A NUMBER THEORETIC FUNCTION AND ITS MEAN VALUE *

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Abstract Let $q \geq 3$ be a fixed positive integer, $e_q(n)$ denotes the largest exponent of power q which divides n. In this paper, we use the elementary method to study the properties of the sequence $e_q(n)$, and give some sharper asymptotic formulas for the mean value $\sum\limits_{n \leq x} e_q^k(n)$.

Keywords: Largest exponent; Asymptotic formula; Mean value.

§1. Introduction

Let $q \geq 3$ be a fixed positive integer, $e_q(n)$ denotes the largest exponent of power q which divides n. It is obvious that $e_q(n) = m$, if $q^m | n$, and $q^{m+1} \dagger n$. In problem 68 of [3], Professor F.Smarandach asked us to study the properties of the sequence $e_q(n)$. About this problem, lv chuan in [2] had given the following result:

If p is a prime, $m \geq 0$ is an integer

$$\sum_{n \leq x} e_p^m(n) = \frac{p-1}{p} a_p(m) x + O\left(\log^{m+1} x\right),$$

where $a_p(m)$ is a computable number.

The author had used the analytic method to consider the special case: p_1 and p_2 are two fixed distinct primes. That is, for any real number $x \geq 1$, we have the asymptotic formula

$$\sum_{n \leq x} e_{p_1 p_2}(n) = \frac{x}{p_1 p_2 - 1} + O\left(x^{1/2+\varepsilon}\right), \tag{1}$$

where ε is any fixed positive number.

In this paper, we use the elementary method to improve the error term of (1), and give some sharper asymptotic formula for the mean value $\sum\limits_{n \leq x} e_q^k(n)$. That is we shall prove the following:

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Theorem 1. *Let $q \geq 3$ be any fixed positive integer, then for any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{n \leq x} e_q(n) = \frac{x}{q-1} + O\left(\log x\right).$$

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Theorem 2. *If $q \geq 3$ is any fixed positive integer, $k \geq 2$ is an integer, then we have the asymptotic formula*

$$\sum_{n \leq x} e_q^k(n) = \frac{q-1}{q} B_q(k) x + O\left(\log^{k+1} x\right),$$

where $B_q(k)$ is given by the recursion formulas: $B_q(0) = \frac{1}{q-1}$,

$$B_q(k) = \frac{1}{q-1}\left(\binom{k}{1} B_q(k-1) + \binom{k}{2} B_q(k-2) + \cdots + \binom{k}{k-1} B_q(1) + B_q(0) + 1\right).$$

Taking $q = p_1 p_2$ in Theorem 1, where p_1, p_2 are two fixed distinct primes, we may immediately obtain the following

Corollary. *For any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{n \leq x} e_{p_1 p_2}(n) = \frac{x}{p_1 p_2 - 1} + O\left(\log x\right).$$

§2. Proof of the theorems

In this section, we shall complete the Theorems.

Let $M = [x]$, the greatest integer $\leq x$, S denotes the set of $\{1, 2, 3, \cdots, M\}$. We distribute the integers of S into disjoint sets as follows. For each integer $m \geq 0$, let

$$A(m) = \{n| \quad e_q(n) = m, 1 \leq n \leq M\}.$$

That is, $A(m)$ contains those elements of S which satisfies: $q^m | n$, but $q^{m+1} \dagger n$.

Therefore if $f(m)$ denotes the number of integers in $A(m)$, we have

$$f(m) = \left[\frac{M}{q^m}\right] - \left[\frac{M}{q^{m+1}}\right]$$

So we have

$$\begin{aligned} \sum_{n \leq x} e_q(n) &= \sum_{n \leq M} e_q(n) = \sum_{m=0}^{\infty} m f(m) \\ &= \sum_{m=1}^{\infty} m \left(\left[\frac{M}{q^m}\right] - \left[\frac{M}{q^{m+1}}\right]\right) = \sum_{m=1}^{\infty} \left[\frac{M}{q^m}\right] \end{aligned}$$

$$
\begin{aligned}
&= \sum_{m=1}^{\infty} \frac{M}{q^m} + O\left(\sum_{m\le \frac{\log M}{\log q}} 1\right) + O\left(\sum_{m> \frac{\log M}{\log q}} \frac{M}{q^m}\right)\\
&= \sum_{m=1}^{\infty} \frac{x}{q^m} + O\left(\sum_{m=1}^{\infty} \frac{1}{q^m}\right) + O(\frac{\log M}{\log q})\\
&= \frac{x}{q-1} + O\left(\log x\right).
\end{aligned}
$$

This completes the proof of Theorem1.

Before proving Theorem 2, we consider the series $B_q(k) = \sum\limits_{m=1}^{\infty} \frac{m^k}{q^m}$, it is easy to show that

$B_q(0) = \sum\limits_{m=1}^{\infty} \frac{1}{q^m} = \frac{1}{q-1}$, and $B_q(k)$ satisfies the recursion formula

$$
B_q(k) = \frac{1}{q-1}\left(\tbinom{k}{1}B_q(k-1) + \tbinom{k}{2}B_q(k-2) + \cdots + \tbinom{k}{k-1}B_q(1) + B_q(0) + 1\right).
$$

Now we complete the proof of theorem2, with the same method as above, we have

$$
\begin{aligned}
&\sum_{n\le x} e_q^k(n) = \sum_{n\le M} e_q^k(n) = \sum_{m=0}^{\infty} m^k f(m)\\
&= \sum_{m=1}^{\infty} m^k \left(\left[\frac{M}{q^m}\right] - \left[\frac{M}{q^{m+1}}\right]\right)\\
&= \sum_{m=1}^{\infty} m^k \left(\frac{M}{q^m} - \frac{M}{q^{m+1}}\right) + O\left(\sum_{m\le \frac{\log M}{\log q}} m^k\right) + O\left(\sum_{m> \frac{\log M}{\log q}} \frac{Mm^k}{q^m}\right)\\
&= \frac{(q-1)M}{q} \sum_{m=1}^{\infty} \frac{m^k}{q^m} + O(\log^{k+1} M) + O\left(\frac{1}{q^{[\frac{\log M}{\log q}]}} \sum_{u=1}^{\infty} \frac{M(\frac{\log M}{\log q} + u)^k}{q^u}\right)\\
&= \frac{(q-1)M}{q} B_q(k) + O(\log^{k+1} M)\\
&\quad + O\left(\left(\frac{\log M}{\log q}\right)^k \sum_{u=1}^{\infty} \frac{1}{q^u} + \tbinom{k}{1}\left(\frac{\log M}{\log q}\right)^{k-1} \sum_{u=1}^{\infty} \frac{u}{q^u} + \cdots + \tbinom{k}{k} \sum_{u=1}^{\infty} \frac{u^k}{q^u}\right)\\
&= \frac{(q-1)M}{q} B_q(k) + O(\log^{k+1} M)\\
&= \frac{q-1}{q} B_q(k)x + O\left(\log^{k+1} x\right).
\end{aligned}
$$

This completes the proof of Theorem2.

References

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