ON A NUMBER SET RELATED TO THE K-FREE NUMBERS

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Abstract Let F_k denotes the set of k-free number. For any positive integers $l \geq 2$, we define a number set $A_{k,l}$ as follows

$$A_{k,l} = \{n : n = m^l + r, m^l \leq n < (m+1)^l, r \in F_k, n \in N\}.$$

In this paper, we study the arithmetical properties of the number set $A_{k,l}$, and give some interesting asymptotic formulae for it.

Keywords: Number set; k-free number; Asymptotic formula.

§1. Introduction

Let $k \geq 2$ be an integer. The k-free numbers set F_k is defined as follows

$$F_k = \{n : \text{if prime } p|n \text{ then } p^k \dagger n, n \in N\}.$$

In problem 31 of [1], Professor F.Smarandache asked us to study the arithmetical properties of the numbers in F_k. About this problem, many authors had studied it, see [2], [3], [4]. For any positive integer n and $l \geq 2$, there exist an integer m such that

$$m^l \leq n \leq (m+1)^l.$$

So we can define the following number set $A_{k,l}$:

$$A_{k,l} = \{n : n = m^l + r, m^l \leq n < (m+1)^l, r \in F_k, n \in N\}.$$

In this paper, we use the elementary methods to study the asymptotic properties of the number of integers in $A_{k,l}$ less than or equal to a fixed real number x, and give some interesting asymptotic formulae. That is, we shall prove the following results:

Theorem 1. Let $k, l \geq 2$ be any integers. Then for any real number $x > 1$, we have the asymptotic formula

$$\sum_{\substack{n \leq x \\ n \in A_{k,l}}} 1 = \frac{x}{\zeta(k)} + O_{k,l}\left(x^{\frac{1}{l}+\frac{1}{k}-\frac{1}{kl}}\right),$$

where $\zeta(s)$ denotes the Riemann zeta function and $O_{k,l}$ means the big Oh constant related to k, l.

Theorem 2. Assuming the Riemann Hypothesis, there holds

$$\sum_{\substack{n\leq x\\ n\in A_{2,2}}} 1 = \frac{6}{\pi^2}x + O\left(x^{\frac{29}{44}+\epsilon}\right),$$

where ϵ is any fixed positive number.

§2. Two Lemmas

Lemma 1. For any real number $x > 1$ and integer $k \geq 2$, we have the asymptotic formula

$$\sum_{\substack{n\leq x\\ n\in F_k}} 1 = \frac{x}{\zeta(k)} + O\left(x^{\frac{1}{k}}\right).$$

Proof. See reference [5].

Lemma 2. Assuming the Riemann Hypothesis, we have

$$\sum_{\substack{n\leq x\\ n\in F_2}} 1 = \frac{6}{\pi^2}x + O\left(x^{\frac{7}{22}} + \epsilon\right).$$

Proof. See reference [6].

§3. Proof of the theorems

In this section, we shall complete the proofs of the theorems. For any real number $x \geq 1$ and integer $l \geq 2$, there exist a positive integer M such that

$$M^l \leq x < (M+1)^l. \tag{1}$$

So from the definition of the number set $A_{k,l}$ and Lemma 1, we can write

$$\begin{aligned}
\sum_{\substack{n\leq x\\ n\in A_{k,l}}} 1 &= \sum_{t=1}^{M-1} \sum_{\substack{m=1\\ m\in F_k}}^{(t+1)^l-t^l} 1 \sum_{\substack{m\leq x-M^l\\ m\in F_k}} 1 \\
&= \sum_{t=1}^{M-1} \frac{(t+1)^l-t^l}{\zeta(k)} + O\left(\sum_{t=1}^{M-1}\left((t+1)^l-t^l\right)^{\frac{1}{k}}\right) \\
&\quad + \frac{x-M^l}{\zeta(k)} + O\left(\left(x-M^l\right)^{\frac{1}{k}}\right) \\
&= \sum_{t=1}^{M-1} \frac{(t+1)^l-t^l}{\zeta(k)} + \frac{x-M^l}{\zeta(k)} + O_{k,l}\left(M^{1+\frac{l-1}{k}}\right) \\
&= \frac{x}{\zeta(k)} + O_{k,l}\left(M^{1+\frac{l-1}{k}}\right),
\end{aligned} \tag{2}$$

On the other hand, from (1) we have the estimates

$$0 \leq x - M^l < (M+1)^l - M^l \ll x^{\frac{l-1}{l}} \tag{3}$$

Now combining (2) and (3), we have

$$\sum_{\substack{n \leq x \\ n \in A_{k,l}}} 1 = \frac{x}{\zeta(k)} + O_{k,l}\left(x^{\frac{1}{l}+\frac{1}{k}-\frac{1}{kl}}\right).$$

This completes the proof of Theorem 1. From the same argue as proving Theorem 1 and Lemma 2, we can get

$$\sum_{\substack{n \leq x \\ n \in A_{2,2}}} 1 = \ = \tfrac{12}{\pi^2} \textstyle\sum_{t=1}^{M-1} t + O\left(\sum_{t=1}^{M-1} t^{\frac{7}{22}+\epsilon}\right) \tag{4}$$

$$= \tfrac{6}{\pi^2} M^2 + O\left(M^{\frac{29}{22}+\epsilon}\right) \tag{5}$$

$$= \tfrac{6}{\pi^2} x + O_{k,l}\left(x^{\frac{29}{44}+\epsilon}\right). \tag{6}$$

This completes the proof of Theorem 2.

References

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