On Algebraic Multi-Ring Spaces

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Abstract: A Smarandache multi-space is a union of n spaces $A_1, A_2, \cdots, A_n$ with some additional conditions holding. Combining Smarandache multi-spaces with rings in classical ring theory, the conception of multi-ring spaces is introduced. Some characteristics of a multi-ring space are obtained in this paper

Key words: ring, multi-space, multi-ring space, ideal subspace chain.

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1. Introduction

The notion of multi-spaces is introduced by Smarandache in [6] under his idea of hybrid mathematics: *combining different fields into a unifying field*([7]), which is defined as follows.

Definition 1.1 *For any integer $i, 1 \leq i \leq n$ let A_i be a set with ensemble of law L_i, and the intersection of k sets $A_{i_1}, A_{i_2}, \cdots, A_{i_k}$ of them constrains the law $I(A_{i_1}, A_{i_2}, \cdots, A_{i_k})$. Then the union of A_i, $1 \leq i \leq n$*

$$\widetilde{A} = \bigcup_{i=1}^{n} A_i$$

is called a multi-space.

As we known, a set R with two binary operation $+$ and $\circ$, denoted by $(R\,;+,\circ)$, is said to be a *ring* if for $\forall x, y \in R$, $x+y \in R$, $x \circ y \in R$, the following conditions hold.

(i) $(R\,;+)$ is an abelian group;
(ii) $(R\,;\circ)$ is a semigroup;
(iii) For $\forall x, y, z \in R$, $x \circ (y+z) = x \circ y + x \circ z$ and $(x+y) \circ z = x \circ z + y \circ z$.

By combining Smarandache multi-spaces with rings, a new kind of algebraic structure called multi-ring space is found, which is defined in the following.

Definition 1.2 *Let $\widetilde{R} = \bigcup\limits_{i=1}^{m} R_i$ be a complete multi-space with double binary operation set $O(\widetilde{R}) = \{(+_i, \times_i), 1 \leq i \leq m\}$. If for any integers $i, j,\ i \neq j, 1 \leq i, j \leq m$, $(R_i; +_i, \times_i)$ is a ring and for $\forall x, y, z \in \widetilde{R}$,*

$$(x +_i y) +_j z = x +_i (y +_j z), \quad (x \times_i y) \times_j z = x \times_i (y \times_j z)$$

and

$$x \times_i (y +_j z) = x \times_i y +_j x \times_i z, \quad (y +_j z) \times_i x = y \times_i x +_j z \times_i x$$

if all their operation results exist, then $\widetilde{R}$ is called a multi-ring space. If for any integer $1 \leq i \leq m$, $(R; +_i, \times_i)$ is a filed, then $\widetilde{R}$ is called a multi-filed space.

For a multi-ring space $\widetilde{R} = \bigcup\limits_{i=1}^{m} R_i$, let $\widetilde{S} \subset \widetilde{R}$ and $O(\widetilde{S}) \subset O(\widetilde{R})$, if $\widetilde{S}$ is also a multi-ring space with double binary operation set $O(\widetilde{S})$, then call $\widetilde{S}$ a *multi-ring subspace* of $\widetilde{R}$. We have the following criterions for the multi-ring subspaces.

The subject of this paper is to find some characteristics of a multi-ring space. For terminology and notation not defined here can be seen in $[1], [5], [12]$ for algebraic terminologies and in $[2], [6] - [11]$ for multi-spaces and logics.

2. **Characteristics of a multi-ring space**

First, we have the following result for multi-ring subspace of a multi-ring space.

Theorem 2.1 *For a multi-ring space $\widetilde{R} = \bigcup\limits_{i=1}^{m} R_i$, a subset $\widetilde{S} \subset \widetilde{R}$ with $O(\widetilde{S}) \subset O(\widetilde{R})$ is a multi-ring subspace of $\widetilde{R}$ if and only if for any integer $k, 1 \leq k \leq m$, $(\widetilde{S} \bigcap R_k; +_k, \times_k)$ is a subring of $(R_k; +_k, \times_k)$ or $\widetilde{S} \bigcap R_k = \emptyset$.*

Proof For any integer $k, 1 \leq k \leq m$, if $(\widetilde{S} \bigcap R_k; +_k, \times_k)$ is a subring of $(R_k; +_k, \times_k)$ or $\widetilde{S} \bigcap R_k = \emptyset$, then since $\widetilde{S} = \bigcup\limits_{i=1}^{m} (\widetilde{S} \bigcap R_i)$, we know that $\widetilde{S}$ is a multi-ring subspace by definition of a multi-ring space.

Now if $\widetilde{S} = \bigcup\limits_{j=1}^{s} S_{i_j}$ is a multi-ring subspace of $\widetilde{R}$ with double binary operation set $O(\widetilde{S}) = \{(+_{i_j}, \times_{i_j}), 1 \leq j \leq s\}$, then $(S_{i_j}; +_{i_j}, \times_{i_j})$ is a subring of $(R_{i_j}; +_{i_j}, \times_{i_j})$. Therefore, for any integer $j, 1 \leq j \leq s$, $S_{i_j} = R_{i_j} \bigcap \widetilde{S}$. But for other integer $l \in \{i; 1 \leq i \leq m\} \setminus \{i_j; 1 \leq j \leq s\}$, $\widetilde{S} \bigcap S_l = \emptyset$. $\natural$

Applying the criterions for subrings of a ring, we get the following result.

Theorem 2.2 *For a multi-ring space $\widetilde{R} = \bigcup\limits_{i=1}^{m} R_i$, a subset $\widetilde{S} \subset \widetilde{R}$ with $O(\widetilde{S}) \subset O(\widetilde{R})$ is a multi-ring subspace of $\widetilde{R}$ if and only if for any double binary operations $(+_j, \times_j) \in O(\widetilde{S})$, $(\widetilde{S} \bigcap R_j; +_j) \prec (R_j; +_j)$ and $(\widetilde{S}; \times_j)$ is complete.*

Proof According to Theorem 2.1, we know that $\widetilde{S}$ is a multi-ring subspace if and only if for any integer $i, 1 \leq i \leq m$, $(\widetilde{S} \bigcap R_i; +_i, \times_i)$ is a subring of $(R_i; +_i, \times_i)$ or $\widetilde{S} \bigcap R_i = \emptyset$. By a well known criterions for subrings of a ring (see also [5]), we know that $(\widetilde{S}\bigcap R_i;+_i,\times_i)$ is a subring of $(R_i;+_i,\times_i)$ if and only if for any double binary operations $(+_j,\times_j)\in O(\widetilde{S})$, $(\widetilde{S}\bigcap R_j;+_j)\prec(R_j;+_j)$ and $(\widetilde{S};\times_j)$ is a complete set. This completes the proof. $\natural$

We use the *ideal subspace chain* of a multi-ring space to characteristic its structure properties. An *ideal subspace* $\widetilde{I}$ of a multi-ring space $\widetilde{R}=\bigcup\limits_{i=1}^{m}R_i$ with double binary operation set $O(\widetilde{R})$ is a multi-ring subspace of $\widetilde{R}$ satisfying the following conditions:

(i) $\widetilde{I}$ is a multi-group subspace with operation set $\{+|\ (+,\times)\in O(\widetilde{I})\}$;

(ii) for any $r\in\widetilde{R}, a\in\widetilde{I}$ and $(+,\times)\in O(\widetilde{I})$, $r\times a\in\widetilde{I}$ and $a\times r\in\widetilde{I}$ if their operation results exist.

Theorem 2.3 *A subset $\widetilde{I}$ with $O(\widetilde{I}), O(\widetilde{I})\subset O(\widetilde{R})$ of a multi-ring space $\widetilde{R}=\bigcup\limits_{i=1}^{m}R_i$ with double binary operation set $O(\widetilde{R})=\{(+_i,\times_i)|\ 1\le i\le m\}$ is an ideal subspace if and only if for any integer $i, 1\le i\le m$, $(\widetilde{I}\bigcap R_i,+_i,\times_i)$ is an ideal of the ring $(R_i,+_i,\times_i)$ or $\widetilde{I}\bigcap R_i=\emptyset$.*

Proof By definition of an ideal subspace, the necessity of the condition is obvious.

For the sufficiency, denote by $\widetilde{R}(+,\times)$ the set of elements in $\widetilde{R}$ with binary operations $+$ and $\times$. If there exists an integer i such that $\widetilde{I}\bigcap R_i\neq\emptyset$ and $(\widetilde{I}\bigcap R_i,+_i,\times_i)$ is an ideal of $(R_i,+_i,\times_i)$, then for $\forall a\in\widetilde{I}\bigcap R_i$, $\forall r_i\in R_i$, we know that

$$r_i\times_i a\in\widetilde{I}\bigcap R_i;\quad a\times_i r_i\in\widetilde{I}\bigcap R_i.$$

Notice that $\widetilde{R}(+_i,\times_i)=R_i$. Therefore, we get that for $\forall r\in\widetilde{R}$,

$$r\times_i a\in\widetilde{I}\bigcap R_i;\ \text{and}\ a\times_i r\in\widetilde{I}\bigcap R_i,$$

if their operation result exist. Whence, $\widetilde{I}$ is an ideal subspace of $\widetilde{R}$. $\natural$

An ideal subspace $\widetilde{I}$ of a multi-ring space $\widetilde{R}$ is said *maximal* if for any ideal subspace $\widetilde{I}'$, if $\widetilde{R}\supseteq\widetilde{I}'\supseteq\widetilde{I}$, then $\widetilde{I}'=\widetilde{R}$ or $\widetilde{I}'=\widetilde{I}$. For any order of the double binary operation set $O(\widetilde{R})$ of a multi-ring space $\widetilde{R}=\bigcup\limits_{i=1}^{m}R_i$, not loss of generality, assume it being $(+_1,\times_1)\succ(+_2,\times_2)\succ\cdots\succ(+_m,\times_m)$, we can define an *ideal subspace chain* of $\widetilde{R}$ by the following programming.

(i) Construct the ideal subspace chain

$$\widetilde{R}\supset\widetilde{R}_{11}\supset\widetilde{R}_{12}\supset\cdots\supset\widetilde{R}_{1s_1}$$

under the double binary operation $(+_1,\times_1)$, where $\widetilde{R}_{11}$ is a maximal ideal subspace of $\widetilde{R}$ and in general, for any integer i, $1\le i\le m-1$, $\widetilde{R}_{1(i+1)}$ is a maximal ideal subspace of $\widetilde{R}_{1i}$.

(ii) If the ideal subspace

$$\widetilde{R} \supset \widetilde{R}_{11} \supset \widetilde{R}_{12} \supset \cdots \supset \widetilde{R}_{1s_1} \supset \cdots \supset \widetilde{R}_{i1} \supset \cdots \supset \widetilde{R}_{is_i}$$

has been constructed for $(+_1, \times_1) \succ (+_2, \times_2) \succ \cdots \succ (+_i, \times_i)$, $1 \leq i \leq m-1$, then construct an ideal subspace chain of $\widetilde{R}_{is_i}$

$$\widetilde{R}_{is_i} \supset \widetilde{R}_{(i+1)1} \supset \widetilde{R}_{(i+1)2} \supset \cdots \supset \widetilde{R}_{(i+1)s_1}$$

under the operations $(+_{i+1}, \times_{i+1})$, where $\widetilde{R}_{(i+1)1}$ is a maximal ideal subspace of $\widetilde{R}_{is_i}$ and in general, $\widetilde{R}_{(i+1)(i+1)}$ is a maximal ideal subspace of $\widetilde{R}_{(i+1)j}$ for any integer $j, 1 \leq j \leq s_i - 1$. Define the ideal subspace chain of $\widetilde{R}$ under $(+_1, \times_1) \succ (+_2, \times_2) \succ \cdots \succ (+_{i+1}, \times_{i+1})$ being

$$\widetilde{R} \supset \widetilde{R}_{11} \supset \cdots \supset \widetilde{R}_{1s_1} \supset \cdots \supset \widetilde{R}_{i1} \supset \cdots \supset \widetilde{R}_{is_i} \supset \widetilde{R}_{(i+1)1} \supset \cdots \supset \widetilde{R}_{(i+1)s_{i+1}}.$$

Similar to a multi-group space([3]), we have the following results for the ideal subspace chain of a multi-ring space.

Theorem 2.4 *For a multi-ring space $\widetilde{R} = \bigcup\limits_{i=1}^{m} R_i$, its ideal subspace chain only has finite terms if and only if for any integer $i, 1 \leq i \leq m$, the ideal chain of the ring $(R_i; +_i, \times_i)$ has finite terms, i.e., each ring $(R_i; +_i, \times_i)$ is an Artin ring.*

Proof Let the order of double operations in $\overrightarrow{O}(\widetilde{R})$ be

$$(+_1, \times_1) \succ (+_2, \times_2) \succ \cdots \succ (+_m, \times_m)$$

and a maximal ideal chain in the ring $(R_1; +_1, \times_1)$ is

$$R_1 \succ R_{11} \succ \cdots \succ R_{1t_1}.$$

Calculate

$$\widetilde{R}_{11} = \widetilde{R} \setminus \{R_1 \setminus R_{11}\} = R_{11} \bigcup (\bigcup_{i=2}^{m}) R_i,$$

$$\widetilde{R}_{12} = \widetilde{R}_{11} \setminus \{R_{11} \setminus R_{12}\} = R_{12} \bigcup (\bigcup_{i=2}^{m}) R_i,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\widetilde{R}_{1t_1} = \widetilde{R}_{1t_1} \setminus \{R_{1(t_1-1)} \setminus R_{1t_1}\} = R_{1t_1} \bigcup (\bigcup_{i=2}^{m}) R_i.$$

According to Theorem 3.10, we know that

$$\widetilde{R} \supset \widetilde{R}_{11} \supset \widetilde{R}_{12} \supset \cdots \supset \widetilde{R}_{1t_1}$$

is a maximal ideal subspace chain of $\widetilde{R}$ under the double binary operation $(+_1, \times_1)$. In general, for any integer $i, 1 \leq i \leq m-1$, assume

$$R_i \succ R_{i1} \succ \cdots \succ R_{it_i}$$

is a maximal ideal chain in the ring $(R_{(i-1)t_{i-1}}; +_i, \times_i)$. Calculate

$$\widetilde{R}_{ik} = R_{ik} \bigcup (\bigcup_{j=i+1}^{m}) \widetilde{R}_{ik} \bigcap R_i$$

Then we know that

$$\widetilde{R}_{(i-1)t_{i-1}} \supset \widetilde{R}_{i1} \supset \widetilde{R}_{i2} \supset \cdots \supset \widetilde{R}_{it_i}$$

is a maximal ideal subspace chain of $\widetilde{R}_{(i-1)t_{i-1}}$ under the double operation $(+_i, \times_i)$ by Theorem 3.10. Whence, if for any integer $i, 1 \leq i \leq m$, the ideal chain of the ring $(R_i; +_i, \times_i)$ has finite terms, then the ideal subspace chain of the multi-ring space $\widetilde{R}$ only has finite terms and if there exists one integer i_0 such that the ideal chain of the ring $(R_{i_0}, +_{i_0}, \times_{i_0})$ has infinite terms, then there must be infinite terms in the ideal subspace chain of the multi-ring space $\widetilde{R}$. $\natural$.

A multi-ring space is called an *Artin multi-ring space* if each ideal subspace chain only has finite terms. We have the following corollary by Theorem 3.11.

Corollary 2.1 *A multi-ring space $\widetilde{R} = \bigcup_{i=1}^{m}$ with double binary operation set $O(\widetilde{R}) = \{(+_i, \times_i) |\ 1 \leq i \leq m\}$ is an Artin multi-ring space if and only if for any integer $i, 1 \leq i \leq m$, the ring $(R_i; +_i, \times_i)$ is an Artin ring.*

For a multi-ring space $\widetilde{R} = \bigcup_{i=1}^{m}$ with double binary operation set $O(\widetilde{R}) = \{(+_i, \times_i) |\ 1 \leq i \leq m\}$, an element e is an *idempotent* element if $e_\times^2 = e \times e = e$ for a double binary operation $(+, \times) \in O(\widetilde{R})$. We define the *directed sum* $\widetilde{I}$ of two ideal subspaces $\widetilde{I}_1$ and $\widetilde{I}_2$ as follows:

(i) $\widetilde{I} = \widetilde{I}_1 \bigcup \widetilde{I}_2$;

(ii) $\widetilde{I}_1 \bigcap \widetilde{I}_2 = \{0_+\}$, or $\widetilde{I}_1 \bigcap \widetilde{I}_2 = \emptyset$, where 0_+ denotes an unit element under the operation $+$.

Denote the directed sum of $\widetilde{I}_1$ and $\widetilde{I}_2$ by

$$\widetilde{I} = \widetilde{I}_1 \bigoplus \widetilde{I}_2.$$

If for any $\widetilde{I}_1, \widetilde{I}_2$, $\widetilde{I} = \widetilde{I}_1 \bigoplus \widetilde{I}_2$ implies that $\widetilde{I}_1 = \widetilde{I}$ or $\widetilde{I}_2 = \widetilde{I}$, then $\widetilde{I}$ is called *non-reducible*. We have the following result for the Artin multi-ring space similar to a well-known result for the Artin ring (see [12]).

Theorem 2.5 *Any Artin multi-ring space* $\widetilde{R} = \bigcup\limits_{i=1}^{m} R_i$ *with double binary operation set* $O(\widetilde{R}) = \{(+_i, \times_i)|\ 1 \leq i \leq m\}$ *is a directed sum of finite non-reducible ideal subspaces, and if for any integer* $i, 1 \leq i \leq m$, $(R_i; +_i, \times_i)$ *has unit* $1_{\times_i}$, *then*

$$\widetilde{R} = \bigoplus_{i=1}^{m}(\bigoplus_{j=1}^{s_i}(R_i \times_i e_{ij})\bigcup(e_{ij} \times_i R_i)),$$

where $e_{ij}, 1 \leq j \leq s_i$ *are orthogonal idempotent elements of the ring* R_i.

Proof Denote by $\widetilde{M}$ the set of ideal subspaces which can not be represented by a directed sum of finite ideal subspaces in $\widetilde{R}$. According to Theorem 3.11, there is a minimal ideal subspace $\widetilde{I}_0$ in $\widetilde{M}$. It is obvious that $\widetilde{I}_0$ is reducible.

Assume that $\widetilde{I}_0 = \widetilde{I}_1 + \widetilde{I}_2$. Then $\widetilde{I}_1 \not\in \widetilde{M}$ and $\widetilde{I}_2 \not\in \widetilde{M}$. Therefore, $\widetilde{I}_1$ and $\widetilde{I}_2$ can be represented by directed sums of finite ideal subspaces. Whence, $\widetilde{I}_0$ can be also represented by a directed sum of finite ideal subspaces. Contradicts that $\widetilde{I}_0 \in \widetilde{M}$.

Now let

$$\widetilde{R} = \bigoplus_{i=1}^{s}\widetilde{I}_i,$$

where each $\widetilde{I}_i, 1 \leq i \leq s$, is non-reducible. Notice that for a double operation $(+, \times)$, each non-reducible ideal subspace of $\widetilde{R}$ has the form

$$(e \times R(\times))\bigcup(R(\times) \times e), \quad e \in R(\times).$$

Whence, we know that there is a set $T \subset \widetilde{R}$ such that

$$\widetilde{R} = \bigoplus_{e\in T,\ \times\in O(\widetilde{R})}(e \times R(\times))\bigcup(R(\times) \times e).$$

For any operation $\times \in O(\widetilde{R})$ and the unit $1_\times$, assume that

$$1_\times = e_1 \oplus e_2 \oplus \cdots \oplus e_l,\ e_i \in T,\ 1 \leq i \leq s.$$

Then

$$e_i \times 1_\times = (e_i \times e_1) \oplus (e_i \times e_2) \oplus \cdots \oplus (e_i \times e_l).$$

Therefore, we get that

$$e_i = e_i \times e_i = e_i^2 \ \text{ and } \ e_i \times e_j = 0_i \ \text{ for } \ i \neq j.$$

That is, $e_i, 1 \leq i \leq l$, are orthogonal idempotent elements of $\widetilde{R}(\times)$. Notice that $\widetilde{R}(\times) = R_h$ for some integer h. We know that $e_i, 1 \leq i \leq l$ are orthogonal idempotent elements of the ring $(R_h, +_h, \times_h)$. Denoted by e_{hj} for e_j, $1 \leq j \leq l$. Consider all units in $\widetilde{R}$, we get that

$$\widetilde{R} = \bigoplus_{i=1}^{m} (\bigoplus_{j=1}^{s_i} (R_i \times_i e_{ij}) \bigcup (e_{ij} \times_i R_i)).$$

This completes the proof. $\natural$

Corollary 2.2([12]) *Any Artin ring* $(R\ ; +, \times)$ *is a directed sum of finite ideals, and if* $(R\ ; +, \times)$ *has unit* $1_\times$*, then*

$$R = \bigoplus_{i=1}^{s} R_i e_i,$$

where $e_i, 1 \leq i \leq s$ *are orthogonal idempotent elements of the ring* $(R; +, \times)$*.*

3. **Open problems for a multi-ring space**

Similar to Artin multi-ring space, we can also define *Noether multi-ring spaces, simple multi-ring spaces, half-simple multi-ring spaces,* $\cdots$*, etc..* The open problems for these new algebraic structure are as follows.

Problem 3.1 Call a ring R a Noether ring if its every ideal chain only has finite terms. Similarly, for a multi-ring space $\widetilde{R}$, if its every ideal multi-ring subspace chain only has finite terms, it is called a Noether multi-ring space. *Whether can we find its structures similar to Corollary* 2.2 *and Theorem* 2.5*?*

Problem 3.2 *Similar to ring theory, define a Jacobson or Brown-McCoy radical for multi-ring spaces and determine their contribution to multi-ring spaces.*

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