Scientiae Mathematicae Japonicae, **64**, No. 3 (2006), 589–594, :**e2006**, 1003–1008

ON PRE-COXETER ALGEBRAS

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Received June 13, 2006

ABSTRACT. In this paper we show that the class of PC-algebras and the class of B-algebras with condition (D) are Smarandache disjoint, and show that an algebra $(X;*,0)$ is a Coxeter algebra if and only if it is a PC-algebra with (N). Moreover, we show that there is no non-trivial quadratic PC-algebras on a field with $|X| \geq 3$.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([5, 6]). Y. B. Jun, E. H. Roh and H. S. Kim ([7]) introduced a new notion, called a BH-algebra, i.e., (I), (II) and (V) $x * y = 0$ and $y * x = 0$ imply $x = y$, which is a generalization of $BCH/BCI/BCK$-algebras. J. Neggers and H. S. Kim ([10]) introduced and investigated a class of algebras, called a *B-algebra* which is related to several classes of algebras of interest such as $BCH/BCI/BCK$-algebras. Furthermore, they demonstrated a rather interesting connection between B-algebras and groups. P. J. Allen et al. ([1]) included several new families of Smarandache-type P-algebras and studied some of their properties in relation to the properties of previously defined Smarandache-types. Recently, Kim et al. ([8]) introduced the notion of a (pre-)Coxeter algebra and showed that a Coxeter algebra is equivalent to an abelian group all of whose elements have order 2, i.e., a Boolean group. Moreover, they proved that the class of Coxeter algebras and the class of B-algebras of odd order are Smarandache disjoint. In this paper we show that the class of PC-algebras and the class of B-algebras with condition (D) are Smarandache disjoint, and show that an algebra $(X;*,0)$ is a Coxeter algebra if and only if it is a PC-algebra with (N). Moreover, we show that there is no non-trivial quadratic PC-algebras on a field with $|X| \geq 3$.

for any $x, y, z \in X$. An example of a Coxeter algebra is a Klein 4-group.

Theorem 2.1. *If $(X;*,0)$ is a Coxeter algebra, then it is an abelian group all of whose elements have order 2, i.e., a Boolean group, and conversely.*

J. Neggers and H. S. Kim introduced and investigated a class of algebras, called a B-algebra, which is related to several classes of algebras such as $BCH/BCI/BCK$-algebras. A *B-algebra* ([10]) is a non-empty set X with a constant 0 and a binary operation "$*$" satisfying the following axioms: (B1), (B2) and (B) $(x * y) * z = x * (z * (0 * y))$, for any $x, y, z \in X$.

Proposition 2.2. *If $(X;*,0)$ is a Coxeter algebra, then it is a B-algebra.*

An algebra $(X;*,0)$ is called a *pre-Coxeter algebra* (shortly, *PC-algebra*) if it satisfies the axioms (B1), (B2), (PC1) $x * y = y * x$, (PC2) $x * y = 0 \Longrightarrow x = y$, for any $x, y \in X$.

Example 2.3. Let $X := [0, \infty)$. If we define $x * y := |x - y|, x, y \in X$, then $(X;*,0)$ is a pre-Coxeter algebra, but not a Coxeter algebra, since $(1 * 2) * 3 = 2$, but $1 * (2 * 3) = 0$.

Proposition 2.4. *Every Coxeter algebra is a pre-Coxeter algebra.*

Proposition 2.5. *Let $(X;*,0)$ be a Coxeter algebra. Then $x * (x * y) = y$, for any $x, y \in Y$.*

3. pre-Coxeter algebras and B-algebras in Smarandache settings

Let $(X,*)$ be a binary system/algebra. Then $(X,*)$ is a *Smarandache-type P-algebra* if it contains a subalgebra $(Y,*)$, where Y is non-trivial, i.e., $|Y| \geq 2$, or Y contains at least two distinct elements, and $(Y,*)$ is itself of type P. Thus, we have *Smarandache-type semigroups* (the type P-algebra is a semigroup), *Smarandache-type groups* (the type P-algebra is a group), *Smarandache-type abelian groups* (the type P-algebra is an abelian group). A Smarandache semigroup in the sense of Kandasamy is in fact a Smarandache-type group (see [11]). Smarandache-type groups are of course a larger class than Kandasamy's Smarandache semigroups since they may include non-associative algebras as well.

Given algebra types $(X,*)$ (type-P_1) and $(X,\circ)$ (type-P_2), we shall consider them to be *Smarandache disjoint* ([1]) if the following two conditions hold:

(A) If $(X,*)$ is a type-P_1-algebra with $|X| > 1$ then it cannot be a Smarandache-type-P_2-algebra $(X,\circ)$;

(B) If $(X,\circ)$ is a type-P_2-algebra with $|X| > 1$ then it cannot be a Smarandache-type-P_1-algebra $(X,*)$.

Theorem 3.1. ([8]) *The class of Coxeter algebras and the class of B-algebras of odd order are Smarandache disjoint.*

Lemma 3.2. *If $(X;*,0)$ is a pre-Coxeter algebra with (B), then $(X;*,0)$ is a Coxeter algebra.*

Proof. For any $x, y, z \in X$, we have

$$\begin{aligned}
(x * y) * z &= x * (z * (0 * y) & [(B)] \\
&= x * (z * (y * 0)) & [(PC1)] \\
&= x * (z * y) & [(B2)] \\
&= x * (y * z), & [(PC1)]
\end{aligned}$$

Theorem 2.3. *If $(X;*,0)$ is a non-trivial super commutative d-algebra, then it cannot contain a BCK-algebra $(A;*,0)$ with $|A| \geq 3$.*

Proof. Note that if $x \in X$, then $x*x = 0*x = 0$ and if $x*0 = x$, then $\{x,0\}$ is a two-element BCK-algebra. Thus, arbitrary d-algebras may easily contain two element subalgebras which are BCK-algebras. Assume that $(A;*,0)$ is a BCK-algebra with $|A| \geq 3$. Let $x, y \in A - \{0\}$ with $x \neq y$. Then $x*y = y*x \neq 0$, since X is super commutative. We claim that $x = x*y, y = y*x$. Suppose that $x \neq x*y$. Then $x*(x*y) = (x*y)*x \neq 0$, since X is super commutative. Since A is a BCK-algebra, we have $(x*y)*x = 0$. Thus $0 = (x*y)*x \neq 0$, a contradiction. Similarly, $y = y*x$. Since X is super commutative, $x = x*y = y*x = y$, a contradiction. Thus X cannot contain a BCK-algebra $(A;*,0)$ with $|A| \geq 3$. □

Let $(X,*)$ be a binary system/algebra. Then $(X,*)$ is a *Smarandache-type P-algebra* if it contains a subalgebra $(Y,*)$, where Y is non-trivial, i.e., $|Y| \geq 2$, or Y contains at least two distinct elements, and $(Y,*)$ is itself of type P. Thus, we have *Smarandache-type semigroups* (the type P-algebra is a semigroup), *Smarandache-type groups* (the type P-algebra is a group), *Smarandache-type abelian groups* (the type P-algebra is an abelian group). Smarandache semigroup in the sense of Kandasamy is in fact a Smarandache-type group (see [2]). Smarandache-type groups are of course a larger class than Kandasamy's Smarandache semigroups since they may include non-associative algebras as well.

Given algebra types $(X,*)$ (type-P_1) and $(X,\circ)$ (type-P_2), we shall consider them to be *Smarandache disjoint* if the following two conditions hold:

(A) If $(X,*)$ is a type-P_1-algebra with $|X| > 1$ then it cannot be a Smarandache-type-P_2-algebra $(X,\circ)$;

(B) If $(X,\circ)$ is a type-P_2-algebra with $|X| > 1$ then it cannot be a Smarandache-type-P_1-algebra $(X,*)$.

A BCK-algebra $(X;*,0)$ is said to be *strict* if $|X| \geq 3$. Putting Theorem 2.2 and 2.3 together we obtain the following conclusion:

Theorem 2.4. *The class of strict BCK-algebras and the class of non-trivial super commutative d-algebras are Smarandache disjoint.*

Another construction of super commutative d-algebras. Consider the set $\underline{n} := \{0,1,2,\cdots,n\}$ and define a product $*$ by $0*0 = 0*k = 0$; $k*0 = k$; $i*j = j*i = |i-j|$. Then $(\underline{n} : *, 0)$ is a super commutative d-algebra. For example, if $n = 2$, then we have a table:

*	0	1	2
0	0	0	0
1	1	0	1
2	2	1	0

which is another non-trivial super commutative d-algebra of order 3, not isomorphic to Example 2.1.

A class $\{(X_i,*)\}$ of algebras is said to be *super Smarandache* if it contains subclasses $\{(A_i,*)\}$ and $\{(B_i,*)\}$ such that these algebras are Smarandache disjoint. As an example, the class of groups and the class of left semigroups are both classes of semigroups which we have shown to be Smarandache disjoint.If we take the class of cyclic groups of prime

4. PC-algebras and $BF/BF_2/Coxeter$-algebras

A. Walendiziak ([12]) introduced the notion of BF-algebras, and showed a very good diagram to understand the address of various algebras which are related to BF-algebras.

Definition 4.1. An algebra $(X;*,0)$ is called a *BF-algebra* ([3]) if it satisfies (B1),(B2) and (BF) $0*(x*y) = y*x$, for any $x, y \in X$.

Proposition 4.2. *Every PC-algebra is a BF-algebra.*

Proof. Let $(X;*,0)$ be a PC-algebra. Then, for any $x, y \in X$, we have

$$\begin{aligned} 0*(x*y) &= (x*y)*0 && [(PC1)] \\ &= x*y && [(B2)] \\ &= y*x, && [(PC1)] \end{aligned}$$

proving that X is a BF-algebra. □

A BF-algebra $(X;*,0)$ is called a *BF_2-algebra* ([12]) if it satisfies (V) $x*y=0$ and $y*x=0$ imply $x=y$.

Corollary 4.3. *If $(X,*)$ is a PC-algebra, then $(X,*)$ is also a BF_2-algebra.*

Proof. It can be easily proved by (PC2). □

Proposition 4.4. *If a PC-algebra $(X;*,0)$ satisfies the condition*

(N) $(x*z)*(y*z) = x*y,$

then it is a Coxeter algebra.

Proof. Given $x, y, z \in X$, we have

$$\begin{aligned} (x*y)*z &= ((x*y)*y)*(z*y) && [(N)] \\ &= ((x*y)*(y*0))*(z*y) && [(B2)] \\ &= ((x*y)*(0*y))*(z*y) && [(PC1)] \\ &= (x*0)*(z*y) && [(N)] \\ &= x*(z*y) && [(B2)] \\ &= x*(y*z) && [(PC1)], \end{aligned}$$

proving the proposition. □

Lemma 4.5. *If $(X;*,0)$ is a Coxeter algebra, then $(x*y)*x = y$ for any $x, y \in X$.*

Proof. Given $x, y \in X$, we have $(x*y)*x = (x*y)*(x*0) = (x*y)*[x*(y*y)] = (x*y)*[(x*y)*y] = [(x*y)*(x*y)]*y = 0*y = y$, proving the lemma. □

Proposition 4.6. *If $(X;*,0)$ is a Coxeter algebra, then it satisfies the condition (N).*

Proof. By applying Lemma 4.5, we have $(x*z)*(y*z) = x*(z*(y*z)) = x*((z*y)*z) = x*y$, proving the proposition. □

Proof. Assume that $(Y;*,0)$ is both a B-algebra with condition (D) and a PC-algebra with $Y \subset X$, $|Y| \geq 2$. Then $0 = a * a = a * (a * 0) = a * (0 * a)$ by applying (B1), (B2) and (PC1), which leads to a contradiction. □

Proposition 3.7. *Let $(X;*,0)$ be a B-algebra with condition (D). Then X can not be a Smarandache-type PC-algebra.*

Proof. Similar to Proposition 3.7. □

It follows from Propositions 3.6 and 3.7 that:

Theorem 3.8. *The class of PC-algebras and the class of B-algebras with condition (D) are Smarandache disjoint.*

proving the lemma. □

Proposition 3.3. *The class of PC-algebras and the class of B-algebras of odd order are Smarandache disjoint.*

Proof. Assume that a PC-algebra $(X;*,0)$ contains a B-algebra $(Y;*,0)$ of odd order where $|Y| \geq 2$. By applying Lemma 3.2, we obtain that Y is a Coxeter algebra. It follows from Theorem 2.3 that Y can not be a non-trivial a B-algebra of odd order, a contradiction.

Conversely, Assume that a B-algebra $(X;*,0)$ of odd order contains a PC-algebra $(Y;*,0)$ where $|Y| \geq 2$. Then $(Y;*,0)$ satisfies the condition (B). By Lemma 3.2 it is a Coxeter algebra, which leads to a contradiction by Theorem 3.1. □

A B-algebra $(X;*,0)$ is said to have the *condition* (D) if $x * (0 * x) \neq 0$ for any $x \neq 0$ in X.

Example 3.4. Let $X := \{0, 1, 2\}$ be a set with the following table:

*	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Then it is a B-algebra ([10]) with condition (D).

Example 3.5. Let X be the set of all real numbers except for a negative integer $-n$. Define a binary operation $*$ on X by

$$x * y := \frac{n(x-y)}{n+y}.$$

Then $(X;*,0)$ is a B-algebra ([10]), but it does not have condition (D), since $(-2n) * (0 * (-2n)) = 0$.

Proposition 3.6. *Let $(X;*,0)$ be a PC-algebra. Then X can not be a Smarandache-type B-algebra with condition (D).*

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