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A Note on Prime and Sequential Labelings of Finite Graphs

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Abstract: A labeling or valuation of a graph G is an assignment f of labels to the vertices of G that induces for each edge xy a label depending on the vertex labels $f(x)$ and $f(y)$. In this paper, we study some classes of graphs and their corresponding labelings.

Key Words: Labeling, sequential graph, harmonious graph, prime graph, Smarandache common k-factor labeling.

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§1. Introduction

Unless mentioned or otherwise, a graph in this paper shall mean a simple finite graph without isolated vertices. For all terminology and notations in Graph Theory, we follow [5] and all terminology regarding to sequential labeling, we follow [3]. Graph labelings where the vertices are assigned values subject to certain conditions have been motivated by practical problems. Labeled graphs serves as useful mathematical models for a broad range of applications such as coding theory, including the design of good radar type codes, synch-set codes, missile guidance codes and convolutional codes with optimal autoconvolutional properties. They facilitate the optimal nonstandard encodings of integers.

Labeled graphs have also been applied in determining ambiguities in X-ray crystallographic analysis, to design a communication network addressing system, in determining optimal circuit layouts and radio astronomy problems etc. A systematic presentation of diverse applications of graph labelings is presented in [1].

Let G be a (p,q)-graph. Let $V(G), E(G)$ denote respectively the vertex set and edge set of G. Consider an injective function $g : V(G) \to X$, where $X = \{0, 1, 2, \cdots, q\}$ if G is a tree and $X = \{0, 1, 2, \cdots, q-1\}$ otherwise. Define the function $f^* : E(G) \to N$, the set of natural numbers such that $f^*(uv) = f(u) + f(v)$ for all edges uv. If $f^*(E(G))$ is a sequence of distinct consecutive integers, say $\{k, k+1, \cdots, k+q-1\}$ for some k, then the function f is said to be *sequential labeling* and the graph which admits such a labeling is called a *sequential graph.*

Another labeling has been suggested by Graham and S Loane [4] named as harmonious

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labeling which is a function $h : V(G) \to Z_q$, q is the number of edges of G such that the induced edge labeling given by $g^*(uv) = [g(u) + g(v)] \pmod{q}$ for any edge uv is injective.

The notion of prime labeling of graphs, was defined in [6]. A graph G with n-vertices is said to have a *prime labeling* if its vertices are labeled with distinct integers $1, 2, \cdots, n$ such that for each edge uv the labels assigned to u and v are relatively prime. Such a graph admitting a prime labeling is known as a *prime graph*. Generally, a *Smarandache common k-factor labeling* is such a labeling with distinct integers $1, 2, \cdots, n$ such that the greatest common factor of labels assigned to u and v is k for $\forall uv \in E(G)$. Clearly, a prime labeling is nothing else but a Smarandache common 1-factor labeling. A graph admitting a Smarandache common k-factor labeling is called a *Smarandache common k-factor graph*. Particularly, a graph admitting a prime labeling is known as a *prime graph* in references.

Notation 1.1 $(a, b) = 1$ means that a and b are relatively prime.

§2. Cycle Related Graphs

In [2], showed that every cycle with a chord is graceful. In [9] proved that a cycle C_n with a chord joining two vertices at a distance 3 is sequential for all odd $n, n \geqslant 5$. Now, we have the following theorems.

Theorem 2.1 *Every cycle C_n, with a chord is prime, for all $n \geqslant 4$.*

Proof Let G be a graph such that $G = C_n$ with a chord joining two non- adjacent vertices of C_n, for all n greater than or equal to 4. Let $\{v_1, v_2, \cdots, v_n\}$ be the vertex set of G. Let the number of vertices of G be n and the edges be $n + 1$. Define a function $f : V(G) \to \{1, 2, \cdots n\}$ such that $f(v_i) = i, i = 1, 2, \cdots, n$. It is obvious that $(f(v_i), f(v_{i+1})) = 1$ for all $i = 1, 2, \cdots, (n - 1)$. Also $(1, n) = 1$ for all n greater than 1. Now select the vertex v_1 and join this to any vertex of C_n, which is not adjacent to v_1, G admits a prime labeling. □

Theorem 2.2 *Every cycle C_n, with* $\left[\dfrac{n-1}{2}\right] - 1$ *chords from a vertex is prime, for all n greater than or equal to 5.*

Proof Let G be a graph such that $G = C_n$, n greater than or equal to 5. Let $\{v_1, v_2, \ldots, v_n\}$ be the vertex set of G. Label the vertices of C_n as in Theorem 2.1. Next select the vertex v_2. By our labeling $f(v_2) = 2$. Now join v_2 to all the vertices of C_n whose f-values are odd. Then it is clear that there exists exactly $[\frac{n-1}{2}] - 1$ chords, and G admits a prime labeling. □

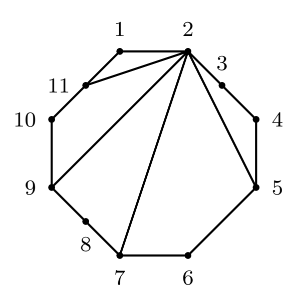

Figure 1

Remark 2.1 From Theorem 2.1, it is clear that there is possible to get $n-3$ chords and Theorem 2.2 tells us there are $[\frac{n-1}{2}]-1$ chords. Thus the bound $n-3$ is best possible and all other possible chords of less than these two bounds.

Example 2.1 Figure 1 gives the prime labeling of C_{11} with 4-chords.

Theorem 2.3 *The graph $C_n + \bar{K}_{1,t}$ is sequential for all odd $n, n \geqslant 3$.*

Proof Let $v_1, v_2, \cdots, v_n$ (n is odd) be the set of vertices of C_n and $u, u_1, u_2, \cdots, u_t$ be the $t+1$ isolated vertices of $\bar{K}_{1,t}$. Let $G = C_n + \bar{K}_{1,t}$ and note that, G has $n+t+1$ vertices and $n(2+t)$ edges.

Define a function $f : V(G) \to \{0, 1, 2, \cdots, \frac{n-1}{2} + tn\}$ such that

$$
\begin{aligned}
f(v_{2i-1}) &= i-1, \text{ for } i = 1, 2, \cdots, \frac{n+1}{2}\\
f(v_{2i}) &= \frac{n-1}{2} + i, \text{ for } i = 1, 2, \cdots, \frac{n-1}{2}\\
f(u_1) &= \frac{3}{2}n - 1
\end{aligned}
$$

and

$$f(u_i) = \frac{n-1}{2} + ni, i = 2, 3, \cdots, t$$

We can easily observe that the above defined f is injective. Hence f becomes a sequential labeling of $C_n + \bar{K}_{1,t}$. Thus $C_n + \bar{K}_{1,t}$ is sequential for all odd n, $n \geqslant 3$. □

Corollary 2.4 *The graph $C_n + \bar{K}_{1,t}$ is harmonious for all odd $n \geqslant 3$.*

Proof Any sequential is harmonius implies that $C_n + \bar{K}_{1,t}$ is harmonius, $n \geqslant 3$. □

Theorem 2.5 *The graph $C_n + \bar{K}_{1,1,t}$ is sequential and harmonius for all odd $n, n \geqslant 3$.*

Theorem 2.6 *The graph $C_n + \bar{K}_{1,1,1,t}$ is sequential and harmonius for all odd n, $n \geqslant 3$.*

Example 2.2 Figure 2 gives the sequential labeling of the graph $C_5 + \bar{K}_{1,1,1,3}$.

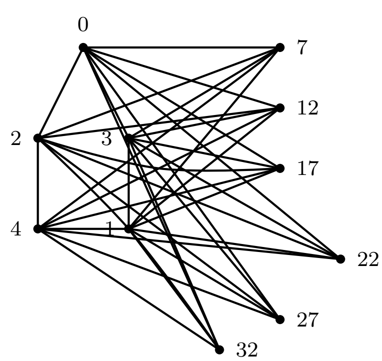

Figure 2

Theorem 2.7 *The graph* $C_n + \bar{K}_{1,1,\ldots,1,t}$ *is sequential as harmonius for odd* n, $n \geqslant 3$.

Theorem 2.8 *The graph* $C_n + \bar{K}_{1,m,n}$ *is sequential and harmonius for all odd* n, $n \geqslant 3, m \geqslant 1$.

§3. On Join of Complete Graphs

In [7], it is shown that $L_n + K_1$ and $B_n + K_1$ are prime and join of any two connected graphs are not odd sequential. Now, we have the following.

Theorem 3.1 *The graph* $K_{1,n} + K_2$ *is prime for* $n \geqslant 4$.

Proof Let $G = K_{1,n} + K_2$. We can notice that G has $(n+3)$-vertices and $(3n+2)$-edges. Let $\{w, v_1, v_2, \cdots, v_n\}$ be the vertices of $K_{1,n}$ and $\{u_1, u_2\}$ be the two vertices of K_2. Assign the first two largest primes less than or equal to $n+3$ to the two vertices of K_2. Assign 1 to w and remaining n values to the n vertices arbitrarily, we can obtain a prime numbering of $K_{1,n} + K_2$. □

Corollary 3.1 *The graph* $K_{1,n} + \bar{K}_2$ *is prime for all* $n \geqslant 4$.

§4. Product Related Graphs

Definition 4.1 *Let* G *and* H *be graphs with* $V(G) = V_1$ *and* $V(H) = V_2$. *The cartesian product of* G *and* H *is the graph* $G \Box H$ *whose vertex set is* $V_1 \times V_2$ *such that two vertices* $u = (x, y)$ *and* $v = (x', y')$ *are adjacent if and only if either* $x = x'$ *and* y *is adjacent to* y' *in* H *or* $y = y'$ *and* x *is adjacent to* x' *in* G. *That is,* u *adj* v *in* $G \Box H$ *whenever* $[x = x'$ *and* y *adj* $y']$ *or* $[y = y'$ *and* x *adj* $x']$.

In [8] A.Nagarajan, A.Nellai Murugan and A.Subramanian proved that $P_n \Box K_2$, $P_n \Box P_n$ are near mean graphs.

Definition 4.2 *Let* P_n *be a path on* n *vertices and* K_4 *be a complete graph on 4 vertices. The Cartesian product* P_n *and* K_4 *is denoted as* $P_n \Box K_4$ *with* $4n$ *vertices and* $10n - 4$ *edges.*

Theorem 4.1 *The graph* $P_n \Box K_4$ *is sequential, for all* $n \geqslant 1$.

Proof Let $G = P_n \Box K_4$. Let $\{v_{i,1}, v_{i,2}, v_{i,3}, v_{i,4} / i = 1, 2, \cdots, n\}$ be the vertex set of G.

Define a function $f: V(G) \to \{0, 1, 2, \cdots, 5n-1\}$ such that

$$f(v_{2i-1,1}) = 10i - 6 \quad ; \quad 1 \leqslant i \leqslant \frac{n}{2} \quad \text{if } n \text{ is even or } 1 \leqslant i \leqslant \frac{n+1}{2} \quad \text{if } n \text{ is odd.}$$
$$f(v_{2i-1,2}) = 10(i-1) \quad ; \quad 1 \leqslant i \leqslant \frac{n}{2} \quad \text{if } n \text{ is even or } 1 \leqslant i \leqslant \frac{n+1}{2} \quad \text{if } n \text{ is odd.}$$
$$f(v_{2i-1,3}) = 10i - 9 \quad ; \quad 1 \leqslant i \leqslant \frac{n}{2} \quad \text{if } n \text{ is even or } 1 \leqslant i \leqslant \frac{n+1}{2} \quad \text{if } n \text{ is odd.}$$
$$f(v_{2i-1,4}) = 10i - 8 \quad ; \quad 1 \leqslant i \leqslant \frac{n}{2} \quad \text{if } n \text{ is even or } 1 \leqslant i \leqslant \frac{n+1}{2} \quad \text{if } n \text{ is odd.}$$
$$f(v_{2i,1}) = 10i - 4 \quad ; \quad 1 \leqslant i \leqslant \frac{n}{2} \quad \text{if } n \text{ is even or } 1 \leqslant i \leqslant \frac{n-1}{2} \quad \text{if } n \text{ is odd.}$$
$$f(v_{2i,2}) = 10i - 1 \quad ; \quad 1 \leqslant i \leqslant \frac{n}{2} \quad \text{if } n \text{ is even or } 1 \leqslant i \leqslant \frac{n-1}{2} \quad \text{if } n \text{ is odd.}$$
$$f(v_{2i,3}) = 10i - 3 \quad ; \quad 1 \leqslant i \leqslant \frac{n}{2} \quad \text{if } n \text{ is even or } 1 \leqslant i \leqslant \frac{n-1}{2} \quad \text{if } n \text{ is odd.}$$

and

$$f(v_{2i,4}) = 10i - 5 \quad ; \quad 1 \leqslant i \leqslant \frac{n}{2} \quad \text{if } n \text{ is even or } 1 \leqslant i \leqslant \frac{n-1}{2} \quad \text{if } n \text{ is odd.}$$

(a) Clearly we can see that f is injective.

(b) Also, $\max_{v \in V} f(v) = \max\{\max_i 10i-6; \max_i 10(i-1); \max_i 10i-9; \max_i 10i-8; \max_i 10i -4; \max_i 10i-1; \max_i 10i-3; \max_i 10i-5\} = 5n-1$. Thus, $f(v) = \{0, 1, 2, \ldots, 5n-1\}$. Finally, it can be easily verified that the labels of the edge values are distinct positive integers in the interval $[1, 10n-4]$. Thus, f is a sequential numbering. Hence, the graph G is sequential. □

Example 4.1 Figure 4 gives the sequential labeling of the graph $P_4 \square K_4$.

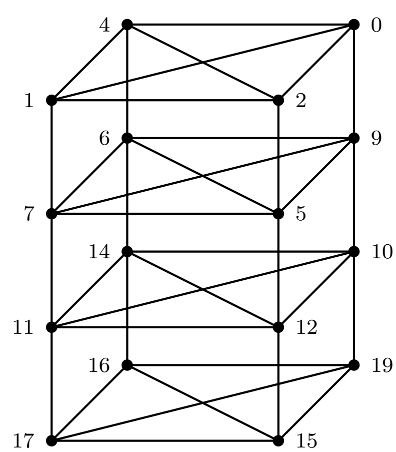

Figure 3

Corollary 4.1 *The graph $P_n \square K_4$ is harmonius, for $n \geqslant 2$.*

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