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A new additive function and the Smarandache divisor product sequences [1]

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Abstract For any positive integer n, we define the arithmetical function $G(n)$ as $G(1) = 0$. If $n > 1$ and $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the prime power factorization of n, then $G(n) = \frac{\alpha_1}{p_1} + \frac{\alpha_2}{p_2} + \cdots + \frac{\alpha_k}{p_k}$. The main purpose of this paper is using the elementary method and the prime distribution theory to study the mean value properties of $G(n)$ in Smarandache divisor product sequences $\{p_d(n)\}$ and $\{q_d(n)\}$, and give two sharper asymptotic formulae for them.

Keywords Additive function, Smarandache divisor product sequences, mean value, elementary method, asymptotic formula.

§1. Introduction and results

In elementary number theory, we call an arithmetical function $f(n)$ as an additive function, if for any positive integers m, n with $(m,\ n) = 1$, we have $f(mn) = f(m) + f(n)$. We call $f(n)$ as a complete additive function, if for any positive integers r and s, $f(rs) = f(r) + f(s)$. There are many arithmetical functions satisfying the additive properties. For example, if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ denotes the prime power factorization of n, then function $\Omega(n) = \alpha_1 + \alpha_2 + \cdots + \alpha_k$ and logarithmic function $f(n) = \ln n$ are two complete additive functions, $\omega(n) = k$ is an additive function, but not a complete additive function. About the properties of the additive functions, there are many authors had studied it, and obtained a series interesting results, see references [1], [2], [5] and [6].

In this paper, we define a new additive function $G(n)$ as follows: $G(1) = 0$; If $n > 1$ and $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ denotes the prime power factorization of n, then $G(n) = \frac{\alpha_1}{p_1} + \frac{\alpha_2}{p_2} + \cdots + \frac{\alpha_k}{p_k}$. It is clear that this function is a complete additive function. In fact if $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ and $n = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k}$, then we have $mn = p_1^{\alpha_1+\beta_1} \cdot p_2^{\alpha_2+\beta_2} \cdots p_k^{\alpha_k+\beta_k}$. Therefore, $G(mn) = \frac{\alpha_1+\beta_1}{p_1} + \frac{\alpha_2+\beta_2}{p_2} + \cdots + \frac{\alpha_k+\beta_k}{p_k} = G(m) + G(n)$. So $G(n)$ is a complete additive function. Now we define the Smarandache divisor product sequences $\{p_d(n)\}$ and $\{q_d(n)\}$ as follows: $p_d(n)$ denotes the product of all positive divisors of n; $q_d(n)$ denotes the product of all positive divisors d of n but n. That is,

$$p_d(n) = \prod_{d|n} d = n^{\frac{d(n)}{2}}; \qquad q_d(n) = \prod_{d|n, d<n} d = n^{\frac{d(n)}{2}-1},$$

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where $d(n)$ denotes the Dirichlet divisor function.

The sequences $\{p_d(n)\}$ and $\{q_d(n)\}$ are introduced by Professor F.Smarandache in references [3], [4] and [9], where he asked us to study the various properties of $\{p_d(n)\}$ and $\{q_d(n)\}$. About this problem, some authors had studied it, and proved some conclusions, see references [7], [8], [10] and [11].

The main purpose of this paper is using the elementary method and the prime distribution theory to study the mean value properties of $G(p_d(n))$ and $G(q_d(n))$, and give two sharper asymptotic formulae for them. That is, we shall prove the following:

Theorem 1. For any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\le x} G(p_d(n)) = B\cdot x\cdot \ln x + (2\gamma\cdot B - D - B)\cdot x + O\left(\sqrt{x}\ln\ln x\right),$$

where $B = \sum_p \frac{1}{p^2}$, $D = \sum_p \frac{\ln p}{p^2}$, γ is the Euler constant, and $\sum_p$ denotes the summation over all primes.

Theorem 2. For any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\le x} G(q_d(n)) = B\cdot x\cdot \ln x + (2\gamma\cdot B - 2B - D)\cdot x + O\left(\sqrt{x}\ln\ln x\right),$$

where B and D are defined as same as in Theorem 1.

§2. Two simple lemmas

In this section, we give two simple lemmas, which are necessary in the proof of the theorems. First we have:

Lemma 1. For any real number $x > 1$, we have the asymptotic formula:

$$\sum_{p\le x}\frac{1}{p} = \ln\ln x + A + O\left(\frac{1}{\ln x}\right),$$

where A be a constant, $\sum_{p\le x}$ denotes the summation over all primes $p \le x$.

Proof. See Theorem 4.12 of reference [6].

Lemma 2. For any real number $x > 1$, we have the asymptotic formulae:

(I) $$\sum_{n\le x} G(n) = B\cdot x + O(\ln\ln x);$$

(II) $$\sum_{n\le x}\frac{G(n)}{n} = B\cdot \ln x + C + O\left(\frac{\ln\ln x}{x}\right),$$

where $B = \sum_p \frac{1}{p^2}$, $C = \gamma\cdot B - \sum_p \frac{\ln p}{p^2}$, γ is the Euler constant, and $\sum_p$ denotes the summation over all primes.

Proof. For any positive integer $n > 1$, from the definition of $G(n)$ we have

$$G(n) = \sum_{p|n} \frac{1}{p}.$$

So from this formula and Lemma 1 we have

$$\begin{aligned}
\sum_{n\le x} G(n) &= \sum_{n\le x}\sum_{p|n}\frac{1}{p} = \sum_{np\le x}\frac{1}{p} = \sum_{p\le x}\frac{1}{p}\sum_{n\le \frac{x}{p}} 1 = \sum_{p\le x}\frac{1}{p}\left[\frac{x}{p}\right] \\
&= x\cdot\sum_{p\le x}\frac{1}{p^2} + O\left(\sum_{p\le x}\frac{1}{p}\right) = B\cdot x + O\left(\ln\ln x\right),
\end{aligned}$$

where $B = \sum_{p} \frac{1}{p^2}$ be a constant. This proves (I) of Lemma 2.

Now we prove (II) of Lemma 2, note that the asymptotic formula

$$\sum_{n\le x}\frac{1}{n} = \ln x + \gamma + O\left(\frac{1}{x}\right),$$

where γ is the Euler constant. So from Lemma 1 and the definition of $G(n)$ we also have

$$\begin{aligned}
\sum_{n\le x}\frac{G(n)}{n} &= \sum_{n\le x}\frac{\sum_{p|n}\frac{1}{p}}{n} = \sum_{np\le x}\frac{1}{p^2 n} = \sum_{p\le x}\frac{1}{p^2}\sum_{n\le\frac{x}{p}}\frac{1}{n} \\
&= \sum_{p\le x}\frac{1}{p^2}\left[\ln x - \ln p + \gamma + O\left(\frac{p}{x}\right)\right] \\
&= \sum_{p\le x}\frac{\ln x}{p^2} - \sum_{p\le x}\frac{\ln p}{p^2} + \sum_{p\le x}\frac{1}{p^2}\gamma + O\left(\frac{1}{x}\sum_{p\le x}\frac{1}{p}\right) \\
&= B\cdot\ln x - \sum_{p}\frac{\ln p}{p^2} + \gamma\cdot B + O\left(\frac{\ln\ln x}{x}\right) \\
&= B\cdot\ln x + C + O\left(\frac{\ln\ln x}{x}\right),
\end{aligned}$$

where $C = \gamma\cdot B - \sum_{p}\frac{\ln p}{p^2}$ is a constant. This proves (II) of Lemma 2.

§3. Proof of the theorems

Now we use the above Lemmas to complete the proof of the theorems. First we prove Theorem 1. Note that the complete additive properties of $G(n)$ and the definition of $p_d(n)$, from (II) of Lemma 2 and Theorem 3.17 of [6] we have

$$\begin{aligned}
\sum_{n\le x} G\left(p_d(n)\right) &= \sum_{n\le x} G\left(n^{\frac{d(n)}{2}}\right) = \frac{1}{2}\sum_{n\le x} d(n)G(n) = \frac{1}{2}\sum_{mn\le x} G(mn) \\
&= \frac{1}{2}\sum_{mn\le x}\left(G(m)+G(n)\right) = \sum_{mn\le x} G(m) \\
&= \sum_{m\le\sqrt{x}}\sum_{n\le\frac{x}{m}} G(m) + \sum_{n\le\sqrt{x}}\sum_{m\le\frac{x}{n}} G(m) - \left(\sum_{m\le\sqrt{x}} G(m)\right)\left(\sum_{n\le\sqrt{x}} 1\right) \\
&= \sum_{m\le\sqrt{x}} G(m)\left[\frac{x}{m}\right] + \sum_{n\le\sqrt{x}}\left[\frac{B\cdot x}{n} + O\left(\ln\ln x\right)\right] \\
&\quad -\left[\sqrt{x}+O(1)\right]\left[B\cdot\sqrt{x}+O(\ln\ln x)\right] \\
&= x\cdot\sum_{m\le\sqrt{x}}\frac{G(m)}{m} + O\left(\sum_{m\le\sqrt{x}} G(m)\right) + B\cdot x\cdot\sum_{n\le\sqrt{x}}\frac{1}{n} \\
&\quad -B\cdot x + O\left(\sqrt{x}\ln\ln x\right) \\
&= x\cdot\left[\frac{1}{2}B\cdot\ln x + C + O\left(\frac{\ln\ln x}{\sqrt{x}}\right)\right] + B\cdot x\cdot\left[\ln\sqrt{x}+\gamma+O\left(\frac{1}{\sqrt{x}}\right)\right] \\
&\quad -B\cdot x + O\left(\sqrt{x}\ln\ln x\right) \\
&= B\cdot x\cdot\ln x + (C+\gamma B - B)\cdot x + O\left(\sqrt{x}\ln\ln x\right) \\
&= B\cdot x\cdot\ln x + (2\gamma B - B - D)\cdot x + O\left(\sqrt{x}\ln\ln x\right),
\end{aligned}$$

where $B=\sum_{p}\frac{1}{p^2}$ and $D=\sum_{p}\frac{\ln p}{p^2}$, γ is the Euler constant. This proves Theorem 1.

From Lemma 2, Theorem 1 and the definition of $q_d(n)$ we can also deduce that

$$\begin{aligned}
\sum_{n\le x} G\left(q_d(n)\right) &= \sum_{n\le x} G\left(n^{\frac{d(n)}{2}-1}\right) = \frac{1}{2}\sum_{n\le x} d(n)G(n) - \sum_{n\le x} G(n) \\
&= B\cdot x\cdot\ln x + (2\gamma B - B - D)\cdot x - B\cdot x + O\left(\sqrt{x}\ln\ln x\right) \\
&= B\cdot x\cdot\ln x + (2\gamma B - 2B - D)\cdot x + O\left(\sqrt{x}\ln\ln x\right).
\end{aligned}$$

This completes the proof of Theorem 2.

§4. Some notes

For any positive integer n and any fixed real number β, we define the general arithmetical function $H(n)$ as $H(1)=0$. If $n>1$ and $n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ be the prime power factorization of n, then $H(n)=\alpha_1\cdot p_1^{\beta}+\alpha_2\cdot p_2^{\beta}+\cdots+\alpha_k\cdot p_k^{\beta}$. It is clear that this function is a complete additive function. If $\beta=0$, then $H(n)=\Omega(n)$. If $\beta=-1$, then $H(n)=G(n)$. Using our method we can also give some asymptotic formulae for the mean vale of $H\left(p_d(n)\right)$ and $H\left(q_d(n)\right)$.

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