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# Monomial geometric programming with fuzzy relation inequality constraints with max-product composition

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ABSTRACT

Monomials function has always been considered as a significant and most extensively used function in real living. Resource allocation, structure optimization and technology management can often apply these functions. In optimization problems the objective functions can be considered by monomials. In this paper, we present monomials geometric programming with fuzzy relation inequalities constraint with max-product composition. Simplification operations have been given to accelerate the resolution of the problem by removing the components having no effect on the solution process. Also, an algorithm and a few practical examples are presented to abbreviate and illustrate the steps of the problem resolution.



## 1. Introduction

Fuzzy relation equations (FRE), fuzzy relation inequalities (FRI), and their connected problems have been investigated by many researchers in both theoretical and applied areas (Czogala, Drewniak, & Pedrycz, 1982; Czogala & Predrycz, 1982; Di Nola, Pedrycz, & Sessa, 1984; Fang & Puthenpura, 1993; Guo, Wang, Di Nola, & Sessa, 1988; Gupta & Qi, 1991; Higashi & Klir, 1984; Han, Song, & Sekiguchi, 1995; Hu, 1998; Li & Fang, 1996; Perfilieva & Novák, 2007; Prevot, 1985; Sheue Shieh, 2007; Wang, 1984, 1991; Zadeh, 1965; Zener, 1971). Sanchez (1977) started the development of the theory and applications of FRE treated as a formalized model for non-precise concepts. Generally, FRE and FRI have a number of properties that make them suitable for formulizing the uncertain information upon which many applied concepts are usually based. The application of FRE and FRI can be seen in many areas, for instance, fuzzy control, fuzzy decision making, systems analysis, fuzzy modeling, fuzzy arithmetic, fuzzy symptom diagnosis and especially fuzzy medical diagnosis, and so on (see Adlassnig, 1986; Berrached et al., 2002; Czogala and Predrycz, 1982; Di Nola and Russo, 2007; Dubois and Prade, 1980; Loia and Sessa, 2005; Nobuhara et al., 2006; Pedrycz, 1981, 1985; Perfilieva and Novák, 2007; Vasantha Kandasamy & Smarandache, 2004, chap. 2; Wenstop, 1976; Zener, 1971).

An interesting extensively investigated kind of such problems is the optimization of the objective functions on the region whose set of feasible solutions have been defined as FRE or FRI constraints (Brouke & Fisher, 1998; Fang & Li, 1999; Fernandez & Gil, 2004; Guo & Xia, 2006; Guu & Wu, 2002; Higashi & Klir, 1984; Khorram & Hassanzadeh, 2008; Loetamonphong & Fang, 2001; Loetamonphong et al., 2002; Zadeh, 2005). Fang and Li solved the linear optimization problem with respect to FRE constraints by considering the max–min composition (Fang & Li, 1999). The max–min composition is commonly used when a system requires conservative solutions, in the sense that the goodness of one value cannot compensate for the badness of another value (Loetamonphong & Fang, 2001). Recent results in the literature, however, show that the min operator is not always the best choice for the intersection operation. Instead, the max-product composition has provided results better than or equivalent to the max–min composition in some applications (Adlassnig, 1986).

The fundamental result for fuzzy relation equations with max-product composition goes back to Pedrycz (1985). A recent study in this regard can be found in Brouke and Fisher (1998). They extended the study of an inverse solution of a system of fuzzy relation equations with max-product composition. They provided theoretical results for determining the complete sets of solutions as well as the conditions for the existence of resolutions. Their results showed that such complete sets of solutions can be characterized by one maximum solution and a number of minimal solutions. An optimization problem was studied by Loetamonfong and Fang with max-product composition (Loetamonphong & Fang, 2001) which was improved by Guu and Wu by shrinking the search region (Guu & Wu, 2002). The linear objective optimization problem with FRI was investigated by Zhang, Dong, and Ren (2003), where the fuzzy operator is considered as the max–min composition.

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Also, Guo and Xia presented an algorithm to accelerate the resolution of this problem (Guo & Xia, 2006).

The geometric programming (GP) theory proposed in 1961 by Zeneret al. for first time (Duffin, Peterson, & Zener, 1967; Peterson, 1976). Business administration, economic analysis, resource allocation and environmental engineering have a large number of applications in GP (Zener, 1971). The fuzzy geometric programming problem proposed by Cao (2001). He considered a few number of power systems problems (Cao, 1999) and also Liu applied it in economic management (Liu, 2004). The fuzzy geometric programming with multi-objective functions has studied by Biswal (1992) and Verma (1990). In order to show importance of geometric programming and the fuzzy relation equation in theory and applications a fuzzy relation geometric programming problem has proposed by Yang and Cao (2005a, 2005b). Furthermore, they discussed optimal solutions with two kinds of objective functions based on the fuzzy max-product operator. Also, they consider monomial geometric programming with fuzzy relation equation (FRI) constraints with max–min composition (Yang & Cao, 2007).

In this paper, we consider the monomial geometric programming of the FRI with the max-product operator. This problem can be formulated as follows:

$$\begin{aligned} &\min\ c\prod_{j=1}^{n} x_j^{\alpha_j} \\ &\text{s.t.}\quad A\cdot x \geqslant d^1 \\ &\qquad B\cdot x \leqslant d^2 \\ &\qquad x\in[0,1]^n \end{aligned} \tag{1}$$

where $c, \alpha_j \in R, c > 0, A = (a_{ij})_{m\times n}, a_{ij} \in [0,1], B = (b_{ij})_{l\times n}, b_{ij} \in [0,1]$, are fuzzy matrices, $d^1 = (d_i^1)_{m\times 1} \in [0,1]^m, d^2 = (d_i^2)_{l\times 1} \in [0,1]^l$ are fuzzy vectors, $x = (x_j)_{n\times 1} \in [0,1]^n 0$ is an unknown vector, and "$\cdot$" denotes the fuzzy max-product operator as defined below. Problem (1) can be rewritten as the following problem in detail:

$$\begin{aligned} &\min\ c\prod_{j=1}^{n} x_j^{\alpha_j} \\ &\text{s.t.}\quad a_i\cdot x \geqslant d_i^1 \quad i\in I^1=\{1,2,\ldots,m\} \\ &\qquad b_i\cdot x \leqslant d_i^2 \quad i\in I^2=\{1,2,\ldots,l\} \\ &\qquad 0\leqslant x_j\leqslant 1 \quad j\in J=\{1,2,\ldots,n\} \end{aligned} \tag{2}$$

where $a_i$ and $b_i$ are the $i$th row of the matrices $A$ and $B$, respectively, and the constraints are expressed by the max-product operator definition as:

$$\begin{aligned} a_i\cdot x &= \max_{j\in J}\{a_{ij}\cdot x_j\} \geqslant d_i^1 \quad \forall i\in I^1 \\ b_i\cdot x &= \max_{j\in J}\{b_{ij}\cdot x_j\} \leqslant d_i^2 \quad \forall i\in I^2 \end{aligned} \tag{3}$$

In Section 2, the set of the feasible solutions of Problem (2) and its properties are studied. A necessary condition and a sufficient condition are given to realize the feasibility of Problem (2). In Section 3, some simplification operations are presented to accelerate the resolution process. Then, in Section 4 an algorithm is introduced to solve the problem by using the results of the previous sections, and a numerical example is also given to illustrate the algorithm in this section. Finally, the conclusion is stated in Section 5.

## 2. The characteristics of the set of feasible solution

**Notation.** We shall use, during the paper, these notations as follows:

$$\begin{aligned} &S(A,d^1)_i = \left\{x\in[0,1]^n : a_i\cdot x \geqslant d_i^1\right\}\text{for each } i\in I^1 \\ &S(B,d^2)_i = \left\{x\in[0,1]^n : b_i\cdot x \leqslant d_i^2\right\}\text{for each } i\in I^2 \\ &S(A,d^1) = \bigcap_{i\in I^1} S(A,d^1)_i = \{x\in[0,1]^n : A\cdot x\geqslant d^1\} \\ &S(B,d^2) = \bigcap_{i\in I^2} S(B,d^2)_i = \{x\in[0,1]^n : B\cdot x\leqslant d^2\} \\ &S(A,B,d^1,d^2) = S(A,d^1)\cap S(B,d^2) = \{x\in[0,1]^n : A\cdot x\geqslant d^1, B\cdot x\leqslant d^2\}. \end{aligned}$$

**Corollary 1.** *$x\in S(A,d^1)_i$ for each $i\in I^1$ if and only if there exists some $j_i\in J$ such that $x_{j_i}\geqslant \frac{d_i^1}{a_{ij_i}}$; similarly, $x\in S(B,d^2)_i$ for each $i\in I^2$ if and only if $x_j\leqslant \frac{d_i^2}{b_{ij}}$, $\forall j\in J$.*

**Proof.** This clearly results from relations (3). □

**Lemma 1.**

(a) *$S(A,d^1)\neq\phi$ if and only if for each $i\in I^1$ there exists some $j_i\in J$ such that $a_{ij_i}\geqslant d_i^1$.*
(b) *If $S(A,d^1)\neq\phi$ then $\bar{1}=[1,1,\ldots,1]^t_{1\times n}$ is the greatest element in set $S(A,d^1)$.*

**Proof.**

(a) Suppose $S(A,d^1)\neq\phi$ and $x\in S(A,d^1)$. Thus, $x\in S(A,d^1)_i, \forall i\in I^1$ and then for each $i\in I^1$ we have $x_{j_i}\geqslant\frac{d_i^1}{a_{ij_i}}$ for some $j_i\in J$ from Corollary (1). Therefore, since $x\in S(A,d^1)$ then $x\in[0,1]^n$ and hence $\frac{d_i^1}{a_{ij_i}}\leqslant 1, \forall i\in I^1$, which implies that there is a $j_i\in J$ such that $a_{ij_i}\geqslant d_i^1, \forall i\in I^1$. Conversely, suppose that there exists some $j_i\in J$ such that $a_{ij_i}\geqslant d_i^1$, $\forall i\in I^1$. Set $x=\bar{1}=[1,1,\ldots,1]^t_{1\times n}$, since $x\in[0,1]^n$ and $x_{j_i}=1\geqslant\frac{d_i^1}{a_{ij_i}}$, $\forall i\in I^1$ then $x\in S(A,d^1)_i$, $\forall i\in I^1$, from Corollary (1), and thus $x\in S(A,d^1)$.
(b) Proof is attained from part (a) and Corollary (1). □

**Lemma 2.**

(a) *$S(B,d^2)\neq\phi$.*
(b) *The smallest element in set $S(B,d^2)$ is $\bar{0}=[0,0,\ldots,0]^t_{1\times n}$.*

**Proof.** Set $x=\bar{0}=[0,0,\ldots,0]^t_{1\times n}$. Since $d_i^2\geqslant 0$ and $b_{ij}\geqslant 0$ (in case $b_i=0$ the problem is always well-defined and it is clear), then $\frac{d_i^2}{b_{ij_i}}\geqslant 0$. Therefore $x_j\leqslant\frac{d_i^2}{b_{ij_i}}, \forall i\in I^2, j\in J$, then Corollary (1) implies that $x\in S(B,d^2)$ and hence parts (a) and (b) are proved. □

**Theorem 1** (*Necessary condition*). *If $S(A,B,d^1,d^2)\neq\phi$, then for each $i\in I^1$ there exist $j\in J$ such that $a_{ij}\geqslant d_i^1$*

**Proof.** Suppose that $S(A,B,d^1,d^2)\neq\phi$, then since $S(A,B,d^1,d^2)=S(A,d^1)\cap S(B,d^2)$, therefore $S(A,d^1)\neq\phi$. Now, the theorem is proved by using part (a) of Lemma (1). □

**Definition 1.** Set $\bar{x}=(\bar{x}_j)_{n\times 1}$ where

$$\bar{x}_j=\begin{cases} 1 & \forall i: b_{ij}\leqslant d_i^2 \\ \min\limits_{i=1,\ldots,l}\left\{\frac{d_i^2}{b_{ij}} : b_{ij} > d_i^2\right\} & \text{otherwise} \end{cases}$$

**Lemma 3.** *If $S(B,d^2)\neq\phi$ then $\bar{x}$ is the greatest element in set $S(B,d^2)$.*

**Proof.** See [19, p. 348]. □

**Corollary 2.** *$S(B,d^2)=\{x\in[0,1]^n : B\cdot x\leqslant d^2\}=[\bar{0},\bar{x}]$, in which $\bar{x}$ and $\bar{0}$ are as defined in Definition (1) and Lemma (2), respectively.*

**Proof.** Since $S(B,d^2)\neq\phi$, then $\bar{0}$ and $\bar{x}$ are the single smallest element and greatest element, respectively, from Lemmas (2) and (3). Let $x\in[\bar{0},\bar{x}]$, then $x\in[0,1]^n$ and $x\leqslant\bar{x}$, Thus, $b_i\cdot x\leqslant b_i\cdot\bar{x}\leqslant d_i^2$, $\forall i\in I^2$ which implies $x\in S(B,d^2)$. Conversely, let $x\in S(B,d^2)$ from part (b) of Lemma (2), $\bar{0}\leqslant x$ and also $x\in S(B,d^2)_i$, $\forall i\in I^2$. Then, Corollary (1) requires $x_j\leqslant\frac{d_i^2}{b_{ij}}$, $\forall i\in I^2$ and $\forall j\in J$. Hence, $x_j\leqslant\bar{x}_j$, $\forall j\in J$, which means $x\leqslant\bar{x}$. Therefore $x\in[\bar{0},\bar{x}]$. □

**Definition 2.** Let $J_i=\{j\in J : a_{ij}\geqslant d_i^1\}$, $\forall i\in I^1$. For each $j\in J_i$, we define $i_{x(j)}=(i_{x(j)_k})_{n\times 1}$ such that

$$i_{x(j)_k}=\begin{cases}\frac{d_i^1}{a_{ij}} & k=j\\ 0 & k\neq j\end{cases}$$

**Lemma 4.** *Consider a fixed $i\in I^1$.*

(a) *If $d_i^1\neq 0$ then the vectors $i_{x(j)}$ are the only minimal elements of $S(A,d^1)_i$ for each $j\in J_i$.*
(b) *If $d_i^1=0$ then $\bar{0}$ is the smallest element in $S(A,d^1)_i$.*

**Proof.**

(a) Suppose $j\in J_i$ and $i\in I^1$. Since $i_{x(j)_j}=\frac{d_i^1}{a_{ij}}$, then $i_{x(j)}\in S(A,d^1)_i$, from Corollary (1). By contradiction, suppose $x\in S(A,d^1)_i$ and $x<i_{x(j)}$. Hence we must have $x_j<\frac{d_i^1}{a_{ij}}$ and $x_k=0$ for $k\in J$ and $k\neq j$. Then $x_j<\frac{d_i^1}{a_{ij}}$, $\forall j\in J$, and hence $x\notin S(A,d^1)_i$ from Corollary (1), which is a contradiction.
(b) It is clear from Corollary (1) and the fact that$x_j\geqslant 0,\forall j\in J$. □

**Corollary 3.** *If $S(A,d^1)_i\neq\phi$, then $S(A,d^1)_i=\{x\in[0,1]^n : a_i\cdot x\geqslant d_i^1\}=\cup_{j\in J_i}[i_x(j),\bar{1}]$, where $i\in I^1$ and $i_{x(j)}$ is as defined in Definition (2).*

**Proof.** If $S(A,d^1)_i\neq\phi$ then from Lemmas (1) and (4), vector $\bar{1}$ is the maximum solution and the vectors $i_{x(j)}$, $\forall j\in J_i$ are the minimal solutions in $S(A,d^1)_i$. Let $x\in\underset{j\in J_i}{\cup}[i_{x(j)},\bar{1}]$. Then $x\in[i_{x(j)},\bar{1}]$ for some $j\in J_i$ and thus $x\in[0,1]^n$ and $x_j\geqslant i_{x(j)_j}=\frac{d_i^1}{a_{ij}}$ from Definition (2), hence, $x\in S(A,d^1)_i$ from Corollary (1). Conversely, let $x\in S(A,d^1)_i$. Then there exits some $j'\in J$ such that $x_{j'}\geqslant\frac{d_i^1}{a_{ij'}}$ from Corollary (1). Since $x\in[0,1]^n$, then $\frac{d_i^1}{a_{ij'}}\leqslant 1$, and thus $j'\in J_i$. Therefore, $i_{x(j')}\leqslant x\leqslant\bar{1}$ which implies $x\in\underset{j\in J_i}{\cup}[i_{x(j)},\bar{1}]$. □

**Definition 3.** Let $e=(e(1),e(2),\dots e(m))\in J_1\times J_2\times\cdots\times J_m$ such that $e(i)=j\in J_i$. We define $x(e)=(x(e)_j)_{n\times 1}$, in which $x(e)_j=\max_{i\in I_j^e}\{i_{x(e(i))_j}\}=\max_{i\in I_j^e}\left\{\frac{d_i^1}{a_{ij}}\right\}$ if $I_j^e\neq\phi$ and $x(e)_j=0$ if $I_j^e=\phi$, where $I_j^e=\{i\in I^1 : e(i)=j\}$.

**Corollary 4.**

(a) *If $d_i^1=0$ for some $i\in I^1$, then we can remove the ith row of matrix $A$ with no effect on the calculation of the vectors $x(e)$ for each $e\in J_I=J_1\times J_2\times\cdots\times J_m$.*
(b) *If $j\notin J_i$, $\forall i\in I^1$, then we can remove the jth column of matrix $A$ before calculating the vectors $x(e)$, $\forall e\in J_I$, and set $x(e)_j=0$ for each $e\in J_I$.*

**Proof.**

(a) It is proved from Definition (3) and part (b) of Lemma (4), because we will get the minimal elements of $S(A,d^1)$.
(b) It is proved by only using Definition (3). □

**Lemma 5.** *Suppose $S(A,d^1)\neq\phi$ then $S(A,d^1)=\underset{X(e)}{\cup}[x(e),\bar{1}]$ where $X(e)=\{x(e) : e\in J_I\}$.*

**Proof.** If $S(A,d^1)\neq\phi$, then $S(A,d^1)_i\neq\phi$, $\forall i\in I^1$. Therefore, we have

$$\begin{aligned}S(A,d^1)&=\underset{i\in I^1}{\cap}S(A,d^1)_i=\underset{i\in I^1}{\cap}[\underset{j\in J_i}{\cup}[i_{x(j)},\bar{1}]]=\underset{i\in I^1}{\cap}[\underset{e(i)\in J_i}{\cup}[i_{x(e(i))},\bar{1}]]\\&=\underset{e\in J_I}{\cup}[\underset{i\in I^1}{\cap}[i_{x(e(i))},\bar{1}]]=\underset{e\in J_I}{\cup}[\max_{i\in I^1}\{i_{x(e(i))}\},\bar{1}]=\underset{e\in J_I}{\cup}[x(e),\bar{1}]\\&=\underset{X(e)}{\cup}[x(e),\bar{1}]\end{aligned}$$

from Corollary (3) and Definition (3).

From Lemma (5), it is obvious that $S(A,d^1)=\underset{X_0(e)}{\cup}[x(e),\bar{1}]$ and $X_0(e)=S_0(A,d^1)$, where $X_0(e)$ and $S_0(A,d^1)$ are the set of minimal solutions in $X(e)$ and $S(A,d^1)$, respectively. □

**Theorem 2.** *If $S(A,B,d^1,d^2)\neq\phi$, then $S(A,B,d^1,d^2)=\underset{X_0(e)}{\cup}[x(e),\bar{x}]$.*

**Proof.** By using Corollary (2) and the result of (5), we have

$$\begin{aligned}S(A,B,d^1,d^2)&=S(A,d^1)\cap S(B,d^2)=\left\{\underset{X_0(e)}{\cup}[x(e),\bar{1}]\right\}\cap[\bar{0},\bar{x}]\\&=\underset{X_0(e)}{\cup}[x(e),\bar{x}]\end{aligned}$$

and the proof is complete. □

**Corollary 5** (*Necessary and sufficient condition*). *$S(A,B,d^1,d^2)\neq\phi$ if and only if $\bar{x}\in S(A,d^1)$. Equivalently, $S(A,B,d^1,d^2)\neq\phi$ if and only if there exists some $e\in J_I$ such that $x(e)\leqslant\bar{x}$.*

**Proof.** Suppose that $S(A,B,d^1,d^2)\neq\phi$, then $S(A,B,d^1,d^2)=\underset{X_0(e)}{\cup}[x(e),\bar{x}]$ by Theorem (2), thus $\bar{x}\in S(A,B,d^1,d^2)$, and hence $\bar{x}\in S(A,d^1)$. Conversely let $\bar{x}\in S(A,d^1)$. Meanwhile we know $\bar{x}\in S(B,d^2)$, therefore $\bar{x}\in S(A,d^1)\cap S(B,d^2)=S(A,B,d^1,d^2)$. □

## 3. Simplification operations and the resolution algorithm

In order to solve Problem (1), we first convert it into the two sub-problems below:

$$\begin{aligned}\min\quad & c\prod_{j\in R^+}x_j^{\alpha_j}\\ \text{s.t.}\quad & A\cdot x=b\\ & x\in[0,1]^n\end{aligned}\tag{4a}$$

$$\begin{aligned}\min\quad & c\prod_{j\in R^-}x_j^{\alpha_j}\\ \text{s.t.}\quad & A\cdot x=b\\ & x\in[0,1]^n\end{aligned}\tag{4b}$$

where $R^+=\{j|\alpha_j\geqslant 0,j\in J\}$ and $R^-=\{j|\alpha_j<0,j\in J\}$.

**Lemma 6.** *The optimal solution of Problem (4b) is $\bar{x}$.*

**Proof.** In objective function (4b) $\alpha_j<0$ therefore, $x_j^{\alpha_j}$ is a monotone decreasing function of $x_j$ in the interval $0\leqslant x_j\leqslant 1$ for each $j\in R^-$. As a result $\prod_{j\in R^-}x_j^{\alpha_j}$ is too. Hence, $\bar{x}$ is the optimal solution because $\bar{x}$ is the greatest element in set $S(A,B,d^1,d^2)$. □

**Lemma 7.** *The optimal solution of Problem (4a) belongs to $X_0(e)$.*

**Proof.** In objective function (4a), $\alpha_j\geqslant 0$; therefore, $x_j^{\alpha_j}$ is a monotone increasing function of $x_j$ in the interval $0\leqslant x_j\leqslant 1$ for each $j\in R^+$. As a result $\prod_{j\in R^+}x_j^{\alpha_j}$ is too. Now, suppose that $y\in S(A,B,d^1,d^2)$ is selected arbitrarily then there exists $x(e_0)\in X_0(e)$ such that $y\geqslant x(e_0)$. Since $\prod_{j\in R^+}x_j^{\alpha_j}$ is a monotone increasing function of $x_j$, then $\prod_{j\in R^+}y_j^{\alpha_j}\geqslant\prod_{j\in R^+}x(e_0)_j^{\alpha_j}$; therefore, one of the elements of $X_0(e)$ is the optimal solution of Problem (4a). □

**Theorem 3.** *Assume that $x(e_0)$ is an optimal solution (not necessary unique) of Problem (4a), then the optimal solution of Problem (1) is $x^*$ defined as follows:*

$$x_j^* = \begin{cases} \bar{x}_j & j \in R^- \\ x(e_0)_j & j \in R^+ \end{cases}$$

**Proof.** Consider $S(A, B, d^1, d^2)$, then by Lemmas (6) and (7) we have

$$\prod_{j \in J} x_j^{\alpha_j} = \prod_{k \in R^+, l \in R^-} x_k^{\alpha_k} \cdot x_l^{\alpha_l} \geqslant \prod_{k \in R^+, l \in R^-} x(e_0)_k^{\alpha_k} \cdot \bar{x}_l^{\alpha_l} = \prod_{j \in J} x_j^{*\alpha_j}$$

Therefore, $x^*$ is the optimal solution of Problem (1) and the proof is completed.

For calculating $x^*$ it is sufficient to find $\bar{x}$ and $x(e_0)$ from Theorem (3). While $\bar{x}$ is easily attained by Definition (1), $x(e_0)$ is usually hard to find. Since $X_0(e)$ is attained by pairwise comparison between the members of set $X(e)$, then the finding process of set $X_0(e)$ is time-consuming if $X(e)$ has many members. Therefore, a simplification operation can accelerate the resolution of Problem (4a) by removing the vectors $e \in J_I$ such that $x(e)$ is not optimal in (4a). One of such operations is given by Corollary (4). Other operations are attained by the theorems below. □

**Theorem 4.** *The set of feasible solutions for Problem (1), namely $S(A, B, d^1, d^2)$, is non-empty if and only if for each $i \in I^1$ set $\bar{J}_i = \left\{ j \in J_i : \frac{d_i^1}{a_{ij}} \leqslant \bar{x}_j \right\}$ is non-empty, where $\bar{x}$ is defined by Definition (1).*

**Proof.** Suppose $S(A, B, d^1, d^2) \neq \phi$. From Corollary (5), $\bar{x} \in S(A, B, d^1, d^2)$ and then we have $\bar{x} \in S(A, d^1)_i$, $\forall i \in I^1$. Thus, for each $i \in I^1$ there exists some $j \in J$ such that $\bar{x}_j \geqslant \frac{d_i^1}{a_{ij}}$ from Corollary (1), which means $\bar{J}_i \neq \phi$, $\forall i \in I^1$. Conversely, suppose $\bar{J}_i \neq \phi$, $\forall i \in I^1$. Then there exists some $j \in J$ such that $\bar{x}_j \geqslant \frac{d_i^1}{a_{ij}}$, $\forall i \in I^1$. Hence, $\bar{x} \in S(A, d^1)_i$, $\forall i \in I^1$ from Corollary (1), which implies $\bar{x} \in S(A, d^1)$. These facts together with Lemma (3) imply $\bar{x} \in S(A, B, d^1, d^2)$, and therefore $S(A, B, d^1, d^2) \neq \phi$. □

**Theorem 5.** *If $S(A, B, d^1, d^2) \neq \phi$, then*
$S(A, B, d^1, d^2) = \underset{\bar{x}(e)}{\cup} [x(e), \bar{x}]$ *where*
$\bar{x}(e) = \{x(e) : e \in \bar{J}_I = \bar{J}_1 \times \bar{J}_2 \times \cdots \times \bar{J}_m\}$.

**Proof.** By considering Theorem (2), it is sufficient to show $x(e) \notin S(A, B, d^1, d^2)$ if $e \notin \bar{J}_I$. Suppose $e \notin \bar{J}_I$. Thus, there exist $i' \in I^1$ and $j' \in J_{i'}$ such that $e(i') = j'$ and $\bar{x}_{j'} < \frac{d_{i'}^1}{a_{i'j'}}$. Then $i' \in I_{j'}^e$ and by Definition (3) we have $x(e)_{j'} = \max_{i \in I_{j'}^e} \left\{ \frac{d_i^1}{a_{ij'}} \right\} \geqslant \frac{d_{i'}^1}{a_{i'j'}} > \bar{x}_{j'}$. Therefore, $x(e) \leqslant \bar{x}$ is not correct, which implies $x(e) \notin S(A, B, d^1, d^2)$ by Theorem (2).

From the defined notation of Theorem (4), $\bar{J}_i \subseteq J_i$, $\forall i \in I^1$, which requires $\overline{X}(e) \subseteq X(e)$. Also, $S_0(A, B, d^1, d^2) \subseteq \overline{X}(e)$ by Theorem (4), in which $S_0(A, B, d^1, d^2)$ is the set of the minimal elements of $S(A, B, d^1, d^2)$, thus Theorem (5) reduces the search region to find set $S_0(A, B, d^1, d^2)$. □

**Definition 4.** Let $j_1, j_2 \in J$, $\alpha_{j_1} > 0$ and $\alpha_{j_2} > 0$. $j_2$ is said to dominate $j_1$ if and only if

(a) $j_1 \in \bar{J}_i$ implies $j_2 \in \bar{J}_i$, $\forall i \in I^1$.
(b) For each $i \in I^1$ we have $\left(\frac{d_i^1}{a_{ij_1}}\right)^{\alpha_{j_1}} \geqslant \left(\frac{d_i^1}{a_{ij_2}}\right)^{\alpha_{j_2}}$, such that $j_1 \in \bar{J}_i$.

**Theorem 6.** *Suppose that $j_2$ dominates $j_1$ for $j_1, j_2 \in J$, then, the minimum value of the objective function is zero.*

**Proof.** Suppose $x(e_0)$ is the optimal solution in (4a), define $e' = (e'(i))_{m \times 1}$ such that

$$e'(i) = \begin{cases} e_0(i) & i \notin I_{j_1}^{e_0} \\ j_2 & i \in I_{j_1}^{e_0}. \end{cases}$$

It is obvious that $I_{j_1}^{e'} = \phi$ and then $x(e')_{j_1} = 0$; also, it is feasible. Since $\prod_{j \in J} x_j^{\alpha_j} \geqslant 0$ for each $x \in S(A, B, d^1, d^2)$, and $\prod_{j \in J} x(e')_j^{\alpha_j} = 0$, therefore $x(e')$ is an optimal solution and the minimum value of the objective function is zero. □

## 4. Algorithm for finding an optimal solution and examples

**Definition 5.** Consider Problem (1). We call $\overline{A} = (\bar{a}_{ij})_{m \times n}$ and $\overline{B} = (\bar{b}_{ij})_{l \times n}$ the characteristic matrices of matrix $A$ and matrix $B$, respectively, where $\bar{a}_{ij} = \frac{d_i^1}{a_{ij}}$ for each $i \in I^1$ and $j \in J$, also $\bar{b}_{ij} = \frac{d_i^2}{b_{ij}}$ for each $i \in I^2$ and $j \in J$. (Set $\frac{0}{0} = 1$ and $\frac{k}{0} = \infty$).

**Algorithm.** Given Problem (2),

(1) Find matrices $\overline{A}$ and $\overline{B}$ by Definition (5).
(2) If there exists $i \in I^1$ such that $\bar{a}_{ij} > 1$, $\forall j \in J$, then stop. Problem (2) is infeasible (see Theorem (1)).
(3) Calculate $\bar{x}$ from $\overline{B}$ by Definition (1).
(4) If there exists $i \in I^1$ such that $d_i^1 = 0$, then remove the $i$'th row of matrix $\overline{A}$ (see part (a) of Corollary (4)).
(5) If $\bar{a}_{ij} > \bar{x}_j$, then set $\bar{a}_{ij} = 0$, $\forall i \in I^1$ and $\forall j \in J$.
(6) If there exists $i \in I^1$ such that $\bar{a}_{ij} = 0$, $\forall j \in J$, then stop. Problem (2) is infeasible (see Theorems (4) and (5))
(7) If there exists $j' \in J$ such that $\bar{a}_{ij'} = 0$, $\forall i \in I^1$, then remove the $j$th column of matrix $\overline{A}$ (see part (b) of Corollary (4)) and set $x(e_0)_{j'} = 0$. If $j' \in R^+$ then $\forall e \in J_I$, $x(e)$ is the optimal solution of (4a) and the minimum value of the objective function is zero. Then stop.
(8) If $j_2$ dominates $j_1$, $(j_1, j_2 \in R^+)$ then remove column $j_1$ from $\overline{A}$, $\forall j_1, j_2 \in J$ (see Theorem (6)), and set $x(e_0)_{j_1} = 0$ then $\forall e \in J_I$, $x(e)$ is the optimal solution of (4a) and the minimum value of the objective function is zero. Then, stop.
(9) Let $J_i^{new} = \{j \in \bar{J}_i : \bar{a}_{ij} \neq 0\}$ and $J_I^{new} = J_1^{new} \times J_2^{new} \times \cdots \times J_m^{new}$. Find the vectors $x(e)$, $\forall e \in J_I^{new}$, by Definition (3) from $\overline{A}$, and $x(e_0)$ by pairwise comparison between the vectors $x(e)$.
(10) Find $x^*$ from Theorem (3).

**Example 1.** Consider the problem below:

$$\min Z = (x_1)^{-2} (x_2) (x_3)^{\frac{1}{2}} (x_4)^{-\frac{1}{2}}$$

$$\begin{bmatrix} 0.5 & 0.8 & 0.35 & 0.25 \\ 0.9 & 0.92 & 0.9 & 1 \\ 0.2 & 1 & 0.45 & 0.4 \\ 0.55 & 0.6 & 0.8 & 0.64 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \geqslant \begin{bmatrix} 0.4 \\ 0.9 \\ 0.8 \\ 0.65 \end{bmatrix}$$

$$\begin{bmatrix} 0.6 & 0.5 & 0.1 & 0.1 \\ 0.2 & 0.6 & 0.6 & 0.5 \\ 0.5 & 0.9 & 0.8 & 0.4 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \leqslant \begin{bmatrix} 0.48 \\ 0.56 \\ 0.72 \end{bmatrix}$$

$$0 \leqslant x_j \leqslant 1, \quad j = 1, 2, 3, 4$$

*Step1:* Matrices $\overline{A}$, $\overline{B}$ are as follows:

$$\overline{A} = \begin{bmatrix} 0.8 & 0.5 & 1.14 & 1.6 \\ 1 & 0.97 & 1 & 0.9 \\ 4 & 0.8 & 1.77 & 2 \\ 1.18 & 1.08 & 0.81 & 1.01 \end{bmatrix}$$

$$\overline{B} = \begin{bmatrix} 0.8 & 0.96 & 4.8 & 4.8 \\ 2.8 & 0.93 & 0.93 & 1.12 \\ 1.44 & 0.8 & 0.9 & 1.8 \end{bmatrix}$$

*Step 2–3:* Vector $\bar{x}$ is as follows:

$\bar{x} = [0.8, 0.8, 0.9, 1]$

*Step 4–5:* By this step, matrix $\overline{A}$ is converted to the following:

$$\overline{A} = \begin{bmatrix} 0.8 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0.9 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.81 & 0 \end{bmatrix}$$

*Step 6–9:* $J_1^{new} = \{1,2\}, J_2^{new} = \{4\}, J_3^{new} = \{2\}$ and $J_4^{new} = \{3\}$Hence, $X_0(e) = \{(0 \quad 0.8 \quad 0.81 \quad 0.9)\}$ and $x(e_0)$, the optimal solution of Problem (4a), is clearly $(0 \quad 0.8 \quad 0.81 \quad 0.9)$.

*Step 10:* The optimal solution of the problem is $x^* = (0.8 \quad 0.8 \quad 0.81 \quad 1)$ and the minimum value of the objective function is $Z = (0.8)^{-2}(0.8)(0.81)^{\frac{1}{2}}(1)^{-\frac{1}{2}} = 1.125$.

**Example 2.** To show high efficiency of the presented algorithm in this example, we consider the problem with fuzzy relation equations (FRI) constraints as follows (see Example 5.1 in Yang & Cao, 2007 that has been solved with max–min composition):

$$\min \quad Z = x_1x_2x_3x_4x_5$$

s.t.

$$A \cdot x = \begin{bmatrix} 0.3391 & 0.4757 & 0.4403 & 0.5857 & 0.4329 \\ 0.3682 & 0.3823 & 0.6001 & 0.4295 & 0.1488 \\ 0.6702 & 0.9954 & 0.6981 & 0.4027 & 0.8493 \\ 0.8195 & 0.6934 & 0.3742 & 0.0096 & 0.7798 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0.5456 \\ 0.5244 \\ 0.8987 \\ 0.7544 \end{bmatrix} = b$$

The above constraint $A \cdot x = b$ is equivalent with $A \cdot x \leqslant b$ and $A \cdot x \geqslant b$, therefore for this problem $A = B$, and then we can use the above algorithm.

*Step 1:*

$$\overline{A} = \overline{B} = \begin{bmatrix} 1.60 & 1.14 & 1.23 & 0.93 & 1.26 \\ 1.42 & 1.37 & 0.87 & 1.22 & 3.52 \\ 1.34 & 0.90 & 1.28 & 2.23 & 1.05 \\ 0.92 & 1.08 & 2.01 & 78.5 & 0.96 \end{bmatrix}$$

*Step 2:*

*Step 3:*

$\bar{x} = (0.92 \quad 0.90 \quad 0.87 \quad 0.93 \quad 0.96)$

*Step 4:*

*Step 5:*

$$\overline{A} = \begin{bmatrix} 0 & 0 & 0 & 0.93 & 0 \\ 0 & 0 & 0.87 & 0 & 0 \\ 0 & 0.90 & 0 & 0 & 0 \\ 0.92 & 0 & 0 & 0 & 0.96 \end{bmatrix}$$

*Step 6:*
*Step 7:*
*Step 8:*

In according to this step first column dominates fifth column, therefore $x(e_0)_5 = 0$, and optimal solution is $\bar{x} = (0.92 \quad 0.90 \quad 0.870.930)$ and the minimum value of the objective function is zero.

We note that Lu and Fang (2001) obtained the objective value this example $f = 0.00000011225$ using a genetic algorithm and after 2004 iterations ( Yang & Cao, 2007). Further; Example 2 shows the presented algorithm achieve the solution easier with respect to that of Yang and Cao (2007).Therefore, the existing approach is superior with respect to other earlier works which has done so far.

**Example 3.** Consider another example as follows:

$$\min \quad Z = \frac{x_1x_2x_4x_6}{x_3x_5}$$

s.t.

$$\begin{bmatrix} 1 & 0.2 & 0.5 & 0.5 & 0.4 & 0.2 \\ 0.4 & 0.2 & 0.2 & 0.4 & 0.5 & 0.5 \\ 0.1 & 0.8 & 0.3 & 0.6 & 0.3 & 0.2 \\ 0.3 & 0.4 & 0.5 & 0.8 & 0.8 & 0.6 \\ 0.3 & 0.2 & 0.4 & 0.5 & 0.4 & 0.3 \\ 0.5 & 0.6 & 0.3 & 0.3 & 0.5 & 0.4 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} \geqslant \begin{bmatrix} 0.4 \\ 0.5 \\ 0.3 \\ 0.6 \\ 0.3 \\ 0.6 \end{bmatrix}$$

$$\begin{bmatrix} 0.3 & 0.2 & 0.4 & 0.5 & 0.5 & 0.3 \\ 0.5 & 0.6 & 0.3 & 0.3 & 0.5 & 0.4 \\ 0.1 & 0.2 & 0.3 & 0.4 & 0.5 & 0.6 \\ 0.6 & 0.5 & 0.4 & 0.3 & 0.2 & 0.1 \\ 0.4 & 0.4 & 0.4 & 0.3 & 0.3 & 0.3 \\ 0.5 & 0.5 & 0.6 & 0.6 & 0.7 & 0.7 \\ 0.8 & 0.8 & 1 & 1 & 1 & 0.8 \\ 0.3 & 0.9 & 0.3 & 0.9 & 0.3 & 0.4 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} \leqslant \begin{bmatrix} 0.3 \\ 0.6 \\ 0.6 \\ 0.6 \\ 0.4 \\ 0.7 \\ 0.8 \\ 0.9 \end{bmatrix}$$

*Step 1:* Matrices $\overline{A}$ and $\overline{B}$ are as follows:

$$\overline{A} = \begin{bmatrix} 0.4 & 2 & 0.8 & 0.8 & 1 & 2 \\ 1.25 & 2.5 & 2.5 & 1.25 & 1 & 1 \\ 3 & 0.375 & 1 & 0.5 & 1 & 1.5 \\ 2 & 1.5 & 1.2 & 0.75 & 0.75 & 1 \\ 1 & 1.5 & 0.75 & 0.6 & 0.75 & 1 \\ 1.2 & 1 & 2 & 2 & 1.2 & 1.5 \end{bmatrix}$$

$$\overline{B} = \begin{bmatrix} 1 & 1.5 & 0.75 & 0.6 & 0.6 & 1 \\ 1.2 & 1 & 2 & 2 & 1.2 & 1.5 \\ 6 & 3 & 2 & 1.5 & 1.2 & 1 \\ 1 & 1.2 & 1.5 & 2 & 3 & 6 \\ 1 & 1 & 1 & 1.33 & 1.33 & 1.33 \\ 1.4 & 1.4 & 1.16 & 1.16 & 1 & 1 \\ 1 & 1 & 0.8 & 0.8 & 0.8 & 1 \\ 3 & 1 & 3 & 1 & 3 & 2.25 \end{bmatrix}$$

*Step 2:*
*Step 3:*

$$\bar{x} = (1 \quad 1 \quad 0.75 \quad 0.6 \quad 0.6 \quad 1)$$

*Step 4:*
*Step 5:*

$$\overline{A} = \begin{bmatrix} 0.4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0.75 & 0.6 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

*Step 6:*
*Step 7:* In according to this step $x(e_0)_5 = 0$ and $\overline{A}$ is converted to

$$\overline{A} = \begin{bmatrix} 0.4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0.75 & 0.6 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

We notice that $5 \notin R^+$, hence we can go to next step.
*Step 8:*
*Step 9:* $J_1^{new} = \{1\}, J_2^{new} = \{6\}, J_3^{new} = \{4\}, J_4^{new} = \{6\}, J_5^{new} = \{1, 3, 4, 6\}$ and $J_6^{new} = \{2\}$. By pairwise comparison between four vectors $(1 \quad 1 \quad 0 \quad 0.5 \quad 0 \quad 1)$, $(0.4 \quad 1 \quad 0.75 \quad 0.5 \quad 0 \quad 1)$, $(0.4 \quad 1 \quad 0 \quad 0.6 \quad 0 \quad 1)$ and $(0.4 \quad 1 \quad 0 \quad 0.5 \quad 0 \quad 1)$, we gain $x(e_0) = (0.4 \quad 1 \quad 0 \quad 0.5 \quad 0 \quad 1)$ as an optimal solution of Problem (4a).
*Step 10:* The optimal solution of the problem is $x^* = (0.4 \quad 1 \quad 0.750.50.61)$ and the minimum value of the objective function is $Z = \frac{(0.4)(1)(0.5)(1)}{(0.75)(0.6)} = 0.444$.

## 5. Conclusion

In this paper, we studied the monomial geometric programming problem with fuzzy relational inequality constraints defined by the max-product operator. Since the difficulty of this problem is finding the minimal solutions optimizing the same problem with the objective function $\prod_{j \in R^+} x_j^{\alpha_j}$, we presented an algorithm together with some simplification operations to accelerate the problem resolution. At last, we gave three numerical examples to illustrate the proposed algorithm. Example (2) shows superiority of the presented approach with respect to preceding methods.

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