand, it is well known that $\frac{S(n)}{n} \leq 1$. For $n = p_k$ (the kth prime), one has $\frac{S(p_k)}{p_k} = 1 \to 1$ as $k \to \infty$, proving the second part.

Remark: With the same proof, we can derive that $\liminf_{n\to\infty} \frac{S(n^r)}{n} = 0$ for all integers r.

—As above $S(2^{kr}) \leq 2kr$, and $\frac{2kr}{2^k} \to 0$ as $k \to \infty$ (r fixed), which gives the result.

2) $\liminf_{n\to\infty} \frac{S(n+1)}{S(n)} = 0$ and $\limsup_{n\to\infty} \frac{S(n+1)}{S(n)} = +\infty$

—Let p_r denote the rth prime. Since $(p_\Lambda ... p_r, 1) = 1$, Dirichlet's theorem on arithmetical progressions assures the existence of a prime p of the form $p = a \cdot p_\Lambda ... p_r - 1$.

Then $S(p+1) = S(ap_\Lambda \cdots p_r) \leq a \cdot S(p_\Lambda \cdots p_r)$ by $S(mn) \leq mS(n)$ (see [1])

But $S(p_\Lambda \cdots p_r) = max\left\{p_\Lambda, \cdots, p_r\right\} = p_r$. Thus $\frac{S(p+1)}{S(p)} \leq \frac{ap_r}{ap_\Lambda \cdots p_r - 1} \leq$

$\frac{p_r}{p_\Lambda \cdots p_r - 1} \to 0$ as $r \to \infty$. This gives the first part.

Let now p be a prime of the form $p = bp_\Lambda \cdots p_r + 1$.

Then $S(p-1) = S(bp_\Lambda \cdots p_r) \le b\, S(p_\Lambda \cdots p_r) = b \cdot p_r$,

and $\frac{S(p-1)}{S(p)} \le \frac{bp_r}{bp_1 \cdots p_r + 1} \le \frac{p_r}{p_\Lambda \cdots p_r} \to 0$ as $r \to \infty$.

3) $\liminf_{n\to\infty} \left[S(n+1) - S(n)\right] = -\infty$ and $\limsup_{m\to\infty} \left[S(n+1) - S(n)\right] = +\infty$

—We have $S(p+1) - S/p) \le \frac{p+1}{2} - p = \frac{-p+1}{2} \to -\infty$ for an odd prime

p (see 1) and 11)$\Big)$. On the other hand, $S(p) - S(p-1) \ge p - \frac{p-1}{2} = \frac{p+1}{2} \to \infty$

(Here $S(p) = p$), where $p - 1$ is odd for $p \ge 5$. This finishes the proof.

4) Let $\sigma(n)$ denotes the sum of divisors of n. Then $\liminf_{n\to\infty} \frac{S(\sigma(n))}{n} = 0$

—This follows by the argument of 2) for $n = p$. Then $\sigma(\varphi) = p + 1$ and $\frac{S(p+1)}{p} \to 0$, where

$\{p\}$ is the sequence constructed there.

5) Let $\varphi(n)$ be the Enter totient function. Then $\liminf_{n\to\infty} \frac{S(\varphi(n))}{n} = 0$

—Let the set of primes $\{p\}$ be defined as in 2). Since $\varphi(n) = p - 1$ and $\frac{S(p-1)}{p} = \frac{S(p-1)}{S(p)} \to 0$,

the assertion is proved. The same result could be obtained by taking $n = z^k$. Then, since

$\varphi(2^k) = 2^{k-1}$, and $\frac{S(2^{k-1})}{2^k} \le \frac{2\cdot(k-1)}{2^k} \to o$ as $k \to \infty$, the assertion follows:

6) $\liminf_{n\to\infty} \frac{S(S(n))}{n} = 0$ and $\max_{n\in\aleph} \frac{S(S(n))}{n} = 1$.

—Let $n = p!$ (p prime). Then, since $S(p!) = p$ and $S(p) = p$, from $\frac{p}{p!} \to 0\ (p \to \infty)$

we get the first result. Now, clearly $\frac{S(S(n))}{n} \le \frac{S(n)}{n} \le 1$. By letting $n = p$ (prime), clearly

one has $\frac{S(S(p))}{p} = 1$, which shows the second relation.

7) $\liminf_{n\to\infty} \frac{\sigma(S(n))}{S(n)} = 1$.

—Clearly, $\frac{\sigma(k)}{k} > 1$. On the other hand, for $n = p$ (prime), $\frac{\sigma\left(S(p)\right)}{S(p)} = \frac{p+1}{p} \to 1$ as $p \to \infty$.

8) Let Q(n) denote the greatest prime power divisor of n. Then $\liminf_{n\to\infty} \frac{\varphi\left(S(n)\right)}{\partial(n)} = 0$.

—Let $n = p_1^k \cdots p_r^k$ ($k > 1$, fixed). Then, clearly $\partial(n) = p_r^k$.

By $S(n) = S(p_r^k)$ $\left(\text{since } S(p_r^k) > S(p_i^k) \text{ for } i < k\right)$ and $S(p_r^k) = j \cdot p_r$, with $j \leq k$ (which is known) and by $\varphi\,(j\,p_k) \leq j \cdot \varphi(p_r) \leq k(p_r - 1)$, we get $\frac{\varphi\left(S(n)\right)}{\partial(n)} \leq \frac{k\cdot(p_r-1)}{p_r^k} \to 0$ as $r \to \infty$ (k fixed).

9) $\lim_{\substack{m\to\infty \\ m \text{ even}}} \frac{S(m^2)}{m^2} = 0$

—By 2) we have $\frac{S(m^2)}{m^2} \leq \frac{1}{m}$ for $m > 4$, even. This clearly inplies the above remark.

Remark. It is known that $\frac{S(m)}{m} \leq \frac{2}{3}$ if $m \neq 4$ is composite. From $\frac{S(m^2)}{m^2} \leq \frac{1}{m} < \frac{2}{3}$ for $m > 4$, for the composite numbers of the perfect squares we have a very strong improvement.

10) $\liminf_{n\to\infty} \frac{\sigma\left(S(n)\right)}{n} = 0$

—By $\sigma(n) = \sum_{d/n} d = n \sum_{d/n} \frac{1}{d} \leq n \sum_{d=1}^{n} \frac{1}{d} < n \cdot (2\ \log n)$, we get $\sigma(n) < 2n \log n$ for $n > 1$. Thus $\frac{\sigma\left(S(n)\right)}{n} < \frac{2\,S(n) \log S(n)}{n}$. For $n = 2^k$ we have $S(2^k) \leq 2k$, and since $\frac{4k \log 2k}{2^k} \to 0$ ($k \to \infty$), the result follows.

11) $\lim_{n\to\infty} \sqrt[n]{S(n)} = 1$

—This simple relation follows by $1 \leq S(n) \leq n$, so $1 \leq \sqrt[n]{S(n)} \leq \sqrt[n]{n}$; and by $\sqrt[n]{n} \to 1$ as $n \to \infty$. However, 11) is one of a (few) limits, which exists for the Smarandache function.

Finally, we shall prove that:

12) $\limsup_{n\to\infty} \frac{\sigma\left(n\,S(n)\right)}{n\,S(n)} = +\infty$.

—We will use the facts that $S(p!) = p$, $\frac{\sigma(p!)}{p!} = \sum_{d|p!} \frac{1}{d} \geq 1 + \frac{1}{2} + \cdots + \frac{1}{p} \to \infty$ as $p \to \infty$, and the inequality $\sigma(ab) \geq a\,\sigma(b)$ (see [2]).

Thus $\frac{\sigma(S(p!)p!}{p! \cdot S(p!)} \geq \frac{S(p!) \cdot \sigma(p!)}{p! \cdot p} = \frac{\sigma(p!)}{p!} \to \infty$. Thus, for the sequence $\{n\} = \{p!\}$, the results follows.

References

[1] J. Sándor. On certain inequalities involving the Smarandache function. Smarandache Notions J. F (1996), 3 - 6;

[2] J. Sándor. On the composition of some arithmetic functions. Studia Univ. Babes-Bolyai, 34 (1989), F - 14.