Thesis for the Doctor of Philosophy

On the structure of general algebras and its applications

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ABSTRACT

On the structure of general algebras and its applications

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In this thesis, we discuss some structural theory of a d-algebra which is a generalization of a BCK-algebra, and we discuss analytic real algebras. We investigate several conditions for analytic real algebras to be d-algebras. Moreover, we introduce the notion of a Smarandacheness to BCI-algebras, and obtain several properties on Smarandache fuzzy BCI-algebras.

1. Introduction

The notions of BCK-algebras and BCI-algebras were introduced by Y. Imai and K. Iséki ([5, 6]). The class of BCK-algebras is a proper subclass of the class of BCI-algebras. We refer useful textbooks for BCK-algebras and BCI-algebras to ([4, 12, 17]). The notion of a d-algebra which is another useful generalization of BCK-algebras was introduced by J. Neggers, Y. B. Jun and H. S. Kim ([14]), and some relations between d-algebras and BCK-algebras as well as several other relations between d-algebras and oriented digraphs were investigated. Several aspects on d-algebras were studied ([1, 3, 10, 11, 13, 14]). Simply, d-algebras can be obtained by deleting two identities as a generalization of BCK-algebras, but it gives more wide ranges of research areas in algebraic structures. Also, J. Neggers, Y. B. Jun and H. S. Kim ([14]) discussed the ideal theory in d-algebras, and introduced the notions of a d-subalgebra, a d-ideal, a $d^{\#}$-ideal and a d^{*}-ideal, and investigated relations among them. Also, a Smarandache structure on a set A means a weak structure W on A such that there exists a proper subset B of A with a strong structure S which is embedded in A. In [16], R. Padilla showed that Smarandache semigroups are very important for the study of congruences. Y. B. Jun ([9]) introduced the notion of Smarandache BCI-algebras, Smarandache fresh and clean ideals of Smarandache BCI-algebras, and obtained many interesting results about them.

In Chapter 2, we study basic facts and useful properties of BCK-algebras, BCI-algebras and d-algebras which are related to the topics. In Chapter 3, we discuss structural properties of quotient d-algebras. We obtain several isomorphism theorems in quotient d-algebras, and we introduce the notion of an obstinate ideal in d-algebras, and obtain its equivalent conditions. In Chapter 4, we introduce the notion of an analytic real algebra, and we obtain some conditions to be a d-algebra. Moreover, we generalize a binary operation on the set $\mathbb{R}$ of real numbers by using real-valued functions, and obtain some conditions to be an edge d-algebra. In Chapter 5, we introduce the notion of a Smarandache concept to BCI-algebras, and discuss Smarandache fuzzy ideals in Smarandache BCI-algebras. Moreover, we discuss Smarandache fuzzy clean ideals and Smarandache fuzzy fresh ideals in Smarandache BCI-algebras.

2. Preliminaries

In this chapter, we provide several definitions and theorems which are useful in the study of d-algebras and Smarandache (fuzzy) BCI-algebras.

2.1. d-algebras

Definition 2.1. ([15]) A *d-algebra* is a non-empty set X with a constant 0 and a binary operation " $*$ " satisfying the following axioms:

(I) $x * x = 0$,

(II) $0 * x = 0$,

(III) $x * y = 0$ and $y * x = 0$ imply $x = y$

for all $x, y \in X$.

For brevity we also call X a *d-algebra.* In X we can define a binary relation " $\leq$ " by $x \leq y$ if and only if $x * y = 0$.

Definition 2.2. ([3]) An algebra $(X, *, 0)$ is said to be a *strong d-algebra* if it satisfies (I), (II) and (III*) for all $x, y \in X$, where

(III*) $x * y = y * x$ implies $x = y$.

Obviously, every strong d-algebra is a d-algebra, but the converse need not be true in general (see [3]).

Example 2.3. ([3]) Let $\mathbb{R}$ be the set of all real numbers and $e \in \mathbb{R}$. Define $x * y := (x - y) \cdot (x - e) + e$ for all $x, y \in \mathbb{R}$ where "$\cdot$" and "$-$" are the ordinary product and subtraction of real numbers. Then it is easy to see that $x * x = e$ and $e * x = e$. If $x * y = y * x = e$ then $(x - y) \cdot (x - e) = 0$, $(y - x) \cdot (y - e) = 0$, and hence $x = y$ or $x = e = y$, i.e., $x = y$, i.e., $(\mathbb{R}, *, e)$ is a d-algebra.
However, $(\mathbb{R}, *, e)$ is not a strong d-algebra. We can easily see that

$$\begin{aligned} x * y = y * x &\Leftrightarrow (x - y) \cdot (x - e) + e = (y - x) \cdot (y - e) + e \\ &\Leftrightarrow (x - y) \cdot (x - e) = -(x - y) \cdot (y - e) \\ &\Leftrightarrow (x - y) \cdot (x - e + y - e) = 0 \\ &\Leftrightarrow (x - y) \cdot (x + y - 2e) = 0 \\ &\Leftrightarrow x = y \ \text{ or } \ x + y = 2e. \end{aligned}$$

If we take $x := e + \alpha$ and $y := e - \alpha$ for some $\alpha \in \mathbb{R}$, then $x + y = 2e$. This shows that $x * y = y * x$, but $x \neq y$. Hence the axiom (III*) does not hold. This shows that $(\mathbb{R}, *, e)$ is a d-algebra, but not a strong d-algebra.

Definition 2.4. ([12]) A *BCK-algebra* is a d-algebra X satisfying the following additional axioms:

(IV) $((x * y) * (x * z)) * (z * y) = 0$,

(V) $(x * (x * y)) * y = 0$

for all $x, y, z \in X$.

Example 2.5. ([14]) Let $X := \{0, 1, 2, 3, 4\}$ be a set with the following table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	3	0
3	3	3	2	0	3
4	4	4	1	1	0

Then $(X, *, 0)$ is a d-algebra which is not a BCK-algebra, since $((2 * 3) * (2 * 4)) * (4 * 3) = (3 * 0) * 1 \neq 0$.

Let X be a d-algebra and $x \in X$. X is said to be *edge* if for any $x \in X$, $x * X = \{x, 0\}$. It is known that if X is an edge d-algebra, then $x * 0 = x$ for any $x \in X$ (see [14]).

Definition 2.6. ([14]) An algebra $(X, *, 0)$ is called a *BCI-algebra* if it satisfies (I), (III), (IV) and (V) for all $x, y, z \in X$

Every BCI-algebra X has the following properties:

(a_1) $x * 0 = x$,

(a_2) $x \leq y$ implies $x * z \leq y * z, z * y \leq z * x$

for all $x, y, z \in X$.

2.2. d-ideals in d-algebras

Definition 2.7. ([14]) Let $(X, *, 0)$ be a d-algebra and $\emptyset \neq I \subseteq X$. I is called a *d-subalgebra* of X if $x*y \in I$ whenever $x \in I$ and $y \in I$. I is called a *BCK-ideal* of X if it satisfies the following conditions:

(D_0) $0 \in I$,

(D_1) $x * y \in I$, $y \in I$ imply $x \in I$ for all $x, y \in X$.

A non-empty subset I is called a *d-ideal* of X if it satisfies (D_1) and

(D_2) $x \in I$ and $y \in X$ imply $x * y \in I$ for all $x, y \in X$.

A d-ideal I of a d-algebra X is called a *$d^{\#}$-ideal* of X if for any $x, y, z \in I$,

(D_3) $x * y \in I$, $y * z \in I$ imply $x * z \in I$.

A $d^{\#}$-ideal I of a d-algebra X is called a *d^{*}-ideal* of X if for any $x, y, z \in X$,

(D_4) $x * y \in I$ and $y * x \in X$ imply $(x * z) * (y * z) \in I$ and $(z * x) * (z * y) \in I$.

Example 2.8. ([14]) Let $X := \{0, a, b, c, d\}$ be a d-algebra which is not a BCK-algebra with the following table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	a
b	b	b	0	c	0
c	c	c	b	0	c
d	c	c	a	a	0

Then $I := \{0, a\}$ is a d-ideal of a d-algebra X.

Example 2.9. ([14]) Let $X := \{0, a, b, c\}$ be a d-algebra which is not a BCK-algebra with the following table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	b
b	b	c	0	0
c	c	c	c	0

Then $I := \{0, a, b\}$ is a BCK-ideal which is not a d-subalgebra of X, while $J := \{0, c\}$ is a d-subalgebra of X which is not a BCK-ideal of X. Moreover, $K := \{0, a\}$ is a d^*-ideal of X.

Clearly, $\{0\}$ is a d-subalgebra of every d-algebra X and every d-ideal of X is a d-subalgebra, but the converse need not be true.

Example 2.10. ([14]) Let $X := \{0, a, b, c\}$ be a d-algebra which is not a BCK-algebra with the following table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	b
b	b	b	0	0
c	c	c	c	0

Then $I := \{0, a\}$ is a d-subalgebra of X, but not a d-ideal of X, since $a * c = b \notin I$.

Lemma 2.11. ([14]) *If I is a d-ideal of a d-algebra X, then $0 \in I$.*

Note that every d-ideal of a d-algebra is a BCK-ideal, but the converse need not be true. In Example 2.10, $I := \{0, a\}$ is a BCK-ideal of X, but not a d-ideal of X.

Proposition 2.12. ([14]) *Let I be a d-ideal of a d-algebra X, If $x \in I$ and $y * x = 0$, then $y \in I$.*

Theorem 2.13. ([14]) *In a d*-algebra, every BCK-ideal is a d-ideal.*

Corollary 2.14. ([14]) *In a d*-algebra, every BCK-ideal is a d-subalgebra.*

Theorem 2.15. ([11]) *If $(X, *, 0)$ is a BCK-algebra, then every BCK-ideal of X is a d*-ideal of X.*

Let $(X, *, 0_X)$ and $(Y, \bullet, 0_Y)$ be d-algebras. A mapping $f : X \to Y$ is called a *homomorphism* if $f(x * y) = f(x) \bullet f(y)$ for all $x, y \in X$. In [13], J. Neggers, A. Dvurečenskij and H. S. Kim used "*d-morphism*", but we change it into "*homomorphism*" for convenience. Note that $f(0_X) = 0_Y$. A d-algebra $(X, *, 0_X)$ is said to be *d-transitive* (see [14]) if $x * z = 0_X$ and $z * y = 0_X$ imply $x * y = 0_X$.

Proposition 2.16. ([14]) *Let $f : X \to Y$ be a homomorphism from a d-algebra X into a d-transitive d-algebra Y. Then Ker f is a d*-ideal of X.*

Let $(X, *, 0)$ be a d-algebra and let I be a d^*-ideal of X. Define a binary relation " $\sim$ " on X by $x \sim y$ if and only if $x * y,\ y * x \in I$. We denote it by "$x \sim y$ (mod I)" or simply "$x \sim y$".

We denote a congruence class containing x by $[x]_I$, i.e., $[x]_I := \{y \in X | x \sim y$ (mod I)$\}$. We see that $x \sim y$ if and only if $[x]_I = [y]_I$. Denote the set of all equivalence classes of X by X/I, i.e., $X/I := \{[x]_I | x \in X\}$.

Lemma 2.17. ([14]) *Let I be a d^*-ideal of a d-algebra $(X, *, 0)$. Then $I = [0]_I$.*

Theorem 2.18. ([14]) *Let $(X, *, 0)$ be a d-algebra and let I be a d^*-ideal of X. If we define $[x]_I * [y]_I := [x * y]_I$ where $x, y \in X$, then $(X/I, *, 0)$ is a d-algebra, called the quotient d-algebra.*

Proposition 2.19. ([14]) *Let I be a d^*-ideal of a d-algebra $(X, *, 0)$. Then the mapping $\pi : X \to X/I$ defined by $\pi(x) := [x]_I$ is a homomorphism of X onto the quotient d-algebra X/I and the kernel of π is precisely the set I.*

Theorem 2.20. ([14]) *If $f : X \to Y$ is a homomorphism from a d-algebra X onto a d-transitive d-algebra Y, then $X/Kerf \cong Y$.*

2.3. Smarandache BCI-algebras

An algebra $(X, *, 0)$ is called a *BCI-algebra* if it satisfies the following conditions:

(BCI-1) $((x * y) * (x * z)) * (z * y) = 0$,

(BCI-2) $(x * (x * y)) * y = 0$,

(BCI-3) $x * x = 0$,

(BCI-4) $x * y = 0$ and $y * x = 0$ imply $x = y$

for all $x, y, z \in X$.

A non-empty subset I of a BCI-algebra X is called a *BCI-ideal* of X if it satisfies the following conditions:

(i) $0 \in I$,

(ii) $x * y \in I$, $y \in I$ imply $x \in I$.

for all $x, y \in X$.

Definition 2.21. ([8]) A BCI-algebra $(X, *, 0)$ is said to be a *Smarandache BCI-algebra* if it contains a proper subset Q of X such that

(i) $0 \in Q$ and $|Q| \geq 2$,

(ii) $(Q, *, 0)$ is a BCK-algebra.

By a *Smarandache positive implicative* (resp., *commutative* and *implicative*) *BCI-algebra*, we mean a BCI-algebra X which has a proper subset Q of X such that

(i) $0 \in Q$ and $|Q| \geq 2$,

(ii) Q is a positive implicative (resp., commutative and implicative) BCK-algebra under the same operation of X.

Let $(X, *, 0)$ be a Smarandache BCI-algebra and H be a subset of X such that $0 \in H$ and $|H| \geq 2$. Then H is called a *Smarandache subalgebra* of X if $(H, *, 0)$ is a Smarandache BCI-algebra (see [14]).

A non-empty subset I of X is called a *Smarandache ideal* of X related to Q if it satisfies the following conditions:

(i) $0 \in I$,

(ii) $x \in Q$, $y \in I$, $x * y \in I$ imply $x \in I$,

where Q is a BCK-algebra contained in X (see [9]). If I is a Smarandache ideal of X related to every BCK-algebra contained in X, we simply say that I is a Smarandache ideal of X.

In what follows, let X and Q denote a Smarandache BCI-algebra and a BCK-algebra which is properly contained in X, respectively.

Definition 2.22. ([9]) A non-empty subset I of X is called a *Smarandache ideal* of X related to Q (or briefly, a Q-*Smarandache ideal*) of X if it satisfies:

(c_1) $0 \in I$,

(c_2) $x \in Q$, $y \in I$, $x * y \in I$ imply $x \in I$.

If I is a Smarandache ideal of X related to every BCK-algebra contained in X, we simply say that I is a Smarandache ideal of X.

Definition 2.23. ([9]) A non-empty subset I of X is called a *Smarandache fresh ideal* of X related to Q (or briefly, a Q-*Smarandache fresh ideal* of X) if it satisfies the conditions (c_1) in Definition 2.22 and

(c_3) $x, y, z \in Q$, $((x * y) * z) \in I$ and $y * z \in I$ imply $x * z \in I$.

Theorem 2.24. ([9]) *Every Q-Smarandache fresh ideal which is contained in Q is a Q-Smarandache ideal.*

The converse of Theorem 2.24 need not be true in general.

Theorem 2.25. ([9]) *Let I and J be Q-Smarandache ideals of X and $I \subset J$. If I is a Q-Smarandache fresh ideal of X, then so is J.*

Definition 2.26. ([9]) A non-empty subset I of X is called a *Smarandache clean ideal* of X related to Q (or briefly, a Q-*Smarandache clean ideal* of X) if it satisfies the conditions (c_1) in Definition 2.22 and

(c_4) $x, y \in Q$, $z \in I$, $(x * (y * x)) * z \in I$ imply $x \in I$.

Theorem 2.27. ([9]) *Every Q-Smarandache clean ideal of X is a Q-Smarandache ideal.*

The converse of Theorem 2.27 need not be true in general.

Theorem 2.28. ([9]) *Every Q-Smarandache clean ideal of X is a Q-Smarandache fresh ideal.*

Theorem 2.29. ([95]) *Let I and J be Q-Smarandache ideals of X and $I \subset J$. If I is a Q-Smarandache clean ideal of X, then so is J.*

A fuzzy set μ in X is called a *fuzzy subalgebra* of a BCI-algebra X if $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$(see [7]).
A fuzzy set μ in X is called a *fuzzy ideal* of X if

(F_1) $\mu(0) \geq \mu(x)$,

(F_2) $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$

for all $x, y \in X$ (see [7]).

Let μ be a fuzzy set in a set X. For $t \in [0, 1]$, the set $\mu_t := \{x \in X | \mu(x) \geq t\}$ is called a *level subset* of μ.

3. Structural properties of quotient d-algebras

3.1. Structures of quotient d-algebras

Let $(X, *, 0)$ be a d-algebra and let I_1, I_2 be d^*-ideals of X with $I_1 \subseteq I_2$. Then $X/I_1 := \{ [x]_{I_1} |\ x \in X\}$ is a quotient d-algebra. We define $I_2/I_1 := \{ [x]_{I_1}^{I_2} \mid x \in I_2 \}$. We claim that each element of I_2/I_1 is an element of X/I_1, i.e., $[x]_{I_1} = [x]_{I_1}^{I_2}$ for all $x \in X$. In fact,

$$\begin{aligned}
[x]_{I_1}^{I_2} &= \{\alpha \in I_2 \mid \alpha \sim x \ (\mathrm{mod}\ I_1)\} \\
&= \{\alpha \in I_2 \mid \alpha * x,\ x * \alpha \in I_1\} \\
&\subseteq \{\alpha \in X \mid \alpha * x,\ x * \alpha \in I_1\} \\
&= [x]_{I_1}.
\end{aligned}$$

If $\beta \in [x]_{I_1}$, then $\beta \sim x$ (mod I_1). It follows that $\beta * x$, $x * \beta \in I_1$. Since $x \in I_1$ and I_1 is a d^*-ideal of X, we obtain $\beta \in I_1$ by (D_1). Since $I_1 \subseteq I_2$, we have $\beta \in I_2$. It follows from $\beta \sim x$ (mod I_1) that $\beta \in [x]_{I_1}^{I_2}$. Hence $[x]_{I_1} \subseteq [x]_{I_1}^{I_2}$. Therefore $[x]_{I_1} = [x]_{I_1}^{I_2}$.

We give an exact analog of Theorem 2.20 without using the notion of a "$d-transitivity$". Usually it is not get known that the kernel of an epimorphism of d-algebras forms a d^*-ideal.

Theorem 3.1. *If $g : (X, *, 0_X) \to (Y, \bullet, 0_Y)$ is an epimorphism of d-algebras and $Ker(g)$ is a d^*-ideal of X, then $X/Ker(g) \cong Y$.*

Proof. Let $I := Ker(g)$ be a d^*-ideal of X. Define $h : X/I \to Y$ by $h([x]_I) := g(x)$ for any $x \in X$. Suppose $[x]_I = [y]_I$. Then $x \sim y \pmod{I}$, i.e., $x * y$, $y * x \in I$. It follows that $g(x) \bullet g(y) = g(x * y) = 0_Y$ and $g(y) \bullet g(x) = g(y * x) = 0_Y$. Since Y is a d-algebra, we obtain $g(x) = g(y)$. Hence h is well-defined. For any $y \in Y$, since g is an onto map, there exists $x \in X$ such that $g(x) = y$. Thus

$$y = g(x) = h([x]_I),$$

which means that $h : X/I \to Y$ is an onto map.

For any $[x]_I, [y]_I \in X/I$ with $h([x]_I) = h([y]_I)$, we have

$$\begin{aligned} g(x) = g(y) &\Rightarrow g(x * y) = 0_Y,\ g(y * x) = 0_Y \\ &\Rightarrow x * y,\ y * x \in Ker(g) = I \\ &\Rightarrow x \sim y \pmod{I} \\ &\Rightarrow [x]_I = [y]_I. \end{aligned}$$

Therefore h is an one-one map. Since

$$h([x]_I * [y]_I) = h([x * y]_I) = g(x * y) = g(x) \bullet g(y) = h([x]_I) * h([y]_I),$$

we obtain $X/Ker(g) \cong Y$. □

Theorem 3.2. *Let $(X, *, 0_X)$ be a d-algebra and let I_1, I_2 be d^*-ideals of X with $I_1 \subseteq I_2$. Then I_2/I_1 is a d^*-ideal of the quotient d-algebra $(X/I_1, *, I_1)$.*

Proof. Suppose $[x]_{I_1} * [y]_{I_1} \in I_2/I_1$, $[y]_{I_1} \in I_2/I_1$. Then $[x * y]_{I_1}$, $[y]_{I_1} \in I_2/I_1$. Since $x * y$, $y \in I_2$ and I_2 is a d^*-ideal of X, we obtain $x \in I_2$. Hence

$$[x]_{I_1} \in I_2/I_1. \qquad \cdots\cdots (D_1)$$

Also, suppose that $[x]_{I_1} \in I_2/I_1$, $[y]_{I_1} \in X/I_1$ where $y \in X$. Then $x \in I_2$. Since I_2 is a d^*-ideal of X, we have $x * y \in I_2$. It follows that

$$[x]_{I_1} * [y]_{I_1} = [x * y]_{I_1} \in I_2/I_1. \qquad \cdots\cdots (D_2)$$

If $[x]_{I_1} * [y]_{I_1} \in I_2/I_1$ and $[y]_{I_1} * [z]_{I_1} \in I_2/I_1$, then $[x * y]_{I_1}$, $[y * z]_{I_1} \in I_2/I_1$. Since $x * y$, $y * z \in I_2$ and I_2 is a d^*-ideal of X, we have $x * z \in I_2$. It follows that

$$[x]_{I_1} * [z]_{I_1} = [x * z]_{I_1} \in I_2/I_1. \qquad \cdots\cdots (D_3)$$

Let $[x]_{I_1} * [y]_{I_1}$, $[y]_{I_1} * [x]_{I_1} \in I_2/I_1$. Then $[x * y]_{I_1}$, $[y * x]_{I_1} \in I_2/I_1$. Since $x * y$, $y * x \in I_2$ and I_2 is a d^*-ideal of X, we obtain $(x * z) * (y * z) \in I_2$ and $(z * x) * (z * y) \in I_2$ for all $z \in X$. It follows that

$$\begin{aligned} [x * z]_{I_1} * [y * z]_{I_1} &= [(x * z) * (y * z)]_{I_1} \in I_2/I_1, \\ [z * x]_{I_1} * [z * y]_{I_1} &= [(z * x) * (z * y)]_{I_1} \in I_2/I_1. \qquad \cdots\cdots (D_4) \end{aligned}$$

Therefore I_2/I_1 is a d^*-ideal of $(X/I_1, *, I_1)$. □

Corollary 3.3. *Let $(X, *, 0_X)$ be a d-algebra and I_1, I_2 be d^*-ideals of X. Then (X/I_1) / (I_2/I_1) is a d-algebra .*

Proof. It follows from Theorem 3.1 and Theorem 2.18 that (X/I_1) / (I_2/I_1) is also a d-algebra. □

In fact, any element of the quotient d-algebra $(X/I_1)/(I_2/I_1)$ can be denoted by $[[x]_{I_1}]_{I_2/I_1}$ where $x \in X$. It is easy to see that

$$\begin{aligned}[][x]_{I_1}]_{I_2/I_1} &= \{[\alpha]_{I_1} \in X/I_1 \mid [\alpha]_{I_1} \sim [x]_{I_1}\} \\ &= \{[\alpha]_{I_1} \in X/I_1 \mid \alpha \sim x \ (\mathrm{mod}\ I_1)\} \\ &= \{[\alpha]_{I_1} \in X/I_1 \mid \alpha * x,\ x * \alpha \in I_1\}.\end{aligned}$$

Hence we conclude that

$$\begin{aligned}(X/I_1)/(I_2/I_1) &= \{\{[\alpha]_{I_1} \in X/I_1 \mid \alpha * x,\ x * \alpha \in I_1\} \mid x \in X\} \\ &= \{[[x]_{I_1}]_{I_2/I_1} \mid x \in X\}.\end{aligned}$$

Theorem 3.4. *Let $(X, *, 0_X)$ be a d-algebra and I_1, I_2 be d^*-ideals of X with $I_1 \subseteq I_2$. Then $(X/I_1)/(I_2/I_1) \cong X/I_2$.*

Proof. Define $g : X/I_1 \to X/I_2$ by $g([x]_{I_1}) := [x]_{I_2}$. Then g is well-defined. Indeed, for any $[x]_{I_1}, [y]_{I_1} \in X/I_1$ with $[x]_{I_1} = [y]_{I_1}$, we have $x * y$, $y * x \in I_1$. Since $I_1 \subseteq I_2$, we obtain $x * y$, $y * x \in I_2$. It follows that $x \sim y$ (mod I_2), which shows that $g([x]_{I_1}) = [x]_{I_2} = [y]_{I_2} = g([y]_{I_1})$. Hence g is well-defined.
Obviously, g is an epimorphism. Also,

$$\begin{aligned}Ker(g) &= \{[x]_{I_1} \in X/I_1 \mid g([x]_{I_1}) = [0_X]_{I_2}\} \\ &= \{[x]_{I_1} \in X/I_1 \mid [x]_{I_2} = [0_X]_{I_2}\} \\ &= \{[x]_{I_1} \in X/I_1 \mid x \sim 0_X \ (\mathrm{mod}\ I_2)\} \\ &= \{[x]_{I_1} \in X/I_1 \mid x * 0_X,\ 0_X * x \in I_2\}.\end{aligned}$$

Since I_2 is a d^*-ideal of X, we have $x \in I_2$ if and only if $x * 0_X$, $0_X * x \in I_2$. This proves that $Ker(g)= \{[x]_{I_1} \in X/I_1 \mid x \in I_2\} = I_2/I_1$.

By applying Theorem 3.1, we obtain

$$(X/I_1)/(I_2/I_1) = (X/I_1)/Ker(g) \cong X/I_2.$$

$\square$

Let $(X, *, 0_X)$ be a d-algebra. Define a relation " $\leq$ " on X by $x \leq y$ if and only if $x * y = 0_X$, where $x, y \in X$. Note that every $BCK(BCI)$-algebra has a partially ordered set (simply, poset), but d-algebras need not have a poset structure in general. Consider the following example.

Example 3.5. Let $X := \{0, 1, 2, 3\}$ be a set with the following table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	2	0	0
3	3	3	3	0

Then $(X, *, 0)$ is a d-algebra. Since $1 * 2 = 2 * 3 = 0$ and $1 * 3 = 1 \neq 0$, we have that $1 \leq 2$, $2 \leq 3$, but $1 \not\leq 3$. This shows that $(X, *, 0)$ has no poset structure.

Note that if $f : (X,\, *,\, 0_X) \to (Y,\, \bullet,\, 0_Y)$ is a homomorphism of d-algebras, then $f(0_X) = 0_Y$. And if $x \leq y$ in X, then $f(x) \leq f(y)$ in Y.

Theorem 3.6. *Let $(X, *, 0_X)$ and $(Y, \bullet, 0_Y)$ be d-algebras and let $f : X \to Y$ be a homomorphism. If B is a d^*-ideal of Y, then $f^{-1}(B)$ is a d^*-ideal of X.*

Proof. Let B be a d^*-ideal of Y. Since $f(0_X) = 0_Y$, we obtain $0_X \in f^{-1}(B)$. If $x * y$, $y \in f^{-1}(B)$, then $f(x) \bullet f(y)$=$f(x * y) \in B$ and $f(y) \in B$. Since B is a d^*-ideal of X, we obtain $f(x) \in B$, i.e.,

$$x \in f^{-1}(B). \qquad \cdots\cdots \ (D_1)$$

If $x \in f^{-1}(B)$, then $f(x) \in B$. Since B is a d^*-ideal of Y, we have $f(x * y) = f(x) \bullet f(y) \in B$ for any $y \in X$. Hence

$$x * y \in f^{-1}(B). \qquad \cdots\cdots \ (D_2)$$

If $x * y$, $y * z \in f^{-1}(B)$, then $f(x * y)$, $f(y * z) \in B$ and hence $f(x) \bullet f(y)$, $f(y) \bullet f(z) \in B$. Since B is a d^*-ideal of Y, we obtain $f(x*z) = f(x) \bullet f(z) \in B$, i.e.,

$$x * z \in f^{-1}(B). \qquad \cdots\cdots \ (D_3)$$

If $x*y$, $y*x \in f^{-1}(B)$, then $f(x) \bullet f(y) = f(x*y)$, $f(y) \bullet f(x) = f(y)*f(x) \in B$. Since B is a d^*-ideal of Y, we have $f((x * z) * (y * z)) = f(x * z) \bullet f(y * z) = (f(x) \bullet f(z)) \bullet (f(y) \bullet f(z)) \in B$ and $f((z * x) * (z * y)) = f(z * x) \bullet f(z * y) = (f(z) \bullet f(x)) \bullet (f(z) \bullet f(y)) \in B$ for all $z \in X$.
Hence $f((x * z) * (y * z))$, $f((z * x) * (z * y)) \in B$. It follows that

$$(x * z) * (y * z),\ (z * x) * (z * y) \in f^{-1}(B). \qquad \cdots\cdots \ (D_4)$$

Thus $f^{-1}(B)$ is a d^*-ideal of X. □

Corollary 3.7. *Let $(X, *, 0_X)$ and $(Y, \bullet, 0_Y)$ be d-algebras and let $f : X \to Y$ be a homomorphism. If B is a d^*-ideal of Y, then $X/f^{-1}(B)$ is a d-algebra.*

Proof. If follows immediately from Theorem 2.18 and Theorem 3.6. □

Theorem 3.8. *Let $g : (X, *, 0_X) \to (Z, \odot, 0_Z)$ be a homomorphism of d-algebras and let $h : (X, *, 0_X) \to (Y, \bullet, 0_Y)$ be an epimorphism of d-algebras such that $Ker(h) \subseteq Ker(g)$. Then there exists a unique homomorphism $f : (Y, \bullet, 0_Y) \to (Z, \odot, 0_Z)$ such that $g = f \circ h$, i.e.,*

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "g"'] & Y \arrow[dl, "f"] \\ Z & \end{tikzcd}$$

the diagram commutes.

Proof. Given y in Y, since h is onto, there exists an x in X such that $y = h(x)$. Define $f : Y \to Z$ by $f(h(x)) := g(x)$. We show that f is well-defined and the diagram commutes. If $h(x_1) = h(x_2) = y$ for some $x_1, x_2 \in X$, then $h(x_1) \bullet h(x_2) = y \bullet y = 0_Y$. Since h is an epimorphism, we have $h(x_1 * x_2) = h(x_1) \bullet h(x_2) = 0_Y$, i.e., $x_1 * x_2 \in Ker(h) \subseteq Ker(g)$. It follows that $0_Z = g(x_1 * x_2) = g(x_1) \odot g(x_2)$. Similarly, $g(x_2) \odot g(x_1) = 0_Z$. Since $(Z, \odot, 0_Z)$ is a d-algebra, we obtain $g(x_1) = g(x_2)$. This shows that $f(h(x_1)) = g(x_1) = g(x_2) = f(h(x_2))$. Hence $f : Y \to Z$ is well-defined and the diagram commutes.

We claim that f is a homomorphism. If $y_1, y_2 \in Y$, since h is an epimorphism, there exist $x_1, x_2 \in X$ such that $g_1 = h(x_1)$, $g_2 = h(x_2)$. It follows that

$$f(y_1 \bullet y_2) = f(h(x_1) \bullet h(x_2))$$

$$
\begin{aligned}
&= f(h(x_1 * x_2)) \\
&= g(x_1 * x_2) \\
&= g(x_1) * g(x_2) \\
&= f(h(x_1)) \odot f(h(x_2)) \\
&= f(y_1) \odot f(y_2).
\end{aligned}
$$

Hence $f : Y \to Z$ is a homomorphism. We show the uniqueness of such a map f.

Let $\hat{f} : Y \to Z$ be a homomorphism such that $\hat{f} \circ h{=}g$. For any $y \in Y$, there exists $x \in X$ such that $h(x) = y$, since h is an epimorphism. It follows that $\hat{f}(y) = \hat{f}(h(x)) = (\hat{f} \circ h)(x) = g(x) = (f \circ h)(x) = f(h(x)) = f(y)$, i.e., $f = \hat{f}$, proving the uniqueness. □

Theorem 3.9. *Let $(X, *, 0_X)$ be a d-algebra, and let $f : (X, *, 0_X) \to (Y, \bullet, 0_Y)$ be an epimorphism. If J is a d^*-ideals of Y, then $X/f^{-1}(J) \cong Y/J$, i.e.,*

$$
\begin{tikzcd}
Y \arrow[r, "\pi_J"] & Y/J \arrow[d, dash, "\cong"] \\
X \arrow[u, "f"] \arrow[ur, "\mu=\pi\circ f"] \arrow[r, "\pi_{Ker(\mu)}"'] & X/Ker(\mu) \arrow[d, phantom, "||"] \\
 & X/f^{-1}(J)
\end{tikzcd}
$$

Proof. Let J be a d^*-ideal of Y and $\pi_J : Y \to Y/J$ be a canonical homomorphism of d-algebras. If we define $\mu := \pi_J \circ f$, the composition of π_J and f, then $\mu : X \to Y/J$ is an epimorphism of d-algebras. If $Ker(\mu)$ is a d^*-ideal of X, then $X/Ker(\mu)$ is isomorphic with Y/J by Theorem 3.1.

In order to show that $Ker(\mu)$ is a d^*-ideal of X, we will show that $Ker(\mu) = f^{-1}(J)$. By Theorem 3.6, if J is a d^*-ideal of Y, then $f^{-1}(J)$ is a d^*-ideal of X. For all $x \in X$, we have

$$\mu(x) = (\pi \circ f)(x) = \pi(f(x)) = [f(x)]_J. \tag{3.1}$$

We claim that $f^{-1}(J) \subseteq Ker(\mu)$. In fact, if $x \in f^{-1}(J)$, then $f(x) \in J$. We need to prove that

$$[f(x)]_J = J. \tag{3.2}$$

If $\alpha \in [f(x)]_J$, then $\alpha \sim f(x)$. It follows that $\alpha \bullet f(x)$, $f(x) \bullet \alpha \in J$. Since $f(x) \in J$ and J is a d^*-ideal of Y, we obtain $\alpha \in J$, i.e., $[f(x)]_J \subseteq J$.

Conversely, if $\beta \in J$, since $f(x) \in J$ and J is a d^*-ideal of Y, we obtain $f(x) \bullet \beta$, $\beta \bullet f(x) \in J$, and hence $\beta \in [f(x)]_J$, i.e., $J \subseteq [f(x)]_J$. So (3.2) holds.

By applying (3.1) and (3.2), we obtain

$$\mu(x) = (\pi \circ f)(x) = \pi(f(x)) = [f(x)]_J = J. \tag{3.3}$$

Since J is a zero in Y/J, we have $x \in Ker(\mu)$ for any $x \in f^{-1}(J)$. This shows that $f^{-1}(J) \subseteq Ker(\mu)$.

Conversely, if $x \in Ker(\mu)$, then $\mu(x) = J$ in Y/J. By (3.1), we have $J = \mu(x) = [f(x)]_J$. It follows that $f(x) \in J$ and $x \in f^{-1}(J)$. Thus $Ker(\mu) \subseteq f^{-1}(J)$. Hence we obtain $Ker(\mu) = f^{-1}(J)$.

By Theorem 3.6, we know that $f^{-1}(J) = Ker(\mu)$ is a d^*-ideal of X. By Theorem 3.1, we conclude

$$X/f^{-1}(J) = X/Ker(\mu) \cong Y/J.$$

□

3.2 Obstinate d-ideals of d-algebras

Definition 3.10. Let $(X, *, 0)$ be a d-algebra and I be a proper d-ideal of X. I is said to be *obstinate* of X if $x, y \notin I$ and $x \neq y$ imply $x * y$, $y * x \in I$.

Example 3.11. Let $X := \{0, 1, 2, 3\}$ be a d-algebra with the following table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	3	2	0	3
3	3	2	2	0

Then I:= $\{0, 1\}$ satisfies the conditions (D_1) and (D_2), but not (D_4) in Definition 2.7, since $0 * 1 = 0$, $1 * 0 = 1 \in I$, $(3 * 0) * (3 * 1) = 3 * 2 = 2 \notin I_1$. Hence I= $\{0, 1\}$ is a d-ideal of X, but not a d^*-ideal of X. Also, since $3, 2 \notin I$, $3 * 2 = 2, 2 * 3 = 3 \notin I$, we see that I is not an obstinate d-ideal of X.

Example 3.12. Let $X := \{0, 1, 2, 3\}$ be a d-algebra with the following table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	3	2	0	0
3	3	3	1	0

Then it is easy to see that $I := \{0, 1\}$ is a d-ideal of X. Since $2, 3 \notin I$ and $2 * 3 = 0$, $3 * 2 = 1$, i.e., $2 * 3$, $3 * 2 \in I$, I is an obstinate d-ideal of X.

Recall that a d-algebra $(X, *, 0_X)$ is said to be *d-transitive* if $x * z = 0_X$ and $z * y = 0_X$ imply $x * y = 0_X$.

Let $\mathbf{J} := \{0, 1\}$ be a set with the following table:

$\bullet$	0	1
0	0	0
1	1	0

Then it is easy to see that $(\mathbf{J}, \bullet, 0)$ is a d-transitive d-algebra.

Proposition 3.13. *Let $(X, *, 0_X)$ be a d-algebra and $f : (X, *, 0_X) \to (\mathbf{J}, \bullet, 0)$ be a homomorphism. Then $Ker(f)$ is an obstinate d^*-ideal of X.*

Proof. By applying Proposition 2.16, we see that $Ker(f)$ is a d^*-ideal of X. If $x, y \notin Ker(f)$, $x \neq y$, then $f(x * y) = f(x) \bullet f(y) = 1 \bullet 1 = 0$.
Also, $f(y * x) = f(y) \bullet f(x) = 1 \bullet 1 = 0$. Thus $x * y$, $y * x \in Ker(f)$. Hence $Ker(f)$ is an obstinate d^**-ideal* of X. □

Theorem 3.14. *Let $(X, *, 0_X)$ be a d-algebra and let I be a proper d-ideal of X. Then, given an edge d-algebra $(Y, \bullet, 0_Y)$, there exists a homomorphism $f : X \to Y$ such that $Ker(f) = I$ if and only if I is an obstinate ideal of X.*

Proof. Let I be an obstinate ideal of X. We define a map $f : X \to Y$ by

$$f(x) := \begin{cases} 0_Y & (x \in I) \\ a & (x \in X \setminus I) \end{cases}$$

where a is a fixed element of Y with $a \neq 0_Y$. We show that f is a homomorphism from X to Y. We consider 4 cases :
Case 1. If x, $y \in I$, then $x * y \in I$ by (D_2) in Definition 2.7. It follows that

$$f(x * y) = 0_Y = 0_Y \bullet 0_Y = f(x) \bullet f(y).$$

Case 2. If x, $y \notin I$, $x \neq y$, since I is obstinate, we obtain $x * y \in I$. It follows that

$$f(x * y) = 0_Y = a \bullet a = f(x) \bullet f(y).$$

Case 3. If $x \notin I$ and $y \in I$, then $x * y \notin I$. In fact, if we assume $x * y \in I$, since $y \in I$ and (D_1) in Definition 2.7, we obtain $x \in I$, a contradiction. Since Y is an edge d-algebra, we obtain

$$f(x * y) = a = a \bullet 0_Y = f(x) \bullet f(y).$$

Case 4. If $x \in I$ and $y \notin I$, then $x * y \in I$ by (D_2) in Definition 2.7. It follows that

$$f(x * y) = 0_Y = 0_Y \bullet a = f(x) \bullet f(y).$$

This shows that $f : X \to Y$ is a homomorphism. Clearly, we have $Ker(f) = I$. Conversely, let $Y := \{0_Y, a\}$ be a set with the following table:

$\bullet$	0_Y	a
0_Y	0_Y	0_Y
a	a	0_Y

Then $(Y, \bullet, 0_Y)$ is an edge d-algebra. By assumption, there exists a homomorphism $f : X \to Y$ such that $Ker(f) = I$. We claim that I is an obstinate ideal of X. If x, $y \notin I$, $x \neq y$, then $f(x) = f(y) = a$, and hence $f(x * y) = f(x) \bullet f(y) = a \bullet a = 0_Y$ and $f(y * x) = f(y) \bullet f(x) = a \bullet a = 0_Y$. It follows that $x * y$, $y * x \in Ker(f) = I$. Hence I is an obstinate ideal of X. $\square$

4. Analytic real algebras and d-algebras

Let $\mathbb{R}$ be the set of all real numbers and let "$*$" be a binary operation on $\mathbb{R}$. Define a map $\lambda : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. We define $x * y := \lambda(x, y)$ for all $x, y \in \mathbb{R}$. Such a groupoid $(\mathbb{R}, *)$ is said to be an *analytic real algebra.*

4.1. Analytic real algebras

Given an analytic real algebra $(\mathbb{R}, *)$, we define

$$tr(*, \lambda) := \int_{-\infty}^{\infty} \lambda(x, x)\ dx$$

We call $tr(*, \lambda)$ a *trace* of λ. Note that the trace $tr(*, \lambda)$ may or may not converge. Given an analytic real algebra $(\mathbb{R}, *)$, where $x * y := \lambda(x, y)$, if $x * x = 0$ for all $x \in \mathbb{R}$, then $tr(*, \lambda) = 0$, but the converse need not be true in general.

Example 4.1. Let $x_0 \in \mathbb{R}$. Define

$$\lambda(x, x) = \begin{cases} 0 & \text{if } x \neq x_0, \\ 1 & \text{otherwise.} \end{cases}$$

Then $tr(*, \lambda) = \int_{-\infty}^{\infty} \lambda(x, x)\ dx = 0$, but $\lambda(x_0, x_0) = 1 \neq 0$, i.e., $x_0 * x_0 \neq 0$.

Proposition 4.2. *Let $(\mathbb{R}, *)$ be an analytic real algebra and let $a, b, c \in \mathbb{R}$, where $x * y := ax + by + c$ for all $x, y \in \mathbb{R}$. If $|tr(*, \lambda)| < \infty$, then $tr(*, \lambda) = 0$ and $x * y = a(x - y)$ for all $x, y \in \mathbb{R}$.*

Proof. Given $x \in \mathbb{R}$, we have $x * x = (a + b)x + c$. Since $|tr(*, \lambda)| < \infty$, we have $|\int_{-\infty}^{\infty} [(a + b)x + c]\, dx| < \infty$. Now $\int_0^A [(a+b)x + c]\ dx = (a + b)\frac{A^2}{2} + cA =$

$A\left[\frac{a+b}{2}A+c\right]$ for a large number A, so that if $|tr(*,\lambda)| < \infty$, then $a+b=0$ and $c=0$, i.e., we have $x*y = a(x-y)$, and thus $x*x=0$ for all $x \in \mathbb{R}$. $\square$

Theorem 4.3. *Let $a,b,c,d,e,f \in \mathbb{R}$. Define a binary operation "$*$" on $\mathbb{R}$ by*

$$x*y := ax^2+bxy+cy^2+dx+ey+f$$

for all $x,y \in \mathbb{R}$. If $|tr(*,\lambda)| < \infty$ and $0*x=0$ for all $x \in \mathbb{R}$, then $x*y = ax(x-y)$ for all $x,y \in \mathbb{R}$.

Proof. Given $x \in \mathbb{R}$, we have $x*x = (a+b+c)x^2+(d+e)x+f$. Let $A := a+b+c$, $B := d+e$. If we assume $|tr(*,\lambda)| < \infty$, then $|\int_{-\infty}^{\infty}(Ax^2+Bx+f)\,dx| < \infty$. Now $\int_0^L(Ax^2+Bx+f)\;dx = \frac{A}{3}L^3+\frac{B}{2}L^2+fL = L\left(\frac{A}{3}L^2+\frac{B}{2}+f\right)$ for a large number L so that $|tr(*,\lambda)| < \infty$ implies $A=B=f=0$, i.e., $a+b+c=0, d+e=0, f=0$. It follows that

$$x*y = (ax-cy+d)(x-y). \tag{4.1}$$

If we assume $0*x=0$ for all $x \in \mathbb{R}$, then, by (4.1), we have

$$\begin{aligned} 0 &= 0*x \\ &= (a0-cx+d)(0-x) \\ &= cx^2-dx \end{aligned}$$

for all $x \in \mathbb{R}$. This shows that $c=d=0$. Hence $x*y = ax(x-y)$ for all $x,y \in \mathbb{R}$. $\square$

Corollary 4.4. *Let $a, b, c, d, e, f \in \mathbb{R}$. Define a binary operation "$*$" on $\mathbb{R}$ by*

$$x * y := ax^2 + bxy + cy^2 + dx + ey + f$$

for all $x, y \in \mathbb{R}$. If $x * x = 0$ and $0 * x = 0$ for all $x \in \mathbb{R}$, then $x * y = ax(x - y)$ for all $x, y \in \mathbb{R}$.

Proof. The condition, $x * x = 0$ for all $x \in \mathbb{R}$, implies $|tr(*, \lambda)| < \infty$. The conclusion follows from Theorem 4.3. $\square$

Proposition 4.5. *Let $a, b, c, d, e, f \in \mathbb{R}$. Define a binary operation "$*$" on $\mathbb{R}$ by*

$$x * y := ax^2 + bxy + cy^2 + dx + ey + f$$

for all $x, y \in \mathbb{R}$. If $|tr(*, \lambda)| < \infty$ and the anti-symmetry law holds for "$*$", then $(ax - cy + d)^2 + (ay - cx + d)^2 > 0$ for $x \neq y$.

Proof. If $|tr(*, \lambda)| < \infty$, then by (4.1) we obtain $x * y = (ax - cy + d)(x - y)$. Assume the anti-symmetry law holds for "$*$". Then either $x * y \neq 0$ or $y * x \neq 0$ for $x \neq y$. It follows that $(x * y)^2 > 0$ or $(y * x)^2 > 0$, and hence $(x * y)^2 + (y * x)^2 > 0$. This shows that $(ax - cy + d)^2 + (ay - cx + d)^2 > 0$. $\square$

Note that in Proposition 4.5 it is clear that if $(ax - cy + d)^2 + (ay - cx + d)^2 > 0$ for $x \neq y$, then the anti-symmetry law holds.

Corollary 4.6. *If we define $x * y := ax(x - y)$ for all $x, y \in \mathbb{R}$ where $a \neq 0$, then $(\mathbb{R}, *)$ is a d-algebra.*

Proof. It is easy to see that $x * x = 0 = 0 * x$ for all $x \in \mathbb{R}$. Assume that $x \neq y$. Since $x * y = ax(x - y) = ax^2 - axy$, by applying Proposition 4.5, we obtain $b = -a, c = 0, d = e = f = 0$. It follows that $(ax - 0y + 0)^2 + (ay - 0x + 0)^2 = a^2x^2 + a^2y^2 = a^2(x^2 + y^2) > 0$ when $a \neq 0$. By Proposition 4.5, $(\mathbb{R}, *)$ is a d-algebra. $\square$

Proposition 4.7. *Let $a, b, c, d, e, f \in \mathbb{R}$. Define a binary operation "$*$" on $\mathbb{R}$ by*

$$x * y := ax^2 + bxy + cy^2 + dx + ey + f$$

for all $x, y \in \mathbb{R}$. If $|tr(*, \lambda)| < \infty$ and $x * 0 = x$ for all $x \in \mathbb{R}$, then $x * y = (1 - cy)(x - y)$ for all $x, y \in \mathbb{R}$.

Proof. If $|tr(*, \lambda)| < \infty$, then by (4.1) we obtain $x * y = (ax - cy + d)(x - y)$ for all $x, y \in \mathbb{R}$. If we let $y := 0$, then $x = x * 0 = (ax + d)x$. It follows that $ax^2 + (d - 1)x = 0$ for all $x \in \mathbb{R}$. This shows that $a = 0, d = 1$. Hence $x * y = (1 - cy)(x - y)$ for all $x, y \in \mathbb{R}$. $\square$

Theorem 4.8. *If we define $x * y := (ax - cy + d)(x - y)$ for all $x, y \in \mathbb{R}$ where $a, c, d \in \mathbb{R}$ with $a + c \neq 0$, then the anti-symmetry law holds.*

Proof. Assume that there exist $x \neq y$ in $\mathbb{R}$ such that $x * y = 0 = y * x$. Then $(ax - cy + d)(x - y) = 0$ and $(ay - cx + d)(y - x) = 0$. Since $x \neq y$, we have

$$ax - cy + d = 0 = ay - cx + d. \tag{4.2}$$

It follows that $(a + c)(x - y) = 0$. Since $a + c \neq 0$, we obtain $x = y$, a contradiction. $\square$

Remark 4.9. The analytic algebra $(\mathbb{R}, *)$, $x * y = ax(x - y)$ for all $x, y \in \mathbb{R}$, was proved to be a d-algebra in Corollary 4.6 by using Proposition 4.5. Since $x * y = ax(x - y) = (ax - 0y + 0)(x - y)$, we know that $a + 0 = a \neq 0$. Hence the algebra $(\mathbb{R}, *)$ can be proved by using Theorem 4.8 also.

Note that the analytic real algebra $(\mathbb{R}, *)$ discussed in Corollary 4.6 need not be an edge d-algebra, since $x * 0 = ax(x - 0) = ax^2 \neq x$.

4.2 Analytic real algebras with functions

Let $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ be real-valued functions. Define a binary operation "$\star$" on $\mathbb{R}$ by

$$x \star y := \alpha(x)x + \beta(y)y + c \tag{4.3}$$

where $c \in \mathbb{R}$.

Proposition 4.10. *Let $(\mathbb{R}, \star)$ be an analytic real algebra defined by (4.3). If $x \star x = 0 = 0 \star x$ for all $x \in \mathbb{R}$, then $x \star y = 0$ for all $x, y \in \mathbb{R}$.*

Proof. Assume that $x \star x = 0$ for all $x \in \mathbb{R}$. Then

$$\begin{aligned} 0 &= x \star x \\ &= \alpha(x)x + \beta(x)x + c \\ &= [\alpha(x) + \beta(x)]x + c. \end{aligned}$$

If we let $x := 0$, then $c = 0$. If $x \neq 0$, then $\alpha(x) + \beta(x) = 0$, i.e., $\beta(x) = -\alpha(x)$ for all $x \neq 0$ in $\mathbb{R}$. It follows that

$$x \star y = \alpha(x)x - \alpha(y)y. \tag{4.4}$$

Assume $0 \star x = 0$ for all $x \in \mathbb{R}$. Then

$$\begin{aligned} 0 &= 0 \star x \\ &= \alpha(0)0 + \beta(x)x + c \\ &= \beta(x)x. \end{aligned}$$

It follows that $\beta(x) = 0$ for all $x \neq 0$ in $\mathbb{R}$. Hence we have $x \star y = 0$ for all $x, y \in \mathbb{R}$. $\square$

Proposition 4.11. *Let* $(\mathbb{R}, \star)$ *be an analytic real algebra defined by (4.3). If* $x \star x = 0$ *and* $x \star 0 = x$ *for all* $x \in \mathbb{R}$*, then* $x \star y = x - y$ *for all* $x, y \in \mathbb{R}$*.*

Proof. If we assume $x \star x = 0$ for all $x \in \mathbb{R}$, then by (4.4) we obtain $x \star y = \alpha(x)x - \alpha(y)y$. Assume that $x \star 0 = x$ for all $x \in \mathbb{R}$. Then $x = x \star 0 = \alpha(x)x - \alpha(0)0 = \alpha(x)x$. This shows that $\alpha(x) = 1$ for any $x \neq 0$ in $\mathbb{R}$. Hence $x \star y = x - y$ for all $x, y \in \mathbb{R}$. □

Let $a, b_1, b_2, c, d, e : \mathbb{R} \to \mathbb{R}$ be real-valued functions and let $f \in \mathbb{R}$. Define a binary operation "$\star$" on $\mathbb{R}$ by

$$x \star y := a(x)x^2 + b_1(x)b_2(y)xy + c(y)y^2 + d(x)x + e(y)y + f \tag{4.5}$$

for all $x, y \in \mathbb{R}$. Assume $0 \star x = 0$ for all $x \in \mathbb{R}$. Then

$$\begin{aligned} 0 &= 0 \star x \\ &= c(x)x^2 + e(x)x + f \\ &= [c(x)x + e(x)]x + f \end{aligned}$$

for all $x \in \mathbb{R}$. It follows that $f = 0$ and $c(x)x + e(x) = 0$ for all $x \neq 0$ in $\mathbb{R}$. Hence $c(y)y^2 + e(y)y = 0$ for all $y \in \mathbb{R}$. Hence

$$x \star y = a(x)x^2 + b_1(x)b_2(y)xy + d(x)x. \tag{4.6}$$

Assume $x \star x = 0$ for all $x \in \mathbb{R}$. Then by (4.6) we obtain

$$\begin{aligned} 0 &= x \star x \\ &= a(x)x^2 + b_1(x)b_2(x)x^2 + d(x)x. \end{aligned}$$

It follows that $d(x)x = -[a(x)x^2 + b_1(x)b_2(x)x^2]$. By (4.6) we obtain

$$x \star y = b_1(x)x[b_2(y)y - b_2(x)x]. \tag{4.7}$$

Theorem 4.12. *Let $b_1, b_2 : \mathbb{R} \to \mathbb{R}$ be real-valued functions. Define a binary operation "$\star$" on $\mathbb{R}$ as in (4.7). If we assume $b_2(x)x \neq b_2(y)y$ and $b_1^2(x)x^2 + b_1^2(y)y^2 > 0$ for any $x \neq y$ in $\mathbb{R}$, then $(\mathbb{R}, \star)$ is a d-algebra.*

Proof. Assume the anti-symmetry law holds. Then it is equivalent to that if $x \neq y$ then $x \star y \neq 0$ or $y \star x \neq 0$, i.e., if $x \neq y$ then $(x \star y)^2 + (y \star x)^2 > 0$. Since $x \star y$ is defined by (4.7), we obtain that if $x \neq y$ then

$$(b_1^2(x)x^2 + b_1^2(y)y^2)(b_2(x)x - b_2(y)y)^2 > 0.$$

By assumption, we obtain that $(\mathbb{R}, \star)$ is a d-algebra. □

Example 4.13. Consider $x \star y := ax(x - y)$ for all $x, y \in \mathbb{R}$. If we compare it with (4.7), then we have $b_1(x) = a, b_2(y) = -1$ and $b_2(x) = -1$ for all $x \in \mathbb{R}$. This shows that $b_2(x)x - b_2(y)y = (-1)x - (-1)y = y - x \neq 0$ when $x \neq y$. Moreover, $b_1^2(x)x^2 + b_1^2(y)y^2 = a^2x^2 + b_1^2(y)y^2 > 0$ since $a \neq 0$. By applying Theorem 4.12, we see that an analytic real algebra $(\mathbb{R}, \star)$ where $x \star y := ax(x-y)$, $a \neq 0$ is a d-algebra.

Example 4.14. Consider $x \star y := x \tan 2x[e^y y - e^x x]$ for all $x, y \in \mathbb{R}$. By comparing it with (4.7), we obtain $b_1(x) = \tan 2x, b_2(y) = e^y$ and $b_2(x) = e^x$. If $x \neq y$, then it is easy to see that $xe^x \neq ye^y$ and $b_1^2(x)x^2 + b_1^2(y)y^2 = (\tan 2x)^2x^2 + (\tan 2y)^2y^2 > 0$ when $x \neq y$. Hence an analytic real algebra $(\mathbb{R}, \star)$ where $x \star y := x \tan 2x[e^y y - e^x x]$ is a d-algebra by Theorem 4.12.

In Theorem 4.12, we obtained some conditions for analytic real algebras to be d-algebras. In addition, we construct an edge d-algebra from Theorem 4.12 as follows.

Theorem 4.15. *If define a binary operation "$\star$" on $\mathbb{R}$ by*

$$x \star y := \begin{cases} x\left[1 - \frac{b_1(x)}{b_1(y)}\right] & \text{if } y \neq 0, \\ x & \text{otherwise,} \end{cases}$$

where $b_1(x)$ is a real-valued function such that $b_1(y) \neq 0$ if $y \neq 0$, then $(\mathbb{R}, \star)$ is an edge d-algebra.

Proof. Define a binary operation "$\star$" on $\mathbb{R}$ as in (4.7) with additional conditions: $b_2(x)x \neq b_2(y)y$ and $b_1^2(x)x^2 + b_1^2(y)y^2 > 0$ for any $x \neq y$ in $\mathbb{R}$. Assume $x \star 0 = x$ for all $x \in \mathbb{R}$. Then

$$\begin{aligned} x &= x \star 0 \\ &= b_1(x)x[b_2(0)0 - b_2(x)x] \\ &= -b_1(x)b_2(x)x^2. \end{aligned}$$

Combining with (4.7) we obtain

$$x \star y = b_1(x)b_2(y)xy - b_1(x)b_2(x)x^2$$

$$
\begin{aligned}
&= b_1(x)b_2(y)xy + x \\
&= x[b_1(x)b_2(y)y + 1].
\end{aligned}
$$

If we let $xy \neq 0$, then

$$
\begin{aligned}
x \star y &= x\left[b_1(x)(-\frac{1}{b_1(y)}) + 1\right] \\
&= x\left[1 - \frac{b_1(x)}{b_1(y)}\right].
\end{aligned}
$$

If we let $x \star y := x$ when $y = 0$, then $(\mathbb{R}, \star)$ is an edge d-algebra. $\square$

Example 4.16. Define a map $b_1(x) := e^{\lambda x}$ for all $x \in \mathbb{R}$. Then $x \star y = x\left[1 - \frac{e^{\lambda x}}{e^{\lambda y}}\right] = x\left(1 - e^{\lambda(x-y)}\right)$ when $y \neq 0$. If we define a binary operation "$\star$" on $\mathbb{R}$ by

$$
x \star y := \begin{cases} x\left[1 - e^{\lambda(x-y)}\right] & \text{if } y \neq 0, \\ x & \text{otherwise,} \end{cases}
$$

then $(\mathbb{R}, \star)$ is an edge d-algebra.

Proposition 4.17. *Suppose that we define a binary operation "$\star$" on $\mathbb{R}$ by*

$$
x \star y := \begin{cases} x\left[1 - \frac{b_1(x)}{b_1(y)}\right] & \textit{if } y \neq 0, \\ x & \textit{otherwise,} \end{cases}
$$

where $b_1(x)$ is a real-valued function such that $b_1(y) \neq 0$ if $y \neq 0$. Assume that if $x \neq y$, then either $b_1(x \star y) = b_1(x)$ or $b_1(x \star (x \star y)) = b_1(y)$. Then

$$
(x \star (x \star y)) \star y = 0 \tag{4.8}
$$

for all $x, y \in \mathbb{R}$.

Proof. By Theorem 4.15 $(\mathbb{R}, \star)$ is an edge d-algebra and hence (4.8) holds for $x \star y = 0$ or $y = 0$. Assume $x \star y \neq 0$ and $y \neq 0$. Then

$$x \star (x \star y) = x \left[1 - \frac{b_1(x)}{b_1(x \star y)} \right].$$

It follows that

$$\begin{aligned}
(x \star (x \star y)) \star y &= [x \star (x \star y)] \left[1 - \frac{b_1(x \star (x \star y))}{b_1(y)} \right] \\
&= x \left[1 - \frac{b_1(x)}{b_1(x \star y)} \right] \left[1 - \frac{b_1(x \star (x \star y))}{b_1(y)} \right] \\
&= 0,
\end{aligned}$$

proving the proposition. $\square$

5. Smarandache fuzzy ideals in BCI-algebras

In this chapter, we discuss a Smarandache fuzzy structure on BCI-algebras and introduce the notion of a Smarandache fuzzy subalgebra (ideal) of a Smarandache BCI-algebra, a Smarandache fuzzy clean (fresh) ideal of a Smarandache BCI-algebra are introduced, and we investigate their properties.

5.1 Smarandache fuzzy ideals

Definition 5.1. Let X be a Smarandache BCI-algebra. A map $\mu : X \to [0,1]$ is called a *Smarandache fuzzy subalgebra* of X if it satisfies

(SF_1) $\mu(0) \geq \mu(x)$ for all $x \in P$,

(SF_2) $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in P$,

where $P \subsetneq X$, P is a BCK-algebra with $|P| \geq 2$. A map $\mu : X \to [0,1]$ is called a *Smarandache fuzzy ideal* of X if it satisfies (SF_1) and (F_2) $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$ for all $x, y \in P$, where $P \subsetneq X$, P is a BCK-algebra with $|P| \geq 2$. This Smarandache fuzzy subalgebra (ideal) is denoted by μ_P, i.e., $\mu_P : P \to [0,1]$ is a fuzzy subalgebra (ideal) of X.

Example 5.2. ([8]) Let $X := \{0, 1, 2, 3, 4, 5\}$ be a Smarandache BCI-algebra with the following table:

$*$	0	1	2	3	4	5
0	0	0	0	3	3	3
1	1	0	1	3	3	3
2	2	2	0	3	3	3
3	3	3	3	0	0	0
4	4	3	4	1	0	0
5	5	3	5	1	1	0

Define a map $\mu : X \to [0, 1]$ by

$$\mu(x) := \begin{cases} 0.5 & \text{if } x \in \{0, 1, 2, 3\}, \\ 0.7 & \text{otherwise.} \end{cases}$$

Clearly μ is a Smarandache fuzzy subalgebra of X. It is verified that μ restricted to a subset $\{0, 1, 2, 3\}$ which is a subalgebra of X is a fuzzy subalgebra of X, i.e., $\mu_{\{0,1,2,3\}} : \{0, 1, 2, 3\} \to [0, 1]$ is a fuzzy subalgebra of X. Thus $\mu : X \to [0, 1]$ is a Smarandache fuzzy subalgebra of X. Note that $\mu : X \to [0, 1]$ is not a fuzzy subalgebra of X, since $\mu(5 * 4) = \mu(1) = 0.5 \not\geq \min\{\mu(5), \mu(4)\} = 0.7$.

Example 5.3. ([8]) Let $X := \{0, 1, 2, 3, 4, 5\}$ be a Smarandache BCI-algebra with the following table:

$*$	0	1	2	3	4	5
0	0	0	0	0	4	4
1	1	0	0	1	4	4
2	2	2	0	2	4	4
3	3	3	3	0	4	4
4	4	4	4	4	0	0
5	5	4	4	5	1	0

Define a map $\mu : X \to [0, 1]$ by

$$\mu(x) := \begin{cases} 0.5 & \text{if } x \in \{0, 1, 2\}, \\ 0.7 & \text{otherwise.} \end{cases}$$

Clearly μ is a Smarandache fuzzy ideal of X. It is verified that μ restricted to a subset $\{0,1,2\}$ which is an ideal of X is a fuzzy ideal of X, i.e., $\mu_{\{0,1,2\}}$: $\{0,1,2\} \to [0,1]$ is a fuzzy ideal of X. Thus $\mu : X \to [0,1]$ is a Smarandache fuzzy ideal of X. Note that $\mu : X \to [0,1]$ is not a fuzzy ideal of X, since $\mu(2) = 0.5 \not\geq \min\{\mu(2*4) = \mu(4), \mu(4)\} = \mu(4) = 0.7$.

Lemma 5.4. *Every Smarandache fuzzy ideal μ_P of a Smarandache BCI-algebra X is order reversing.*

Proof. Let P be a BCI-algebra with $P \subsetneq X$ and $|P| \geq 2$. If $x, y \in P$ with $x \leq y$, then $x * y = 0$. Hence we have $\mu(x) \geq \min\{\mu(x*y), \mu(y)\} = \min\{\mu(0), \mu(y)\} = \mu(y)$. □

Theorem 5.5. *Every Smarandache fuzzy ideal μ_P of a Smarandache BCI-algebra X is a Smarandache fuzzy subalgebra of X.*

Proof. Let P be a BCI-algebra with $P \subsetneq X$ and $|P| \geq 2$. Since $x * y \leq x$ for any $x, y \in P$, it follows from Lemma 5.4 that $\mu(x) \leq \mu(x*y)$, so by (SF_2) we obtain $\mu(x*y) \geq \mu(x) \geq \min\{\mu(x*y), \mu(y)\} \geq \min\{\mu(x), \mu(y)\}$. This shows that μ is a Smarandache fuzzy subalgebra of X, proving the theorem. □

Proposition 5.6. *Let μ_P be a Smarandache fuzzy ideal of a Smarandache BCI-algebra X. If the inequality $x * y \leq z$ holds in P where BCI-algebra P with $P \subsetneq X$ and $|P| \geq 2$, then $\mu(x) \geq \min\{\mu(x), \mu(z)\}$ for all $x, y, z \in P$.*

Proof. If $x * y \leq z$ in P, then $(x * y) * z = 0$. Hence we have $\mu(x * y) \geq \min\{\mu((x * y) * z),\ \mu(z)\} = \min\{\mu(0),\ \mu(z)\} = \mu(z)$. It follows that $\mu(x) \geq \min\{\mu(x * y),\ \mu(y)\} \geq \min\{\mu(y),\ \mu(z)\}$. □

Theorem 5.7. *Let X be a Smarandache BCI-algebra. A Smarandache fuzzy subalgebra μ_P of X is a Smarandache fuzzy ideal of X if and only if for all $x, y \in P$ where BCI-algebra P with $P \subsetneq X$ and $|P| \geq 2$, the inequality $x * y \leq z$ implies $\mu(x) \geq \min\{\mu(y), \mu(z)\}$.*

Proof. Suppose that μ_P is a Smarandache fuzzy subalgebra of X satisfying the condition $x * y \leq z$ implies $\mu(x) \geq \min\{\mu(y), \mu(z)\}$. Since $x * (x * y) \leq y$ for all $x, y \in P$, it follows that $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$. Hence μ_P is a Smarandache fuzzy ideal of X. The converse follows from Proposition 5.6. □

5.2 Smarandache fuzzy clean ideals

Definition 5.8. Let X be a Smarandache BCI-algebra. A map $\mu : X \to [0,1]$ is called a *Smarandache fuzzy clean ideal* of X if it satisfies (SF_1) and

(SF_3) $\mu(x) \geq \min\{\mu(x * (y * x)) * z), \mu(z)\}$ for all $x, y, z \in P$,

where $P \subsetneq X$ and P is a BCK-algebra with $|P| \geq 2$. This Smarandache fuzzy clean ideal is denoted by μ_P, i.e., $\mu_P : P \to [0,1]$ is a Smarandache fuzzy clean ideal of X.

Example 5.9. ([9]) Let $X := \{0, 1, 2, 3, 4, 5\}$ be a Smarandache BCI-algebra with the following table:

$*$	0	1	2	3	4	5
0	0	0	0	0	0	5
1	1	0	0	0	0	5
2	2	1	0	1	0	5
3	3	4	4	4	0	5
4	4	4	4	4	0	5
5	5	5	5	5	5	0

Define a map $\mu : X \to [0,1]$ by

$$\mu(x) := \begin{cases} 0.4 & \text{if } x \in \{0, 1, 2, 3\}, \\ 0.8 & \text{otherwise.} \end{cases}$$

Clearly μ is a Smarandache fuzzy clean ideal of X, but μ is not a fuzzy clean ideal of X, since $\mu(3) = 0.4 \not\geq \min\{\mu((3*(0*3))*5), \mu(5)\} = \min\{\mu(5), \mu(5)\} = \mu(5) = 0.8$.

Theorem 5.10. *Let X be a Smarandache BCI-algebra. Every Smarandache fuzzy clean ideal μ_P of X is a Smarandache fuzzy ideal of X.*

Proof. Let X be a BCI-algebra with $P \subsetneq X$ and $|P| \geq 2$. Let $\mu_P : P \to [0,1]$ be a Smarandache fuzzy clean ideal of X. If we let $y := x$ in (SF_3), then $\mu(x) \geq \min\{\mu((x*(x*x))*z), \mu(z)\} = \min\{\mu((x*0)*z), \mu(z)\} = \min\{\mu(x*z), \mu(z)\}$, for all $x, y, z \in P$. This shows that μ satisfies (SF_2). Combining (SF_1), we get μ_P is a Smarandache fuzzy ideal of X, proving the theorem. □

Corollary 5.11. *Every Smarandache fuzzy clean ideal μ_P of a Smarandache BCI-algebra X is a Smarandache fuzzy subalgebra of X.*

Proof. It follows from Theorem 5.5 and Theorem 5.10. □

Example 5.12. Let $X := \{0, 1, 2, 3, 4, 5\}$ be a Smarandache BCI-algebra with the following table:

$*$	0	1	2	3	4	5
0	0	0	0	0	0	5
1	1	0	1	0	0	5
2	2	2	0	0	0	5
3	3	3	3	0	0	5
4	4	3	4	1	0	5
5	5	5	5	5	5	0

Let μ_P be a fuzzy set in $P = \{0, 1, 2, 3, 4\}$ defined by $\mu(0) = \mu(2) = 0.8$ and $\mu(1) = \mu(3) = \mu(4) = 0.3$. It is easy to check that μ_P is a fuzzy ideal of X. Hence $\mu : X \to [0,1]$ is a Smarandache fuzzy ideal of X. But it is not a Smarandache fuzzy clean ideal of X since $\mu(1) = 0.3 \not\geq \min\{\mu((1*(3*1))*2), \mu(2)\} = \min\{\mu(0), \mu(2)\} = 0.8$.

Theorem 5.13. *Let X be a Smarandache implicative BCI-algebra. Every Smarandache fuzzy ideal μ_P of X is a Smarandache fuzzy clean ideal of X.*

Proof. Let P be a BCI-algebra with $P \subsetneq X$ and $|P| \geq 2$. Since X is a Smarandache implicative BCI-algebra, we have $x = x * (y * x)$ for all $x, y \in P$. Let μ_P be a Smarandache fuzzy ideal of X. It follows from (SF_2) that $\mu(x) \geq \min\{\mu(x * z), \mu(z)\} \geq \min\{\mu((x * (y * x)) * z), \mu(z)\}$, for all $x, y, z \in P$. Hence μ_P is a Smarandache clean ideal of X. The proof is complete. □

In what follows, we give characterizations of fuzzy implicative ideals.

Theorem 5.14. *Let X be a Smarandache BCI-algebra. Suppose that μ_P is a Smarandache fuzzy ideal of X. Then the following equivalent:*

(i) *μ_P is Smarandache fuzzy clean,*

(ii) *$\mu(x) \geq \mu(x * (y * x))$ for all $x, y \in P$,*

(iii) *$\mu(x) = \mu(x * (y * x))$ for all $x, y \in P$.*

Proof. (i) $\Rightarrow$ (ii): Let μ_P be a Smarandache fuzzy clean ideal of X. It follows from (SF_3) that $\mu(x) \geq \min\{\mu((x * (y * x)) * 0), \mu(0)\} = \min\{\mu(x * (y * x)), \mu(0)\} = \mu(x * (y * x))$, for all $x, y \in P$. Hence the condition (ii) holds.
(ii) $\Rightarrow$ (iii): Since X is a Smarandache BCI-algebra, we have $x * (y * x) \leq x$ for all $x, y \in P$. It follows from Lemma 5.4 that $\mu(x) \leq \mu(x * (y * x))$. By (ii), $\mu(x) \geq \mu(x * (y * x))$. Thus the condition (iii) holds.

(iii) $\Rightarrow$ (i): Suppose that the condition (iii) holds. Since μ_P is a Smarandache fuzzy ideal, by (SF_2), we have $\mu(x*(y*x)) \geq \min\{\mu((x*(y*x))*z), \mu(z)\}$. By assumption, we obtain $\mu(x) \geq \min\{\mu((x*(y*x))*z), \mu(z)\}$. Hence μ satisfies the condition (SF_3). Obviously, μ satisfies (SF_1). Therefore μ is a fuzzy clean ideal of X. Hence the condition (i) holds. The proof is complete. $\square$

For any fuzzy sets μ and ν in X, we write $\mu \leq \nu$ if and only if $\mu(x) \leq \nu(x)$ for any $x \in X$.

Definition 5.15. Let X be a Smarandache BCI-algebra and let $\mu_P : P \to [0,1]$ be a Smarandache fuzzy BCI-algebra of X. For $t \leq \mu(0)$, the set $\mu_t := \{x \in P | \mu(x) \geq t\}$ is called a *level subset* of μ_P.

Theorem 5.16. *A fuzzy set μ in P is a Smarandache fuzzy clean ideal of X if and only if, for all $t \in [0,1]$, μ_t is either empty or a Smarandache clean ideal of X.*

Proof. Suppose that μ_P is a Smarandache fuzzy clean ideal of X and $\mu_t \neq \emptyset$ for any $t \in [0,1]$. It is clear that $0 \in \mu_t$ since $\mu(0) \geq t$. Let $\mu((x*(y*x))*z) \geq t$ and $\mu(z) \geq t$. It follows from (SF_3) that $\mu(x) \geq \min\{\mu((x*(y*x))*z), \mu(z)\} \geq t$, namely, $x \in \mu_t$. This shows that μ_t is a Smarandache clean ideal of X.

Conversely, assume that for each $t \in [0,1]$, μ_t is either empty or a Smarandache clean ideal of X. For any $x \in P$, let $\mu(x) = t$. Then $x \in \mu_t$. Since $\mu_t (\neq \emptyset)$ is a Smarandache clean ideal of X, $0 \in \mu_t$ and hence $\mu(0) \geq \mu(x) = t$. Thus $\mu(0) \geq \mu(x)$ for all $x \in P$. Now we show that μ satisfies (SF_3). If not, then there exist $x', y', z' \in P$ such that $\mu(x') < \min\{\mu((x' * (y' * z')) * z'), \mu(z')\}$. Taking $t_0 := \frac{1}{2}\{\mu(x') + \min\{\mu((x' * (y' * z')) * z'), \mu(z')\}\}$, we have $\mu(x') < t_0 < \min\{\mu((x' * (y' * z')) * z'), \mu(z')\}$. Hence $x' \notin \mu_{t_0}$, $(x' * (y' * x')) * z \in \mu_{t_0}$, and $z' \in \mu_{t_0}$, i.e., μ_{t_0} is not a Smarandache clean of X, which is a contradiction. Therefore, μ_P is a Smarandache fuzzy clean ideal, completing the proof. $\square$

Theorem 5.17. *([9]) (Extension Property) Let X be a Smarandache BCI-algebra. Let I and J be Q-Smarandache ideals of X and $I \subseteq J \subseteq Q$. If I is a Q-Smarandache clean ideal of X, then so is J.*

Next we give the extension theorem of Smarandache fuzzy clean ideals.

Theorem 5.18. *Let X be a Smarandache BCI-algebra. Let μ and ν be Smarandache fuzzy ideals of X such that $\mu \leq \nu$ and $\mu(0) = \nu(0)$. If μ is a Smarandache fuzzy clean ideal of X, then so is ν.*

Proof. It suffices to show that for any $t \in [0, 1]$, ν_t is either empty or a Smarandache clean ideal of X. If the level subset ν_t is non-empty, then $\mu_t \neq \emptyset$ and $\mu_t \subseteq \nu_t$. In fact, if $x \in \mu_t$, then $t \leq \mu(x)$; hence $t \leq \nu(x)$, i.e, $x \in \nu_t$. So $\mu_t \subseteq \nu_t$. By the hypothesis, since μ is a Smarandache fuzzy clean ideal of X, μ_t is a Smarandache clean of X by Theorem 5.16. It follows from Theorem 5.17 that ν_t is a Smarandache clean ideal of X. Hence ν is a Smarandache fuzzy clean of X. The proof is complete. $\square$

5.3 Smarandache fuzzy fresh ideals

Definition 5.19. Let X be a Smarandache BCI-algebra. A map $\mu : X \to [0,1]$ is called a *Smarandache fuzzy fresh ideal* of X if it satisfies (SF_1) and

(SF_4) $\mu(x * z) \geq \min\{\mu((x * y) * z), \mu(y * z)\}$ for all $x, y, z \in P$,

where P is a BCK-algebra with $P \subsetneq X$ and $|P| \geq 2$. This Smarandache fuzzy ideal is denoted by μ_P, i.e., $\mu_P : P \to [0,1]$ is a Smarandache fuzzy fresh ideal of X.

Example 5.20. ([9]) Let $X := \{0,1,2,3,4,5\}$ be a Smarandache BCI-algebra with the following table:

$*$	0	1	2	3	4	5
0	0	0	0	0	0	5
1	1	0	1	0	1	5
2	2	2	0	2	0	5
3	3	1	3	0	3	5
4	4	4	4	4	0	5
5	5	5	5	5	5	0

Define a map $\mu : X \to [0,1]$ by

$$\mu(x) := \begin{cases} 0.5 & \text{if } x \in \{0,1,3\}, \\ 0.9 & \text{otherwise.} \end{cases}$$

Clearly μ is a Smarandache fuzzy fresh ideal of X. But it is not a fuzzy fresh ideal of X, since $\mu(2*4) = \mu(0) = 0.5 \not\geq \min\{\mu(\ (2*5)*4),\ \mu(5*4)\} = \mu(5) = 0.9$.

Theorem 5.21. *Every Smarandache fuzzy fresh ideal of a Smarandache BCI-algebra X is a Smarandache fuzzy ideal of X.*

Proof. Taking $z := 0$ in (SF_4) and $x * 0 = x$, we have $\mu(x * 0) \geq \min\{\mu((x * y) * 0), \mu(y * 0)\}$. Hence $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$. Thus (SF_2) holds. □

The converse of Theorem 5.21 need not be true in general.

Example 5.22. ([9]) Let $X := \{0, 1, 2, 3, 4, 5\}$ be a Smarandache BCI-algebra with the following table:

$*$	0	1	2	3	4	5
0	0	0	0	0	0	5
1	1	0	0	0	1	5
2	2	1	0	1	2	5
3	3	1	1	0	3	5
4	4	4	4	4	0	5
5	5	5	5	5	5	0

Define a map $\mu : X \to [0, 1]$ by

$$\mu(x) := \begin{cases} 0.5 & \text{if } x \in \{0, 4\}, \\ 0.4 & \text{otherwise.} \end{cases}$$

Clearly $\mu(x)$ is a Smarandache fuzzy ideal of X. But $\mu(x)$ is not a Smarandache fuzzy fresh ideal of X, since $\mu(2 * 3) = \mu(1) = 0.4 \ngeq \min\ \{\mu((2 * 1) * 3), \mu(1 * 3)\} = \min\{\mu(1 * 3),\ \mu(0)\} = \mu(0) = 0.5$.

Proposition 5.23. *Let X be a Smarandache BCI-algebra. A Smarandache fuzzy ideal μ_P of X is a Smarandache fuzzy fresh ideal of X if and only if it satisfies the condition $\mu(x * y) \geq \mu((x * y) * y)$ for all $x, y \in P$.*

Proof. Assume that μ_P is a Smarandache fuzzy fresh ideal of X. Putting $z := y$ in (SF_4), we have $\mu(x*y) \geq \min\{\mu((x*y)*y), \mu(y*y)\} = \min\{\mu((x*y)*y), \mu(0)\} = \mu((x*y)*y)$, for all $x, y \in P$.

Conversely, let μ_P be a Smarandache fuzzy ideal of X such that $\mu(x*y) \geq \mu((x*y)*y)$. Since, for all $x, y, z \in P$, $((x*z)*z)*(y*z) \leq (x*z)*y = (x*y)*z$, we have $\mu((x*y)*z) \leq \mu(((x*z)*z)*(y*z))$. Hence $\mu(x*z) \geq \mu((x*z)*z) \geq \min\{\mu(((x*z)*z)*(y*z)), \mu(y*z)\} \geq \min\{\mu((x*y)*z), \mu(y*z)\}$. This completes the proof. □

Since $(x*y)*y \leq x*y$, it follows from Lemma 5.4 that $\mu(x*y) \leq \mu((x*y)*y)$. Thus we have the following theorem.

Theorem 5.24. *Let X be a Smarandache BCI-algebra. A Smarandache fuzzy ideal μ_P of X is Smarandache fuzzy fresh if and only if it satisfies the equality*

$$\mu(x*y) = \mu((x*y)*y), \quad for\ all\ x, y \in X.$$

We give an equivalent condition for which a Smarandache fuzzy subalgebra of a Smarandache BCI-algebra to be a Smarandache fuzzy clean ideal of X.

Theorem 5.25. *A Smarandache fuzzy subalgebra μ_P of X is a Smarandache fuzzy clean ideal of X if and only if it satisfies*

$$(x*(y*x))*z \leq u \text{ implies } \mu(x) \geq \min\{\mu(z), \mu(u)\} \text{ for all } x, y, z, u \in P. \quad (**)$$

Proof. Assume that μ_P is a Smarandache fuzzy clean ideal of X. Let $x, y, z, u \in P$ be such that $(x * (y * x)) * z \leq u$. Since μ is a Smarandache fuzzy ideal of X, we have $\mu(x*(y*x)) \geq \min\{\mu(z), \mu(u)\}$ by Theorem 5.7. By Theorem 5.14 (iii), we obtain $\mu(x) \geq \min\{\mu(z), \mu(u)\}$.

Conversely, suppose that μ_P satisfies $(**)$. Obviously, μ_P satisfies (SF_1), since $(x * \ (y * x)) * ((x * (y * x)) * z) \leq z$, by $(\dagger)$, we obtain $\mu(x) \geq \min\{\mu\ ((x*(y*x))*z),\ \mu(z)\}$, which shows that μ_P satisfies (SF_3). Hence μ_P is a Smarandache fuzzy clean ideal of X. The proof is complete. $\square$

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Analytic real algebras

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Abstract
In this paper we construct some real algebras by using elementary functions, and discuss some relations between several axioms and its related conditions for such functions. We obtain some conditions for real-valued functions to be a (edge) d-algebra.

Keywords: Analytic real algebra, Trace, d-algebra, BCK-algebra

Mathematics Subject Classification: 26A09, 06F35

Background

The notions of BCK-algebras and BCI-algebras were introduced by Iséki and Iséki and Tanaka (1980, 1978). The class of BCK-algebras is a proper subclass of the class of BCI-algebras. We refer useful textbooks for BCK-algebras and BCI-algebras (Lorgulescu 2008); Meng and Jun (1994); Yisheng (2006). The notion of d-algebras which is another useful generalization of BCK-algebras was introduced by Neggers and Kim (1999), and some relations between d-algebras and BCK-algebras as well as several other relations between d-algebras and oriented digraphs were investigated. Several aspects on d-algebras were studied (Allen et al. 2007; Han et al. 2010; Kim et al. 2012; Lee and Kim 1999; Neggers et al. 1999, 2000). Simply d-algebras can be obtained by deleting two identities as a generalization of BCK-algebras, but it gives more wide ranges of research areas in algebraic structures. Allen et al. (2007) developed a theory of companion d-algebras in sufficient detail to demonstrate considerable parallelism with the theory of BCK-algebras as well as obtaining a collection of results of a novel type. Han et al. (2010) defined several special varieties of d-algebras, such as strong d-algebras, (weakly) selective d-algebras and pre-d-algebras, and they showed that the squared algebra $(X, \square, 0)$ of a pre-d-algebra $(X, *, 0)$ is a strong d-algebra if and only if $(X, *, 0)$ is strong. Allen et al. (2011) introduced the notion of deformations in d / BCK-algebras. Using such deformations, d-algebras were constructed from BCK-algebras. Kim et al. (2012) studied properties of d-units in d-algebras, and they showed that the d-unit is the greatest element in bounded BCK-algebras, and it is equivalent to the greatest element in bounded commutative BCK-algebras. They obtained several properties related with the notions of weakly associativity, d-integral domain, left injective in d-algebras also.

In this paper we construct some real algebras by using elementary functions, and discuss some relations between several axioms and its related conditions for such functions. We obtain some conditions for real-valued functions to be a (edge) d-algebra.

Preliminaries

A *d-algebra* (Neggers and Kim 1999) is a non-empty set X with a constant 0 and a binary operation "$*$" satisfying the following axioms:

(I) $x * x = 0$,
(II) $0 * x = 0$,
(III) $x * y = 0$ and $y * x = 0$ imply $x = y$ for all $x, y \in X$.

For brevity we also call X a *d-algebra*. In X we can define a binary relation "$\leq$" by $x \leq y$ if and only if $x * y = 0$.

An algebra $(X, *, 0)$ of type (2,0) is said to be a *strong d-algebra* (Han et al. 2010) if it satisfies (I), (II) and (III*) hold for all $x, y \in X$, where

(III*) $x * y = y * x$ implies $x = y$.

Obviously, every strong d-algebra is a d-algebra, but the converse need not be true (Han et al. 2010).

Example 1 (Han et al. 2010) Let $\mathbf{R}$ be the set of all real numbers and $e \in \mathbf{R}$. Define $x * y := (x - y) \cdot (x - e) + e$ for all $x, y \in \mathbf{R}$ where "$\cdot$" and "$-$" are the ordinary product and subtraction of real numbers. Then $x * x = e$; $e * x = e$; $x * y = y * x = e$ yields $(x - y) \cdot (x - e) = 0$, $(y - x) \cdot (y - e) = 0$ and $x = y$ or $x = e = y$, i.e., $x = y$, i.e., $(\mathbf{R}, *, e)$ is a d-algebra.

However, $(\mathbf{R}, *, e)$ is not a strong d-algebra. If $x * y = y * x \Leftrightarrow (x - y) \cdot (x - e) + e = (y - x) \cdot (y - e) + e \Leftrightarrow (x - y) \cdot (x - e) = -(x - y) \cdot (y - e) \Leftrightarrow (x - y) \cdot (x - e + y - e) = 0 \Leftrightarrow (x - y) \cdot (x + y - 2e) = 0 \Leftrightarrow (x = y$ or $x + y = 2e)$, then there exist $x = e + \alpha$ and $y = e - \alpha$ such that $x + y = 2e$, i.e., $x * y = y * x$ and $x \neq y$. Hence, axiom (III*) fails and thus the d-algebra $(\mathbf{R}, *, e)$ is not a strong d-algebra.

A *BCK-algebra* is a d-algebra X satisfying the following additional axioms:

(IV) $((x * y) * (x * z)) * (z * y) = 0$,
(V) $(x * (x * y)) * y = 0$ for all $x, y, z \in X$.

Example 2 (Neggers et al. 1999) Let $X := \{0, 1, 2, 3, 4\}$ be a set with the following table:

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	3	0
3	3	3	2	0	3
4	4	4	1	1	0

Then $(X, *, 0)$ is a d-algebra which is not a BCK-algebra.

Let X be a d-algebra and $x \in X$. X is said to be *edge* if for any $x \in X$, $x * X = \{x, 0\}$. It is known that if X is an edge d-algebra, then $x * 0 = x$ for any $x \in X$ (Neggers et al. 1999).

Analytic real algebras

Let $\mathbf{R}$ be the set of all real numbers and let "$*$" be a binary operation on $\mathbf{R}$. Define a map $\lambda : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$. If we define $x * y := \lambda(x, y)$ for all $x, y \in \mathbf{R}$, then we call such a groupoid $(\mathbf{R}, *)$ an *analytic real algebra*.

Given an analytic groupoid $(\mathbf{R}, *)$, we define

$$tr(*, \lambda) := \int_{-\infty}^{\infty} \lambda(x, x)\ dx$$

We call $tr(*, \lambda)$ a *trace* of λ. Note that the trace $tr(*, \lambda)$ may or may not converge. Given an analytic groupoid $(\mathbf{R}, *)$, where $x * y := \lambda(x, y)$, if $x * x = 0$ for all $x \in \mathbf{R}$, then $tr(*, \lambda) = 0$, but the converse need not be true in general.

Example 3 Let $x_0 \in \mathbf{R}$. Define

$$\lambda(x, x) = \begin{cases} 0 & \text{if } x \neq x_0, \\ 1 & \text{otherwise} \end{cases}$$

Then $tr(*, \lambda) = \int_{-\infty}^{\infty} \lambda(x, x)\ dx = 0$, but $\lambda(x_0, x_0) = 1 \neq 0$, i.e., $x_0 * x_0 \neq 0$.

Proposition 4 *Let $(\mathbf{R}, *)$ be an analytic real algebra and let $a, b, c \in \mathbf{R}$, where $x * y := ax + by + c$ for all $x, y \in \mathbf{R}$. If $|tr(*, \lambda)| < \infty$, then $tr(*, \lambda) = 0$ and $x * y = a(x - y)$ for all $x, y \in \mathbf{R}$.*

Proof Given $x \in \mathbf{R}$, we have $x * x = (a + b)x + c$. Since $|tr(*, \lambda)| < \infty$, we have $|\int_{-\infty}^{\infty}[(a + b)x + c]dx| < \infty$. Now $\int_0^A[(a + b)x + c]\ dx = (a + b)\frac{A^2}{2} + cA = A[\frac{a+b}{2}A + c]$ for a large number A, so that if $|tr(*, \lambda)| < \infty$, then $a + b = 0$ and $c = 0$, i.e., we have $x * y = a(x - y)$, and thus $x * x = 0$ for all $x \in \mathbf{R}$. □

Theorem 5 *Let $a, b, c, d, e, f \in \mathbf{R}$. Define a binary operation "$*$" on $\mathbf{R}$ by*

$$x * y := ax^2 + bxy + cy^2 + dx + ey + f$$

for all $x, y \in \mathbf{R}$. If $|tr(*, \lambda)| < \infty$ and $0 * x = 0$ for all $x \in \mathbf{R}$, then $x * y = ax(x - y)$ for all $x, y \in \mathbf{R}$.

Proof Given $x \in \mathbf{R}$, we have $x * x = (a + b + c)x^2 + (d + e)x + f$. Let $A := a + b + c$, $B := d + e$. If we assume $|tr(*, \lambda)| < \infty$, then $|\int_{-\infty}^{\infty}(Ax^2 + Bx + f)\,dx| < \infty$. Now $\int_0^L(Ax^2 + Bx + f)\ dx = \frac{A}{3}L^3 + \frac{B}{2}L^2 + fL = L(\frac{A}{3}L^2 + \frac{B}{2} + f)$ for a large number L so that $|tr(*, \lambda)| < \infty$ implies $A = B = f = 0$, i.e., $a + b + c = 0, d + e = 0, f = 0$. It follows that

$$x * y = (ax - cy + d)(x - y) \tag{1}$$

If we assume $0 * x = 0$ for all $x \in \mathbf{R}$, then, by (1), we have

$$\begin{aligned} 0 &= 0 * x \\ &= (a0 - cx + d)(0 - x) \\ &= cx^2 - dx, \end{aligned}$$

for all $x \in \mathbf{R}$. This shows that $c = d = 0$. Hence $x * y = ax(x - y)$ for all $x, y \in \mathbf{R}$. □

Corollary 6 *Let $a, b, c, d, e, f \in \mathbf{R}$. Define a binary operation "$*$" on $\mathbf{R}$ by*

$$x * y := ax^2 + bxy + cy^2 + dx + ey + f$$

for all $x, y \in \mathbf{R}$. If $x * x = 0$ and $0 * x = 0$ for all $x \in \mathbf{R}$, then $x * y = ax(x - y)$ for all $x, y \in \mathbf{R}$.

Proof The condition, $x * x = 0$ for all $x \in \mathbf{R}$, implies $|tr(*, \lambda)| < \infty$. The conclusion follows from Theorem 5. □

Proposition 7 *Let $a, b, c, d, e, f \in \mathbf{R}$. Define a binary operation "$*$" on $\mathbf{R}$* by

$$x * y := ax^2 + bxy + cy^2 + dx + ey + f$$

for all $x, y \in \mathbf{R}$. If $|tr(*, \lambda)| < \infty$ and the anti-symmetry law holds for "$*$", then $(ax - cy + d)^2 + (ay - cx + d)^2 > 0$ for $x \neq y$.

Proof If $|tr(*, \lambda)| < \infty$, then by (1) we obtain $x * y = (ax - cy + d)(x - y)$. Assume the anti-symmetry law holds for "$*$". Then either $x * y \neq 0$ or $y * x \neq 0$ for $x \neq y$. It follows that $(x * y)^2 > 0$ or $(y * x)^2 > 0$, and hence $(x * y)^2 + (y * x)^2 > 0$. This shows that $(ax - cy + d)^2 + (ay - cx + d)^2 > 0$. □

Note that in Proposition 7 it is clear that if $(ax - cy + d)^2 + (ay - cx + d)^2 > 0$ for $x \neq y$, then the anti-symmetry law holds.

Corollary 8 *If we define $x * y := ax(x - y)$ for all $x, y \in \mathbf{R}$ where $a \neq 0$, then $(\mathbf{R}, *)$ is a d-algebra.*

Proof It is easy to see that $x * x = 0 = 0 * x$ for all $x \in \mathbf{R}$. Assume that $x \neq y$. Since $x * y = ax(x - y) = ax^2 - axy$, by applying Proposition 7, we obtain $b = -a, c = 0$, $d = e = f = 0$. It follows that $(ax - 0y + 0)^2 + (ay - 0x + 0)^2 = a^2x^2 + a^2y^2 = a^2(x^2 + y^2) > 0$ when $a \neq 0$. By Proposition 7, $(\mathbf{R}, *)$ is a d-algebra. □

Proposition 9 *Let $a, b, c, d, e, f \in \mathbf{R}$. Define a binary operation "$*$" on $\mathbf{R}$ by*

$$x * y := ax^2 + bxy + cy^2 + dx + ey + f$$

for all $x, y \in \mathbf{R}$. If $|tr(*, \lambda)| < \infty$ and $x * 0 = x$ for all $x \in \mathbf{R}$, then $x * y = (1 - cy)(x - y)$ for all $x, y \in \mathbf{R}$.

Proof If $|tr(*, \lambda)| < \infty$, then by (1) we obtain $x * y = (ax - cy + d)(x - y)$ for all $x, y \in \mathbf{R}$. If we let $y := 0$, then $x = x * 0 = (ax + d)x$. It follows that $ax^2 + (d - 1)x = 0$ for all $x \in \mathbf{R}$. This shows that $a = 0, d = 1$. Hence $x * y = (1 - cy)(x - y)$ for all $x, y \in \mathbf{R}$. □

Theorem 10 *If we define $x * y := (ax - cy + d)(x - y)$ for all $x, y \in \mathbf{R}$ where $a, c, d \in \mathbf{R}$ with $a + c \neq 0$, then the anti-symmetry law holds.*

Proof Assume that there exist $x \neq y$ in $\mathbf{R}$ such that $x * y = 0 = y * x$. Then $(ax - cy + d)(x - y) = 0$ and $(ay - cx + d)(y - x) = 0$. Since $x \neq y$, we have

$$ax - cy + d = 0 = ay - cx + d \tag{2}$$

It follows that $(a + c)(x - y) = 0$. Since $a + c \neq 0$, we obtain $x = y$, a contradiction. □

Remark The analytic algebra $(\mathbf{R}, *)$, $x * y = ax(x - y)$ for all $x, y \in \mathbf{R}$, was proved to be a d-algebra in Corollary 8 by using Proposition 7. Since $x * y = ax(x - y) = (ax - 0y + 0)(x - y)$, we know that $a + 0 = a \neq 0$. Hence the algebra $(\mathbf{R}, *)$ can be proved by using Theorem 10 also.

Note that the analytic real algebra $(\mathbf{R}, *)$ discussed in Corollary 8 need not be an edge d-algebra, since $x * 0 = ax(x - 0) = ax^2 \neq x$.

Analytic real algebras with functions

Let $\alpha, \beta : \mathbf{R} \to \mathbf{R}$ be real-valued functions. Define a binary operation "$*$" on $\mathbf{R}$ by

$$x * y := \alpha(x)x + \beta(y)y + c \tag{3}$$

where $c \in \mathbf{R}$.

Proposition 11 *Let $(\mathbf{R}, *)$ be an analytic real algebra defined by (3). If $x * x = 0 = 0 * x$ for all $x \in \mathbf{R}$, then $x * y = 0$ for all $x, y \in \mathbf{R}$.*

Proof Assume that $x * x = 0$ for all $x \in \mathbf{R}$. Then

$$\begin{aligned} 0 &= x * x \\ &= \alpha(x)x + \beta(x)x + c \\ &= [\alpha(x) + \beta(x)]x + c \end{aligned}$$

If we let $x := 0$, then $c = 0$. If $x \neq 0$, then $\alpha(x) + \beta(x) = 0$, i.e., $\beta(x) = -\alpha(x)$ for all $x \neq 0$ in $\mathbf{R}$. It follows that

$$x * y = \alpha(x)x - \alpha(y)y \tag{4}$$

Assume $0 * x = 0$ for all $x \in \mathbf{R}$. Then

$$\begin{aligned} 0 &= 0 * x \\ &= \alpha(0)0 + \beta(x)x + c \\ &= \beta(x)x \end{aligned}$$

It follows that $\beta(x) = 0$ for all $x \neq 0$ in $\mathbf{R}$. Hence we have $x * y = 0$ for all $x, y \in \mathbf{R}$. □

Proposition 12 *Let $(\mathbf{R}, *)$ be an analytic real algebra defined by (3). If $x * x = 0$ and $x * 0 = x$ for all $x \in \mathbf{R}$, then $x * y = x - y$ for all $x, y \in \mathbf{R}$.*

Proof If we assume $x * x = 0$ for all $x \in \mathbf{R}$, then by (4) we obtain $x * y = \alpha(x)x - \alpha(y)y$. Assume that $x * 0 = x$ for all $x \in \mathbf{R}$. Then $x = x * 0 = \alpha(x)x - \alpha(0)0 = \alpha(x)x$. This shows that $\alpha(x) = 1$ for any $x \neq 0$ in $\mathbf{R}$. Hence $x * y = x - y$ for all $x, y \in \mathbf{R}$. □

Let $a, b_1, b_2, c, d, e : \mathbf{R} \to \mathbf{R}$ be real-valued functions and let $f \in \mathbf{R}$. Define a binary operation "$*$" on $\mathbf{R}$ by

$$x * y := a(x)x^2 + b_1(x)b_2(y)xy + c(y)y^2 + d(x)x + e(y)y + f \tag{5}$$

for all $x, y \in \mathbf{R}$. Assume $0 * x = 0$ for all $x \in \mathbf{R}$. Then

$$\begin{aligned} 0 &= 0 * x \\ &= c(x)x^2 + e(x)x + f \\ &= [c(x)x + e(x)]x + f \end{aligned}$$

for all $x \in \mathbf{R}$. It follows that $f = 0$ and $c(x)x + e(x) = 0$ for all $x \neq 0$ in $\mathbf{R}$. Hence $c(y)y^2 + e(y)y = 0$ for all $y \in \mathbf{R}$. Hence

$$x * y = a(x)x^2 + b_1(x)b_2(y)xy + d(x)x \tag{6}$$

Assume $x * x = 0$ for all $x \in \mathbf{R}$. Then by (6) we obtain

$$\begin{aligned} 0 &= x * x \\ &= a(x)x^2 + b_1(x)b_2(x)x^2 + d(x)x \end{aligned}$$

It follows that $d(x)x = -[a(x)x^2 + b_1(x)b_2(x)x^2]$. By (6) we obtain

$$x * y = b_1(x)x[b_2(y)y - b_2(x)x] \tag{7}$$

Theorem 13 *Let $b_1, b_2 : \mathbf{R} \to \mathbf{R}$ be real-valued functions. Define a binary operation "$*$" on $\mathbf{R}$ as in (7). If we assume $b_2(x)x \neq b_2(y)y$ and $b_1^2(x)x^2 + b_1^2(y)y^2 > 0$ for any $x \neq y$ in $\mathbf{R}$, then $(\mathbf{R}, *)$ is a d-algebra.*

Proof Assume the anti-symmetry law holds. Then it is equivalent to that if $x \neq y$ then $x * y \neq 0$ or $y * x \neq 0$, i.e., if $x \neq y$ then $(x * y)^2 + (y * x)^2 > 0$. Since $x * y$ is defined by (7), we obtain that if $x \neq y$ then

$$(b_1^2(x)x^2 + b_1^2(y)y^2)(b_2(x)x - b_2(y)y)^2 > 0$$

By assumption, we obtain that $(\mathbf{R}, *)$ is a d-algebra. □

Example 14 Consider $x * y := ax(x - y)$ for all $x, y \in \mathbf{R}$. If we compare it with (7), then we have $b_1(x) = a, b_2(y) = -1$ and $b_2(x) = -1$ for all $x \in \mathbf{R}$. This shows that $b_2(x)x - b_2(y)y = (-1)x - (-1)y = y - x \neq 0$ when $x \neq y$. Moreover, $b_1^2(x)x^2 + b_1^2(y)y^2 = a^2x^2 + b_1^2(y)y^2 > 0$ since $a \neq 0$. By applying Theorem 13, we see that an analytic real algebra $(\mathbf{R}, *)$ where $x * y := ax(x - y)$, $a \neq 0$ is a d-algebra.

Example 15 Consider $x * y := x \tan 2x[e^y y - e^x x]$ for all $x, y \in \mathbf{R}$. By comparing it with (7), we obtain $b_1(x) = \tan 2x, b_2(y) = e^y$ and $b_2(x) = e^x$. If $x \neq y$, then it is easy to see that $xe^x \neq ye^y$ and $b_1^2(x)x^2 + b_1^2(y)y^2 = (\tan 2x)^2x^2 + (\tan 2y)^2y^2 > 0$ when $x \neq y$.

Hence an analytic real algebra $(\mathbf{R}, *)$ where $x * y := x \tan 2x[e^y y - e^x x]$ is a d-algebra by Theorem 13.

In Theorem 13, we obtained some conditions for analytic real algebras to be d-algebras. In addition, we construct an edge d-algebra from Theorem 13 as follows.

Theorem 16 *If we define a binary operation "$*$" on* $\mathbf{R}$ *by*

$$x * y := \begin{cases} x[1 - \frac{b_1(x)}{b_1(y)}] & \text{if } y \neq 0, \\ x & \text{otherwise} \end{cases}$$

where $b_1(x)$ is a real-valued function such that $b_1(y) \neq 0$ if $y \neq 0$. Then $(\mathbf{R}, *)$ is an edge d-algebra.

Proof Define a binary operation "$*$" on $\mathbf{R}$ as in (7) with additional conditions: $b_2(x)x \neq b_2(y)y$ and $b_1^2(x)x^2 + b_1^2(y)y^2 > 0$ for any $x \neq y$ in $\mathbf{R}$. Assume $x * 0 = x$ for all $x \in \mathbf{R}$. Then

$$\begin{aligned} x &= x * 0 \\ &= b_1(x)x[b_2(0)0 - b_2(x)x] \\ &= -b_1(x)b_2(x)x^2 \end{aligned}$$

Combining with (7) we obtain

$$\begin{aligned} x * y &= b_1(x)b_2(y)xy - b_1(x)b_2(x)x^2 \\ &= b_1(x)b_2(y)xy + x \\ &= x[b_1(x)b_2(y)y + 1] \end{aligned}$$

If we let $xy \neq 0$, then

$$\begin{aligned} x * y &= x\left[b_1(x)(-\frac{1}{b_1(y)}) + 1\right] \\ &= x\left[1 - \frac{b_1(x)}{b_1(y)}\right] \end{aligned}$$

If we let $x * y := x$ when $y = 0$, then $(\mathbf{R}, *)$ is an edge d-algebra. □

Example 17 Define a map $b_1(x) := e^{\lambda x}$ for all $x \in \mathbf{R}$. Then $x * y = x[1 - \frac{e^{\lambda x}}{e^{\lambda y}}] = x(1 - e^{\lambda(x-y)})$ when $y \neq 0$. If we define a binary operation "$*$" on $\mathbf{R}$ by

$$x * y := \begin{cases} x(1 - e^{\lambda(x-y)}) & \text{if } y \neq 0, \\ x & \text{otherwise,} \end{cases}$$

then $(\mathbf{R}, *)$ is an edge d-algebra.

Proposition 18 *If we define a binary operation "$*$" on* $\mathbf{R}$ *by*

$$x * y := \begin{cases} x[1 - \frac{b_1(x)}{b_1(y)}] & \text{if } y \neq 0, \\ x & \text{otherwise} \end{cases}$$

where $b_1(x)$ is a real-valued function such that $b_1(y) \neq 0$ if $y \neq 0$. Assume that if $x \neq y$, then either $b_1(x * y) = b_1(x)$ or $b_1(x * (x * y)) = b_1(y)$. Then

$$(x * (x * y)) * y = 0 \tag{8}$$

for all $x, y \in$ $\mathbf{R}$.

Proof By Theorem 16, $(\mathbf{R}, *)$ is an edge d-algebra and hence (8) holds for $x * y = 0$ or $y = 0$. Assume $x * y \neq 0$ and $y \neq 0$. Then

$$x * (x * y) = x\left[1 - \frac{b_1(x)}{b_1(x * y)}\right]$$

It follows that

$$\begin{aligned}(x * (x * y)) \star y &= [x * (x * y)]\left[1 - \frac{b_1(x * (x * y))}{b_1(y)}\right] \\ &= x\left[1 - \frac{b_1(x)}{b_1(x * y)}\right]\left[1 - \frac{b_1(x * (x * y))}{b_1(y)}\right] \\ &= 0,\end{aligned}$$

proving the proposition. □

Conclusions

We constructed some algebras on the set of real numbers by using elementary functions. The notions of (edge) d-algebras were developed from BCK-algebras, and widened the range of research areas. It is useful to find linear (quadratic) polynomial real algebras by using the real functions. In "Analytic real algebras" section, we obtained some linear (quadratic) algebras related to some algebraic axioms, and found suitable binary operations for (edge) d-algebras. In "Analytic real algebras with functions" section, we developed the idea of analytic methods, and obtained necessary conditions for the real valued function so that the real algebra is an edge d-algebra. We may apply the analytic method discussed here to several algebraic structures, and it may useful for find suitable conditions to construct several algebraic structures and many examples.

Authors' contributions
All authors read and approved the final manuscript.

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Smarandache fuzzy BCI-algebras

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Abstract. The notions of a Smarandache fuzzy subalgebra (ideal) of a Smarandache BCI-algebra, a Smarandache fuzzy clean(fresh) ideal of a Smarandache BCI-algebra are introduced. Examples are given, and several related properties are investigated.

1. Introduction

Generally, in any human field, a Smarandache structure on a set A means a weak structure W on A such that there exists a proper subset B of A with a strong structure S which is embedded in A. In [4], R. Padilla showed that Smarandache semigroups are very important for the study of congruences. Y. B. Jun ([1,2]) introduced the notion of Smarandache BCI-algebras, Smarandache fresh and clean ideals of Smarandache BCI-algebras, and obtained many interesting results about them.

In this paper, we discuss a Smarandache fuzzy structure on BCI-algebras and introduce the notions of a Smarandache fuzzy subalgebra (ideal) of a Smarandache BCI-algebra, a Smarandache fuzzy clean (fresh) ideal of a Smarandache BCI-algebra are introduced, and we investigate their properties.

2. Preliminaries

An algebra $(X;*,0)$ of type (2,0) is called a *BCI-algebra* if it satisfies the following conditions:

(I) $(\forall x,y,z\in X)(((x*y)*(x*z))*(z*y)=0)$,

(II) $(\forall x,y\in X)((x*(x*(x*y))*y=0)$,

(III) $(\forall x\in X)((x*x=0)$,

(IV) $(\forall x,y\in X)(x*y=0$ and $y*x=0$ imply $x=y)$.

If a BCI-algebra X satisfies the following identity;

(V) $(\forall x\in X)(0*x=0)$,

then X is said to be a *BCK-algebra*. We can define a partial order "$\le$" on X by $x\le y$ if and only if $x*y=0$.

Every BCI-algebra X has the following properties:

(a_1) $(\forall x\in X)(x*0=x)$,

(a_1) $(\forall x,y,z\in X)(x\le y$ implies $x*z\le y*z, z*y\le z*x)$.

A non-empty subset I of a BCI-algebra X is called an *ideal* of X if it satisfies the following conditions:

(i) $0\in I$,

(ii) $(\forall x\in X)(\forall y\in I)(x*y\in I$ implies $x\in I)$.

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Definition 2.1. ([1]) A *Smarandache BCI-algebra* is defined to be a BCI-algebra X in which there exists a proper subset Q of X such that

(i) $0 \in Q$ and $|Q| \geq 2$,

(ii) Q is a BCK-algebra under the same operation of X.

By a *Smarandache positive implicative* (resp. *commutative* and *implicative*) *BCI-algebra*, we mean a BCI-algebra X which has a proper subset Q of X such that

(i) $0 \in Q$ and $|Q| \geq 2$,

(ii) Q is a positive implicative (resp. commutative and implicative) BCK-algebra under the same operation of X.

Let $(X; *, 0)$ be a Smarandache BCI-algebra and H be a subset of X such that $0 \in H$ and $|H| \geq 2$. Then H is called a *Smarandache subalgebra* of X if $(H; *, 0)$ is a Smarandache BCI-algebra.

A non-empty subset I of X is called a *Smarandache ideal* of X related to Q if it satisfies:

(i) $0 \in I$,

(ii) $(\forall x \in Q)(\forall y \in I)(x * y \in I$ implies $x \in I)$,

where Q is a BCK-algebra contained in X. If I is a Smarandache ideal of X related to every BCK-algebra contained in X, we simply say that I is a Smarandache ideal of X.

In what follows, let X and Q denote a Smarandache BCI-algebra and a BCK-algebra which is properly contained in X, respectively.

Definition 2.2. ([2]) A non-empty subset I of X is called a *Smarandache ideal* of X related to Q (or briefly, a *Q-Smarandache ideal*) of X if it satisfies:

(c_1) $0 \in I$,

(c_2) $(\forall x \in Q)(\forall y \in I)(x * y \in I$ implies $x \in I)$.

If I is a Smarandache ideal of X related to every BCK-algebra contained in X, we simply say that I is a Smarandache ideal of X.

Definition 2.3. ([2]) A non-empty subset I of X is called a *Smarandache fresh ideal* of X related to Q (or briefly, a *Q-Smarandache fresh ideal* of X) if it satisfies the conditions (c_1) and

(c_3) $(\forall x, y, z \in Q)(((x * y) * z) \in I$ and $y * z \in I$ imply $x * z \in I)$.

Theorem 2.4. ([2]) *Every Q-Smarandache fresh ideal which is contained in Q is a Q-Smarandache ideal.*

The converse of Theorem 2.4 need not be true in general.

Theorem 2.5. ([2]) *Let I and J be Q-Smarandache ideals of X and $I \subset J$. If I is a Q-Smarandache fresh ideal of X, then so is J.*

Definition 2.6. ([2]) A non-empty subset I of X is called a *Smarandache clean ideal* of X related to Q (or briefly, a *Q-Smarandache clean ideal* of X) if it satisfies the conditions (c_1) and

(c_4) $(\forall x, y \in Q)(z \in I)((x * (y * x)) * z \in I$ implies $x \in I)$.

Theorem 2.7. ([2]) *Every Q-Smarandache clean ideal of X is a Q-Smarandache ideal.*

The converse of Theorem 2.7 need not be true in general.

Theorem 2.8. ([2]) *Every Q-Smarandache clean ideal of X is a Q-Smarandache fresh ideal.*

Theorem 2.9. ([2]) *Let I and J be Q-Smarandache ideals of X and $I \subset J$. If I is a Q-Smarandache clean ideal of X, then so is J.*

A fuzzy set μ in X is called a *fuzzy subalgebra* of a BCI-algebra X if $\mu(x*y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$. A fuzzy set μ in X is called a *fuzzy ideal* of X if

(F_1) $\mu(0) \geq \mu(x)$ for all $x \in X$,

(F_2) $\mu(x) \geq \min\{\mu(x*y), \mu(y)\}$ for all $x, y \in X$.

Let μ be a fuzzy set in a set X. For $t \in [0, 1]$, the set $\mu_t := \{x \in X | \mu(x) \geq t\}$ is called a *level subset* of μ.

3. Smarandache fuzzy ideals

Definition 3.1. Let X be a Smarandache BCI-algebra. A map $\mu : X \to [0, 1]$ is called a *Smarandache fuzzy subalgebra* of X if it satisfies

(SF_1) $\mu(0) \geq \mu(x)$ for all $x \in P$,

(SF_2) $\mu(x*y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in P$,

where $P \subsetneq X$, P is a BCK-algebra with $|P| \geq 2$.
A map $\mu : X \to [0, 1]$ is called a *Smarandache fuzzy ideal* of X if it satisfies (SF_1) and

(SF_2) $\mu(x) \geq \min\{\mu(x*y), \mu(y)\}$ for all $x, y \in P$,

where $P \subsetneq X$, P is a BCK-algebra with $|P| \geq 2$. This Smarandache fuzzy subalgebra (ideal) is denoted by μ_P, i.e., $\mu_P : P \to [0, 1]$ is a fuzzy subalgebra(ideal) of X.

Example 3.2. Let $X := \{0, 1, 2, 3, 4, 5\}$ be a Smarandache BCI-algebra ([1]) with the following Cayley table:

*	0	1	2	3	4	5
0	0	0	0	3	3	3
1	1	0	1	3	3	3
2	2	2	0	3	3	3
3	3	3	3	0	0	0
4	4	3	4	1	0	0
5	5	3	5	1	1	0

Define a map $\mu : X \to [0, 1]$ by

$$\mu(x) := \begin{cases} 0.5 & \text{if } x \in \{0, 1, 2, 3\}, \\ 0.7 & \text{otherwise} \end{cases}$$

Clearly μ is a Samrandache fuzzy subalgebra of X. It is verified that μ restricted to a subset $\{0, 1, 2, 3\}$ which is a subalgebra of X is a fuzzy subalgebra of X, i.e., $\mu_{\{0,1,2,3\}} : \{0, 1, 2, 3\} \to [0, 1]$ is a fuzzy subalgebra of X. Thus $\mu : X \to [0, 1]$ is a Smarandache fuzzy subalgebra of X. Note that $\mu : X \to [0, 1]$ is not a fuzzy subalgebra of X, since $\mu(5*4) = \mu(0) = 0.5 \not\geq \min\{\mu(5), \mu(4)\} = 0.7$.

Example 3.3. Let $X := \{0,1,2,3,4,5\}$ be a Smarandache BCI-algebra ([1]) with the following Cayley table:

$*$	0	1	2	3	4	5
0	0	0	0	0	4	4
1	1	0	0	1	4	4
2	2	2	0	2	4	4
3	3	3	3	0	4	4
4	4	44	4	0	0	
5	5	4	4	5	1	0

Define a map $\mu : X \to [0,1]$ by

$$\mu(x) := \begin{cases} 0.5 & \text{if } x \in \{0,1,2\} \\ 0.7 & \text{otherwise} \end{cases}$$

Clearly μ is a Samrandache fuzzy ideal of X. It is verified that μ restricted to a subset $\{0,1,2\}$ which is an ideal of X is a fuzzy ideal of X, i.e., $\mu_{\{0,1,2\}} : \{0,1,2\} \to [0,1]$ is a fuzzy ideal of X. Thus $\mu : X \to [0,1]$ is a Smarandache fuzzy ideal of X. Note that $\mu : X \to [0,1]$ is not a fuzzy ideal of X, since $\mu(2) = 0.5 \not\geq \min\{\mu(2*4) = \mu(4), \mu(4)\} = \mu(4) = 0.7$.

Lemma 3.4. *Every Smarandache fuzzy ideal μ_P of a Smarandache BCI-algebra X is order reversing.*

Proof. Let P be a BCK-algebra with $P \subsetneq X$ and $|P| \geq 2$. If $x, y \in P$ with $x \leq y$, then $x * y = 0$. Hence we have $\mu(x) \geq \min\{\mu(x*y), \mu(y)\} = \min\{\mu(0), \mu(y)\} = \mu(y)$. □

Theorem 3.5. *Any Smarandache fuzzy ideal μ_P of a Smarandache BCI-algebra X must be a Smarandache fuzzy subalgebra of X.*

Proof. Let P be a BCK-algebra with $P \subsetneq X$ and $|X| \geq 2$. Since $x * y \leq x$ for any $x, y \in P$, it follows from Lemma 3.4 that $\mu(x) \leq \mu(x*y)$, so by (SF_2) we obtain $\mu(x*y) \geq \mu(x) \geq \min\{\mu(x*y), \mu(y)\} \geq \min\{\mu(x), \mu(y)\}$. This shows that μ is a Smarandache fuzzy subalgebra of X, proving the theorem. □

Proposition 3.6. *Let μ_P be a Smarandache fuzzy ideal of a Smarandache BCI-algebra X. If the inequality $x * y \leq z$ holds in P, then $\mu(x) \geq \min\{\mu(x), \mu(z)\}$ for all $x, y, z \in P$.*

Proof. Let P be a BCK-algebra with $P \subsetneq X$ and $|P| \geq 2$. If $x * y \leq z$ in P, then $(x*y)*z = 0$. Hence we have $\mu(x*y) \geq \min\{\mu((x*y)*z), \mu(z)\} = \min\{\mu(0), \mu(z)\} = \mu(z)$. It follows that $\mu(x) \geq \min\{\mu(x*y), \mu(y)\} \geq \min\{\mu(y), \mu(z)\}$. □

Theorem 3.7. *Let X be a Smarandache BCI-algebra. A Smarandache fuzzy subalgebra μ_P of X is a Smarandache fuzzy ideal of X if and only if for all $x, y \in P$, the inequality $x * y \leq z$ implies $\mu(x) \geq \min\{\mu(y), \mu(z)\}$.*

Proof. Suppose that μ_P is a Smarandache fuzzy subalgebra of X satisfying the condition $x * y \leq z$ implies $\mu(x) \geq \min\{\mu(y), \mu(z)\}$. Since $x * (x*y) \leq y$ for all $x, y \in P$, it follows that $\mu(x) \geq \min\{\mu(x*y), \mu(y)\}$. Hence μ_P is a Smarandache fuzzy ideal of X. The converse follows from Proposition 3.6. □

Definition 3.8. Let X be a Smarandache BCI-algebra. A map $\mu : X \to [0,1]$ is called a *Smarandache fuzzy clean ideal* of X if it satisfies (SF_1) and

(SF_3) $\mu(x) \geq \min\{\mu(x*(y*x))*z), \mu(z)\}$ for all $x, y, z \in P$, where $P \subsetneq X$ and P is a BCK-algebra with $|P| \geq 2$. This Smarandache fuzzy clean ideal is denoted by μ_P, i.e., $\mu_P : P \to [0,1]$ is a Smarandache fuzzy clean ideal of X.

Example 3.9. Let $X := \{0,1,2,3,4,5\}$ be a Smarandache BCI-algebra ([2]) with the following Cayley table:

$*$	0	1	2	3	4	5
0	0	0	0	0	0	5
1	1	0	0	0	0	5
2	2	1	0	1	0	5
3	3	4	4	4	0	5
4	4	4	4	4	0	5
5	5	5	5	5	5	0

Define a map $\mu : X \to [0,1]$ by

$$\mu(x) := \begin{cases} 0.4 & \text{if } x \in \{0,1,2,3\} \\ 0.8 & \text{otherwise} \end{cases}$$

Clearly μ is a Samrandache fuzzy clean ideal of X, but μ is not a fuzzy clean ideal of X, since $\mu(3) = 0.4 \not\geq \min\{\mu((3*(0*3))*5), \mu(5)\} = \min\{\mu(5), \mu(5)\} = \mu(5) = 0.8$.

Theorem 3.10. *Let X be a Smarandache BCI-algebra. Any Smarandache fuzzy clean ideal μ_P of X must be a Smarandache fuzzy ideal of X.*

Proof. Let X be a BCK-algebra with $P \subsetneq X$ and $|P| \geq 2$. Let $\mu_P : P \to [0,1]$ be a Smarndache fuzzy clean ideal of X. If we let $y := x$ in (SF_3), then $\mu(x) \geq \min\{\mu((x*(x*x))*z), \mu(z)\} = \min\{\mu((x*0)*z), \mu(z)\} = \min\{\mu(x*z), \mu(z)\}$, for all $x, y, z \in P$. This shows that μ satisfies (SF_2). Combining (SF_1), μ_P is a Smarandache fuzzy ideal of X, proving the theorem. □

Corollary 3.11. *Every Smarandache fuzzy clean ideal μ_P of a Smarndache BCI-algebra X must be a Smarandache fuzzy subalgebra of X.*

Proof. It follows from Theorem 3.5 and Theorem 3.10. □

The converse of Theorem 3.10 may not be true as shown in the following example.

Example 3.12. Let $X := \{0,1,2,3,4,5\}$ be a Smarandache BCI-algebra with the following Cayley table:

$*$	0	1	2	3	4	5
0	0	0	0	0	0	5
1	1	0	1	0	0	5
2	2	2	0	0	0	5
3	3	3	3	0	0	5
4	4	3	4	1	0	5
5	5	5	5	5	5	0

Let μ_P be a fuzzy set in $P = \{0,1,2,3,4\}$ defined by $\mu(0) = \mu(2) = 0.8$ and $\mu(1) = \mu(3) = \mu(4) = 0.3$. It is easy to check that μ_P is a fuzzy ideal of X. Hence $\mu : X \to [0,1]$ is a Smarandache fuzzy ideal of X. But it is not a Smarandache fuzzy clean ideal of X since $\mu(1) = 0.3 \not\geq \min\{\mu((1*(3*1))*2), \mu(2)\} = \min\{\mu(0), \mu(2)\} = 0.8$.

Theorem 3.13. *Let X be a Smarandache implicative BCI-algebra. Every Smarandache fuzzy ideal μ_P of X is a Smarandache fuzzy clean ideal of X.*

Proof. Let P be a BCK-algebra with $P \subsetneq X$ and $|P| \geq 2$. Since X is a Smarandache implicative BCI-algebra, we have $x = x * (y * x)$ for all $x, y \in P$. Let μ_P be a Smarandache fuzzy ideal of X. It follows from (SF_2) that $\mu(x) \geq \min\{\mu(x * z), \mu(z)\} \geq \min\{\mu((x * (y * x)) * z), \mu(z)\}$, for all $x, y, z \in P$. Hence μ_P is a Smarandache clean ideal of X. The proof is complete. □

In what follows, we give characterizations of fuzzy implicative ideals.

Theorem 3.14. *Let X be a Smarandache BCI-algebra. Suppose that μ_P is a Smarandache fuzzy ideal of X. Then the following equivalent:*

(i) *μ_P is Smarandache fuzzy clean,*
(ii) *$\mu(x) \geq \mu(x * (y * x))$ for all $x, y \in P$,*
(iii) *$\mu(x) = \mu(x * (y * x))$ for all $x, y \in P$.*

Proof. (i) $\Rightarrow$ (ii): Let μ_P be a Smarandache fuzzy clean ideal of X. It follows from (SF_3) that $\mu(x) \geq \min\{\mu((x * (y * x)) * 0), \mu(0)\} = \min\{\mu(x * (y * x)), \mu(0)\} = \mu(x * (y * x))$, $\forall x, y \in P$. Hence the condition (ii) holds.
(ii) $\Rightarrow$ (iii): Since X is a Smarnadache BCI-algebra, we have $x * (y * x) \leq x$ for all $x, y \in P$. It follows from Lemma 3.4 that $\mu(x) \leq \mu(x * (y * x))$. By (ii), $\mu(x) \geq \mu(x * (y * x))$. Thus the condition (iii) holds.
(iii) $\Rightarrow$ (i): Suppose that the condition (iii) holds. Since μ_P is a Smarandache fuzzy ideal, by (SF_2), we have $\mu(x * (y * x)) \geq \min\{\mu((x * (y * x)) * z), \mu(z)\}$. Combining (iii), we obtain $\mu(x) \geq \min\{\mu((x * (y * x)) * z), \mu(z)\}$. Hence μ satisfies the condition (SF_3). Obviously, μ satisfies (SF_1). Therefore μ is a fuzzy clean ideal of X. Hence the condition (i) holds. The proof is complete. □

For any fuzzy sets μ and ν in X, we write $\mu \leq \nu$ if and only if $\mu(x) \leq \nu(x)$ for any $x \in X$.

Definition 3.15. Let X be a Smarandache BCI-algebra and let $\mu_P : P \to [0, 1]$ be a Smarandache fuzzy BCI-algebra of X. For $t \leq \mu(0)$, the set $\mu_t := \{x \in P | \mu(x) \geq t\}$ is called a *level subset* of μ_P.

Theorem 3.16. *A fuzzy set μ in P is a Smarandache fuzzy clean ideal of X if and only if, for all $t \in [0, 1]$, μ_t is either empty or a Smarandache clean ideal of X.*

Proof. Suppose that μ_P is a Smarandache fuzzy clean ideal of X and $\mu_t \neq \emptyset$ for any $t \in [0, 1]$. It is clear that $0 \in \mu_t$ since $\mu(0) \geq t$. Let $\mu((x * (y * x)) * z) \geq t$ and $\mu(z) \geq t$. It follows from (SF_3) that $\mu(x) \geq \min\{\mu((x * (y * x)) * z), \mu(z)\} \geq t$, namely, $x \in \mu_t$. This shows that μ_t is a Smarandache clean ideal of X.

Conversely, assume that for each $t \in [0, 1]$, μ_t is either empty or a Smaranadche clean ideal of X. For any $x \in P$, let $\mu(x) = t$. Then $x \in \mu_t$. Since $\mu_t (\neq \emptyset)$ is a Smarandache clean ideal of X, therefore $0 \in \mu_t$ and hence $\mu(0) \geq \mu(x) = t$. Thus $\mu(0) \geq \mu(x)$ for all $x \in P$. Now we show that μ satisfies (SF_3). If not, then there exist $x', y', z' \in P$ such that $\mu(x') < \min\{\mu((x' * (y' * z')) * z'), \mu(z')\}$. Taking $t_0 := \frac{1}{2}\{\mu(x') + \min\{\mu((x' * (y' * z')) * z'), \mu(z')\}\}$, we have $\mu(x') < t_0 < \min\{\mu((x' * (y' * z')) * z'), \mu(z')\}$. Hence $x' \notin \mu_{t_0}$, $(x' * (y' * x')) * z \in \mu_{t_0}$, and $z' \in \mu_{t_0}$, i.e., μ_{t_0} is not a Smaraqndache clean of X, which is a contradiction. Therefore, μ_P is a Smarnadche fuzzy clean ideal, completing the proof. □

Theorem 3.17. ([2]) (Extension Property) *Let X be a Smarandache BCI-algebra. Let I and J be Q-Smarandache ideals of X and $I \subseteq J \subseteq Q$. If I is a Q-Smarandache clean ideal of X, then so is J.*

Next we give the extension theorem of Smarandache fuzzy clean ideals.

Theorem 3.18. *Let X be a Smarandache BCI-algebra. Let μ and ν be Smarandache fuzzy ideals of X such that $\mu \leq \nu$ and $\mu(0) = \nu(0)$. If μ is a Smarndache fuzzy clean ideal of X, then so is ν.*

Proof. It suffices to show that for any $t \in [0,1]$, ν_t is either empty or a Smarandache clean ideal of X. If the level subset ν_t is non-empty, then $\mu_t \neq \emptyset$ and $\mu_t \subseteq \nu_t$. In fact, if $x \in \mu_t$, then $t \leq \mu(x)$; hence $t \leq \nu(x)$, i.e, $x \in \nu_t$. So $\mu_t \subseteq \nu_t$. By the hypothesis, since μ is a Smarandache fuzzy clean ideal of X, μ_t is a Smarandache clean of X by Theorem 3.16. It follows from Theorem 3.17 that ν_t is a Smarandache clean ideal of X. Hence ν is a Smarandache fuzzy clean of X. The proof is complete. □

Definition 3.19. Let X be a Smarandache BCI-algebra. A map $\mu : X \to [0,1]$ is called a *Smarandache fuzzy fresh ideal* of X if it satisfies (SF_1) and

(SF_4) $\mu(x * z) \geq \min\{\mu((x * y) * z), \mu(y * z)\}$ for all $x, y, z \in P$,

where P is a BCK-algebra with $P \subsetneq X$ and $|P| \geq 2$. This Smarandache fuzzy ideal is denoted by μ_P, i.e., $\mu_P : P \to [0,1]$ is a Smarandache fuzzy fresh ideal of X.

Example 3.20. Let $X := \{0, 1, 2, 3, 4, 5\}$ be a Smarandache BCI-algebra ([2]) with the following Cayley table:

$*$	0	1	2	3	4	5
0	0	0	0	0	0	5
1	1	0	1	0	1	5
2	2	2	0	2	0	5
3	3	1	3	0	3	5
4	4	4	4	4	0	5
5	5	5	5	5	5	0

Define a map $\mu : X \to [0,1]$ by

$$\mu(x) := \begin{cases} 0.5 & \text{if } x \in \{0,1,3\}, \\ 0.9 & \text{otherwise} \end{cases}$$

Clearly μ is a Samrandache fuzzy fresh ideal of X. But it is not a fuzzy fresh ideal of X, since $\mu(2 * 4) = \mu(0) = 0.5 \not\geq \min\{\mu((2 * 5) * 4), \mu(5 * 4)\} = \mu(5) = 0.9$.

Theorem 3.21. *Any Smarandache fuzzy fresh ideal of a Smarandache BCI-algebra X must be a Smarandache fuzzy ideal of X.*

Proof. Taking $z := 0$ in (SF_4) and $x * 0 = x$, we have $\mu(x * 0) \geq \min\{\mu((x * y) * 0), \mu(y * 0)\}$. Hence $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$. Thus (SF_2) holds. □

The converse of Theorem 3.21 may not be true as show in the following example.

Example 3.22. Let $X := \{0, 1, 2, 3, 4, 5\}$ be a Smarandache BCI-algebra ([2]) with the following Cayley table:

$*$	0	1	2	3	4	5
0	0	0	0	0	0	5
1	1	0	0	0	1	5
2	2	1	0	1	2	5
3	3	1	1	0	3	5
4	4	4	4	4	0	5
5	5	5	5	5	5	0

Define a map $\mu : X \to [0,1]$ by

$$\mu(x) := \begin{cases} 0.5 & \text{if } x \in \{0,4\}, \\ 0.4 & \text{otherwise} \end{cases}$$

Clearly $\mu(x)$ is a Samrandache fuzzy ideal of X. But $\mu(x)$ is not a Samrandache fuzzy fresh ideal of X, since $\mu(2*3) = \mu(1) = 0.4 \not\geq \min\{\mu((2*1)*3), \mu(1*3)\} = \min\{\mu(1*3), \mu(0)\} = \mu(0) = 0.5$.

Proposition 3.23. *Let X be a Smarandache BCI-algebra. A Smarandache fuzzy ideal μ_P of X is a Smarandache fuzzy fresh ideal of X if and only if it satisfies the condition $\mu(x*y) \geq \mu((x*y)*y)$ for all $x, y \in P$.*

Proof. Assume that μ_P is a Smarandache fuzzy fresh ideal of X. Putting $z := y$ in (SF_4), we have $\mu(x*y) \geq \min\{\mu((x*y)*y), \mu(y*y)\} = \min\{\mu((x*y)*y), \mu(0)\} = \mu((x*y)*y)$, $\forall x, y \in P$.

Conversely, let μ_P be Smarandache fuzzy ideal of X such that $\mu(x*y) \geq \mu((x*y)*y)$. Since, for all $x, y, z \in P$, $((x*z)*z)*(y*z) \leq (x*z)*y = (x*y)*z$, we have $\mu((x*y)*z) \leq \mu(((x*z)*z)*(y*z))$. Hence $\mu(x*z) \geq \mu((x*z)*z) \geq \min\{\mu(((x*z)*z)*(y*z)), \mu(y*z)\} \geq \min\{\mu((x*y)*z), \mu(y*z)\}$. This completes the proof. □

Since $(x*y)*y \leq x*y$, it follows from Lemma 3.4 that $\mu(x*y) \leq \mu((x*y)*y)$. Thus we have the following theorem.

Theorem 3.24. *Let X be a Smarandache BCI-algebra. A Smarandache fuzzy ideal μ_P of X is a Smarandache fuzzy fresh if and only if it satisfies the identity*

$$\mu(x*y) = \mu((x*y)*y), \ textfor \ all \ x, y \in X.$$

We give an equivalent condition for which a Smarandache fuzzy subalgebra of a Smarandache BCI-algebra to be a Smarandache fuzzy clean ideal of X.

Theorem 3.25. *A Smarandache fuzzy subalghebra μ_P of X is a Smarandache fuzzy clean ideal of X if and only if it satisfies*

$$(x*(y*x))*z \leq u \text{ implies } \mu(x) \geq \min\{\mu(z), \mu(u)\} \text{ for all} x, y, z, u \in P. \qquad (*)$$

Proof. Assume that μ_P is a Smarandache fuzzy clean ideal of X. Let $x, y, z, u \in P$ be such that $(x*(y*x))*z \leq u$. Since μ is a Smarandache fuzzy ideal of X, we have $\mu(x*(y*x)) \geq \min\{\mu(z), \mu(u)\}$ by Theorem 3.7. By Theorem 3.14-(iii), we obtain $\mu(x) \geq \min\{\mu(z), \mu(u)\}$.

Conversely, suppose that μ_P satisfies $(*)$. Obviously, μ_P satisfies (SF_1), since $(x*(y*x))*((x*(y*x))*z) \leq z$, by $(*)$, we obtain $\mu(x) \geq \min\{\mu((x*(y*x))*z), \mu(z)\}$, which shows that μ_P satisfies (SF_3). Hence μ_P is a Smarandache fuzzy clean ideal of X. The proof is complete. □

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국 문 요 지

대수구조와 그 응용에 관한 연구
(On the structure of general
algebras and its applications)

한양대학교 대학원

수학과

서 영 주

본 논문에서는 BCK-대수의 일반화된 대수 구조인 d-대수의 구조적 이해와 해석적 실 대수가 d-대수가 되는 과정을 규명하였으며, BCI-대수 상에서 Smarandache 개념을 도입하여 퍼지이론을 전개하였다. 먼저, 상 d-대수에서 두 가지 동형정리를 증명하고, obstinate d-이데알의 개념을 도입하여, 그 동치조건을 구하였다. 또한, 실공간 위에서 함수로서 정의되는 이항연산을 정의하여, 그것이 d-대수가 될 수 있는 조건들을 구하였다. 마지막으로 Smarandache BCI-대수 위에 Smarandache 퍼지 부분대수(이데알)의 개념을 도입하여 여러 동치가 되는 조건들을 구하였고, 기존 퍼지 개념을 재정립 하였다.

연구 윤리 서약서

본인은 한양대학교 대학원생으로서 이 학위논문 작성 과정에서 다음과 같이 연구 윤리의 기본 원칙을 준수하였음을 서약합니다.

첫째, 지도교수의 지도를 받아 정직하고 엄정한 연구를 수행하여 학위논문을 작성한다.

둘째, 논문 작성시 위조, 변조, 표절 등 학문적 진실성을 훼손하는 어떤 연구 부정행위도 하지 않는다.

셋째, 논문 작성시 논문유사도 검증시스템 "카피킬러"등을 거쳐야 한다.

2019년06월11일

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학과 : 수학과

지도교수 : 김희식

성명 : 서영주 (서명)

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