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Negation Switching Equivalence in Signed Graphs

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Abstract: A *Smarandachely k-signed graph* (*Smarandachely k-marked graph*) is an ordered pair $S = (G, \sigma)$ ($S = (G, \mu)$) where $G = (V, E)$ is a graph called *underlying graph of* S and $\sigma : E \rightarrow (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k)$ ($\mu : V \rightarrow (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k)$) is a function, where each $\overline{e}_i \in \{+, -\}$. Particularly, a Smarandachely 2-signed graph or Smarandachely 2-marked graph is called abbreviated a *signed graph* or a *marked graph*. In this paper, we establish a new graph equation $L^2(G) \cong L^k(G)$, where $L^2(G)$ & $L^k(G)$ are second iterated line graph and k^{th} iterated line graph respectively. Further, we characterize signed graphs S for which $L^2(S) \sim L^k(S)$ and $\eta(S) \sim L^k(S)$, where $\sim$ denotes switching equivalence and $L^2(S)$, $L^k(S)$ and $\eta(S)$ are denotes the second iterated line signed graph, k^{th} iterated line signed graph and negation of S respectively.

Key Words: Smarandachely k-signed graphs, Smarandachely k-marked graphs, signed graphs, marked graphs, balance, switching, line signed graphs, negation.

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§1. Introduction

Unless mentioned or defined otherwise, for all terminology and notion in graph theory the reader is refer to [8]. We consider only finite, simple graphs free from self-loops.

A *Smarandachely k-signed graph* (*Smarandachely k-marked graph*) is an ordered pair $S = (G, \sigma)$ ($S = (G, \mu)$) where $G = (V, E)$ is a graph called *underlying graph of* S and $\sigma : E \rightarrow (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k)$ ($\mu : V \rightarrow (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k)$) is a function, where each $\overline{e}_i \in \{+, -\}$. Particularly, a Smarandachely 2-signed graph or Smarandachely 2-marked graph is called abbreviated a *signed graph* or a *marked graph*. Cartwright and Harary [5] considered graphs in which vertices represent persons and the edges represent symmetric dyadic relations amongst persons each of which designated as being positive or negative according to whether the nature of the relationship is positive (friendly, like, etc.) or negative (hostile, dislike, etc.). Such a network

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S is called a *signed graph* (Chartrand [6]; Harary et al. [11]).

Signed graphs are much studied in literature because of their extensive use in modeling a variety socio-psychological process (e.g., see Katai and Iwai [14], Roberts [16] and Roberts and Xu [17]) and also because of their interesting connections with many classical mathematical systems (Zaslavsky [25]).

A cycle in a signed graph S is said to be *positive* if the product of signs of its edges is positive. A cycle which is not positive is said to be *negative*. A signed graph is then said to be *balanced* if every cycle in it is positive (Harary [9]). Harary and Kabell [12] developed a simple algorithm to detect balance in signed graphs as also enumerated them.

A *marking* of S is a function $\mu : V(G) \rightarrow \{+, -\}$; A signed graph S together with a marking μ is denoted by S_μ. Given a signed graph S one can easily define a marking μ of S as follows: For any vertex $v \in V(S)$,

$$\mu(v) = \prod_{uv \in E(S)} \sigma(uv),$$

the marking μ of S is called *canonical marking* of S.

The following characterization of balanced signed graphs is well known.

Theorem 1(E. Sampathkumar, [18]) *A signed graph $S = (G, \sigma)$ is balanced if, and only if, there exists a marking μ of its vertices such that each edge uv in S satisfies $\sigma(uv) = \mu(u)\mu(v)$.*

The idea of switching a signed graph was introduced in [1] in connection with structural analysis of social behavior and also its deeper mathematical aspects, significance and connections may be found in [25].

Switching S with respect to a marking μ is the operation of changing the sign of every edge of S to its opposite whenever its end vertices are of opposite signs. The signed graph obtained in this way is denoted by $\mathcal{S}_\mu(S)$ and is called *μ-switched signed graph* or just *switched signed graph*. Two signed graphs $S_1 = (G, \sigma)$ and $S_2 = (G', \sigma')$ are said to be *isomorphic*, written as $S_1 \cong S_2$ if there exists a graph isomorphism $f : G \rightarrow G'$ (that is a bijection $f : V(G) \rightarrow V(G')$ such that if uv is an edge in G then $f(u)f(v)$ is an edge in G') such that for any edge $e \in G$, $\sigma(e) = \sigma'(f(e))$. Further a signed graph $S_1 = (G, \sigma)$ *switches* to a signed graph $S_2 = (G', \sigma')$ (or that S_1 and S_2 are *switching equivalent*) written $S_1 \sim S_2$, whenever there exists a marking μ of S_1 such that $\mathcal{S}_\mu(S_1) \cong S_2$. Note that $S_1 \sim S_2$ implies that $G \cong G'$, since the definition of switching does not involve change of adjacencies in the underlying graphs of the respective signed graphs.

Two signed graphs $S_1 = (G, \sigma)$ and $S_2 = (G', \sigma')$ are said to be *weakly isomorphic* (see [23]) or *cycle isomorphic* (see [23]) if there exists an isomorphism $\phi : G \rightarrow G'$ such that the sign of every cycle Z in S_1 equals to the sign of $\phi(Z)$ in S_2. The following result is well known (See [24]).

Theorem 2(T. Zaslavsky, [24]) *Two signed graphs S_1 and S_2 with the same underlying graph are switching equivalent if, and only if, they are cycle isomorphic.*

§2. Negation switching equivalence in signed graphs

One of the important operations on signed graphs involves changing signs of their edges. From socio-psychological point of view, if a signed graph represents the structure of a social system in which vertices represent persons in a social group, edges represent their pair-wise (dyadic) interactions and sign on each edge represents the qualitative nature of interaction between the corresponding members in the dyad classified as being positive or negative then according to social balance theory, the social system is defined to be in a balanced state if every cycle in the signed graph contains an even number of negative edges [9]; otherwise, the social system is said to be in an unbalanced state. The term balance used here refers to the real-life situation in which the individuals in a social group experience a state of cognitive stability in the sense that there is no psychological tension amongst them that demands a change in the pattern of their ongoing in- terpersonal interactions. For instance, as pointed out by Heider [13], any situation in which a person is forced to maintain a positive relation simultaneously with two other persons who are in conflict with each other is an unbalanced state of the triad consisting of the three persons. Thus, when the social system is found to be in an unbalanced state it is desired to bring it into a balanced state by means of forcing some positive (negative) relationships change into negative (positive) relationships; such sets of edges in the corresponding signed graph model are called balancing sets (see Katai & Iwai [14]). Such a change (which may be regarded as a unary operation transforming the given signed graph) is called *negation*, which has other implications in social psychology too (see Acharya & Joshi [2]). Thus, formally, the negation $\eta(S)$ of S is a signed graph obtained from S by negating the sign of every edge of S; that is, by changing the sign of each edge to its opposite [10].

Behzad and Chartrand [4] introduced the notion of line signed graph $L(S)$ of a given signed graph S as follows: Given a signed graph $S = (G, \sigma)$ its *line signed graph* $L(S) = (L(G), \sigma')$ is the signed graph whose underlying graph is $L(G)$, the line graph of G, where for any edge e_ie_j in $L(S)$, $\sigma'(e_ie_j)$ is negative if, and only if, both e_i and e_j are adjacent negative edges in S. Another notion of line signed graph introduced in [7] is as follows: The *line signed graph* of a signed graph $S = (G, \sigma)$ is a signed graph $L(S) = (L(G), \sigma')$, where for any edge ee' in $L(S)$, $\sigma'(ee') = \sigma(e)\sigma(e')$. In this paper, we follow the notion of line signed graph defined by M. K. Gill [7] (See also E. Sampathkumar et al. [19,20]).

Theorem 3(M. Acharya, [3]) *For any signed graph $S = (G, \sigma)$, its line signed graph $L(S) = (L(G), \sigma')$ is balanced.*

Hence, we shall call a given signed graph S a *line signed graph* if it is isomorphic to the line signed graph $L(S')$ of some signed graph S'. In [20], the authors obtained a structural characterization of line signed graphs as well as line signed digraphs.

For any positive integer k, the k^{th} iterated line graph, $L^k(G)$ of G (k^{th} iterated line signed graph, $L^k(S)$ of S) is defined as follows:

$$L^0(G) = G,\ L^k(G) = L(L^{k-1}(G))\ (L^0(S) = S,\ L^k(S) = L(L^{k-1}(S)))$$

Corollary 4(P. Siva Kota Reddy & M. S. Subramanya, [22]) *For any signed graph $S = (G, \sigma)$*

and for any positive integer k, $L^k(S)$ is balanced.

The following result is well known.

Theorem 5(V. V. Menon, [15]) *For a graph G, $G \cong L^k(G)$ for any integers $k \geq 1$ if, and only if, G is 2-regular.*

Proposition 6(D. Sinha, [21]) *For a connected graph $G = (V, E)$, $L(G) \cong L^2(G)$ if, and only if, G is cycle or $K_{1,3}$.*

From the above results we have the following result for graphs.

Theorem 7 *For any graph G, $L^2(G) \cong L^k(G)$ for some $k \geq 3$, if, and only if, G is either a cycle or $K_{1,3}$.*

Proof Suppose that $L^2(G) \cong L^k(G)$ for some $k \geq 3$. We observe that $L^k(G) = L^{k-2}(L^2(G))$. Hence, by Proposition 6, $L^2(G)$ must be a cycle. But for any graph G, $L(G)$ is a cycle if, and only if, G is either cycle or $K_{1,3}$. Since $K_{1,3}$ is a forbidden to line graph and $L(G)$ is a line graph, $G \neq K_{1,3}$. Hence $L(G)$ must be a cycle. Finally $L(G)$ is a cycle if, and only if, G is either a cycle or $K_{1,3}$.

Conversely, if G is a cycle C_r , of length r, $r \geq 3$ then for any $k \geq 2$, $L^k(G)$ is a cycle and if $G = K_{1,3}$ then for any $k \geq 2$, $L^k(G) = C_3$. This implies, $L^2(G) = L^k(G)$ for any $k \geq 3$. This completes the proof. □

We now characterize those second iterated line signed graphs that are switching equivalent to their k^{th} iterated line signed graphs.

Proposition 8 *For any signed graph $S = (G, \sigma)$, $L^2(S) \sim L^k(S)$ if, and only if, G is either a cycle or $K_{1,3}$.*

Proof Suppose $L^2(S) \sim L^k(S)$. This implies, $L^2(G) \cong L^k(G)$ and hence by Theorem 7, we see that the graph G must be isomorphic to either a cycle or $K_{1,3}$.

Conversely, suppose that G is a cycle or $K_{1,3}$. Then $L^2(G) \cong L^k(G)$ by Theorem 7. Now, if S any signed graph on any of these graphs, By Corollary 4, $L^2(S)$ and $L^k(S)$ are balanced and hence, the result follows from Theorem 2. □

We now characterize those negation signed graphs that are switching equivalent to their line signed graphs.

Proposition 9 *For any signed graph $S = (G, \sigma)$, $\eta(S) \sim L^k(S)$ if, and only if, S is an unbalanced signed graph and G is 2-regular with odd length.*

Proof Suppose $\eta(S) \sim L^k(S)$. This implies, $G \cong L^k(G)$ and hence G is 2-regular. Now, if S is any signed graph with underlying graph as 2-regular, Corollary 4 implies that $L^k(S)$ is balanced. Now if S is an unbalanced signed graph with underlying graph $G = C_n$, where n is even, then clearly $\eta(S)$ is unbalanced. Next, if S is unbalanced signed graph with underlying graph $G = C_n$, where n is odd, then $\eta(S)$ is unbalanced. Hence, if $\eta(S)$ is unbalanced and its line signed graph $L^k(S)$ being balanced can not be switching equivalent to S in accordance with Theorem 2. Therefore, S must be unbalanced and G is 2-regular with odd length.

Conversely, suppose that S is an unbalanced signed graph and G is 2-regular with odd length. Then, since $L^k(S)$ is balanced as per Corollary 4 and since $G \cong L^k(G)$, the result follows from Theorem 2 again. □

Corollary 10 *For any signed graph $S = (G, \sigma)$, $\eta(S) \sim L(S)$ if, and only if, S is an unbalanced signed graph and G is 2-regular with odd length.*

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