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On Pathos Lict Subdivision of a Tree

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Abstract: Let G be a graph and $E_1 \subset E(G)$. A Smarandachely E_1-lict graph $n^{E_1}(G)$ of a graph G is the graph whose point set is the union of the set of lines in E_1 and the set of cutpoints of G in which two points are adjacent if and only if the corresponding lines of G are adjacent or the corresponding members of G are incident.Here the lines and cutpoints of G are member of G. Particularly, if $E_1 = E(G)$, a Smarandachely $E(G)$-lict graph $n^{E(G)}(G)$ is abbreviated to *lict graph of* G and denoted by $n(G)$. In this paper, the concept of pathos lict sub-division graph $P_n[S(T)]$ is introduced. Its study is concentrated only on trees. We present a characterization of those graphs, whose lict sub-division graph is planar, outerplanar, maximal outerplanar and minimally nonouterplanar. Further, we also establish the characterization for $P_n[S(T)]$ to be eulerian and hamiltonian.

Key Words: pathos, path number, Smarandachely lict graph, lict graph, pathos lict sub-division graphs, Smarandache path k-cover, pathos point.

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§1. Introduction

The concept of *pathos* of a graph G was introduced by Harary [1] as a collection of minimum number of line disjoint open paths whose union is G. The path number of a graph G is the number of paths in a pathos. Stanton [7] and Harary [3] have calculated the path number for certain classes of graphs like trees and complete graphs. The *subdivision of a graph* G is obtained by inserting a point of degree 2 in each line of G and is denoted by $S(G)$. The path number of a subdivision of a tree $S(T)$ is equal to K, where $2K$ is the number of odd degree point of $S(T)$. Also, the end points of each path of any pathos of $S(T)$ are odd points. The *lict graph* $n(G)$ of a graph G is the graph whose point set is the union of the set of lines and the set of cutpoints of G in which two points are adjacent if and only if the corresponding lines of G are adjacent or the corresponding members of G are incident.Here the lines and cutpoints of G are member of G.

For any integer $k \geq 1$, a Smarandache path k-cover of a graph G is a collection ψ of paths in G such that each edge of G is in at least one path of ψ and two paths of ψ have at most

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k vertices in common. Thus if $k = 1$ and every edge of G is in exactly one path in ψ, then a Smarandache path k-cover of G is a simple path cover of G. See [8].

By a graph we mean a finite, undirected graph without loops or multiple lines. We refer to the terminology of [1]. The *pathos lict subdivision of a tree* T is denoted as $P_n[S(T)]$ and is defined as the graph, whose point set is the union of set of lines, set of paths of pathos and set of cutpoints of $S(T)$ in which two points are adjacent if and only if the corresponding lines of $S(T)$ are adjacent and the line lies on the corresponding path P_i of pathos and the lines are incident to the cutpoints. Since the system of path of pathos for a $S(T)$ is not unique, the corresponding pathos lict subdivision graph is also not unique. The pathos lict subdivision graph is defined for a tree having at least one cutpoint.

In Figure 1, a tree T and its subdivision graph $S(T)$, and their pathos lict subdivision graphs $P_n[S(T)]$ are shown.

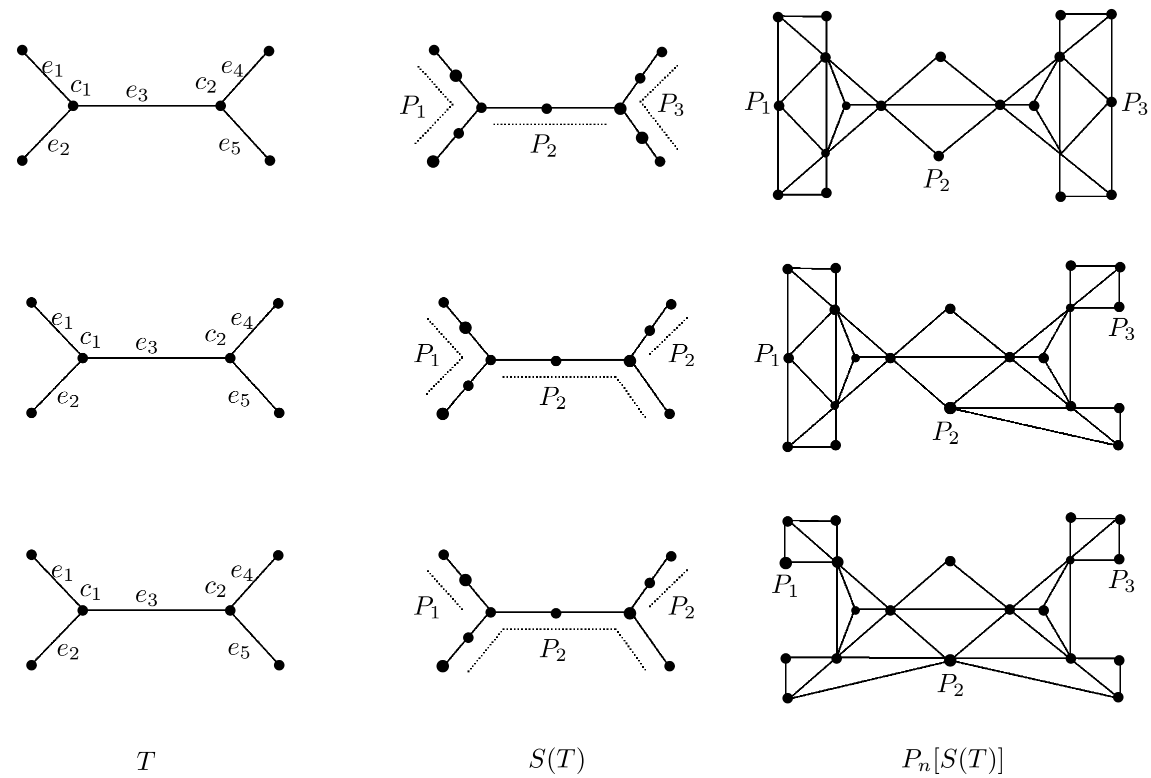

Figure 1

The *line degree* of a line uv in $S(T)$ is the sum of the degrees of u and v. The pathos length is the number of lines which lies on a particular path P_i of pathos of $S(T)$. A pendant pathos is a path P_i of pathos having unit length which corresponds to a pendant line in $S(T)$. A pathos point is a point in $P_n[S(T)]$ corresponding to a path of pathos of $S(T)$. If G is planar graph,the *innerpoint number* $i(G)$ of a graph G is the minimum number of vertices not belonging to the boundary of the exterior region in any embedding of the plane. A graph is said to be minimally nonouterplanar if $i(G) = 1$ was given by [4].

We need the following for immediate use.

Remark 1.1 *For any tree T, $n[S(T)]$ is a subgraph of $P_n[S(T)]$.*

Remark 1.2 *For any tree T, $T \subseteq S(T)$.*

Remark 1.3 *If the line degree of a nonpendant line in $S(T)$ is odd(even), the correspondig point in $P_n[S(T)]$ is of even(odd) degree.*

Remark 1.4 *The pendant line in $S(T)$ is always odd degree and the corresponding point in $P_n[S(T)]$ is of odd degree.*

Remark 1.5 *For any tree T with C cutpoints, the number of cutpoints in $n[S(T)]$ is equal to sum of the lines incident to C in T.*

Remark 1.6 *For any tree T, the number of blocks in $n[S(T)]$ is equal to the sum of the cutpoints and lines of T.*

Remark 1.7 *$n[S(T)]$ is connected if and only if T is connected.*

Theorem 1.1([5]) *If G is a non trivial connected (p, q) graph whose points have degree d_i and l_i be the number of lines to which cutpoint C_i belongs in G, then lict graph $n(G)$ has $q + \sum C_i$ points and $-q + \sum[\frac{d_i^2}{2} + l_i]$ lines.*

Theorem 1.2([5]) *The lict graph $n(G)$ of a graph G is planar if and only if G is planar and the degree of each point is atmost 3.*

Theorem 1.3([2]) *Every maximal outerplanar graph G with p points has $2p - 3$ lines.*

Theorem 1.4([6]) *A graph is a nonempty path if and only if it is a connected graph with $p \geq 2$ points and $\sum d_i^2 - 4p + 6 = 0$.*

Theorem 1.5([2]) *A graph G is eulerian if and only if every point of G is of even degree.*

§2. Pathos Lict Subdivision Graph

In the following Theorem we obtain the number of points and lines of $P_n[S(T)]$.

Theorem 2.1 *For any (p, q) graph T, whose points have degree d_i and cutpoints C have degree C_j, then the pathos lict sub-division graph $P_n[S(T)]$ has $(3q + C + P_i)$ points and $\frac{1}{2}\sum d_i^2 + 4q + \sum C_j$ lines.*

Proof By Theorem 1.1, $n(T)$ has $q + \sum c$ points by subdivision of T $n(S(T))$contains $2q + q + \sum c$ points and by Remark 1.1, $P_nS(T)$ will contain $3q + \sum c + P_i$ points, where P_i is the path number. By the definition of $n(T)$, it follows that $L(T)$ is a subgraph of $n(T)$. Also, subgraphs of $L(T)$ are line-disjoint subgraphs of $n[S(T)]$ whose union is $L(T)$ and the cutpoints c of T having degree C_j are also the members of $n[s(T)]$. Hence this implies that $n[s(T)]$ contains $-q + \frac{1}{2}\sum d_i^2 + \sum c_j$ lines. Apart from these lines every subdivision of T generates a line and a cutpoint c of degree 2. This creates $q + 2q$ lines in $n[s(T)]$. Thus $n[S(T)]$ has $\frac{1}{2}\sum d_i^2 + \sum c_j + 2q$ lines. Further, the pathos contribute $2q$ lines to $P_nS(T)$. Hence $P_n[S(T)]$ contains $\frac{1}{2}\sum d_i^2 + \sum c_j + 4q$lines. □

Corollary 2.1 *For any (p, q) graph T, the number of regions in $P_n[S(T)]$ is $2(p + q) - 3$.*

§3. Planar Pathos Lict Sub-division Graph

In this section we obtain the condition for planarity of pathos.

Theorem 3.1 *$P_n[S(T)]$ of a tree T is planar if and only if $\Delta(T) \leq 3$.*

Proof Suppose $P_n[S(T)]$ is planar. Assume $\Delta(T) \leq 4$. Let v be a point of degree 4 in T. By Remark 1.1, $n(S(T))$ is a subgraph of $P_n[S(T)]$ and by Theorem 1.2, $P_n[S(T)]$ is non-planar. Clearly, $P_n[S(T)]$ is non-planar, a contradiction.

Conversely, suppose $\Delta(T) \leq 3$. By Theorem 1.2, $n[S(T)]$ is planar. Further each block of $n[S(T)]$ is either K_3 or K_4. The pathos point is adjacent to atmost two vertices of each block of $n[S(T)]$. This gives a planar $P_n[S(T)]$. □

We next give a characterization of trees whose pathos lict subdivision of trees are outerplanar and maximal outerplanar.

Theorem 3.2 *The pathos lict sub-division graph $P_n[S(T)]$ of a tree T is outerplanar if and only if $\Delta(T) \leq 2$.*

Proof Suppose $P_n[S(T)]$ is outerplanar. Assume T has a point v of degree 3. The lines incident to v and the cut-point v form $\langle K_4\rangle$ as a subgraph in $n[S(T)]$. Hence $P_n[S(T)]$ is non-outerplanar, a contradiction.

Conversely, suppose T is a path P_m of length $m \geq 1$, by definition each block of $n[S(T)]$ is K_3 and $n[S(T)]$ has $2m - 1$ blocks. Also, $S(T)$ has exactly one path of pathos and the pathos point is adjacent to atmost two points of each block of $n[S(T)]$. The pathos point together with each block form $2m - 1$ number of $\langle K_4 - x\rangle$ subgraphs in $P_n[S(T)]$. Hence $P_n[S(T)]$ is outerplanar. □

Theorem 3.3 *The pathos lict sub-division graph $P_n[S(T)]$ of a tree T is maximal outerplanar if and only if.*

Proof Suppose $P_n[S(T)]$ is maximal outerplanar. Then $P_n[S(T)]$ is connected. Hence by Remark 1.7, T is connected. Suppose $P_n[S(T)]$ is $K_4 - x$, then clearly, T is K_2. Let T be any connected tree with $p > 2$ points, q lines and having path number k and C cut-points. Then clearly, $P_n[S(T)]$ has $3q+k+C$ points and $\frac{1}{2}\sum d_i^2+4q+\sum C_j$ lines. Since $P_n[S(T)]$ is maximal outerplanar, by Theorem 1.3, it has $[2(3q+k+C)-3]$ lines. Hence

$$\begin{aligned}\frac{1}{2}\sum d_i^2+4q+\sum C_j &= [2(3q+k+C)-3]\\ &= [2(3(p-1)+k+C)-3]\\ &= 6p-6+2k+2C-3\\ &= 6p+2k+2C-9.\end{aligned}$$

But for $k=1$,

$$\sum d_i^2+8q+2\sum C_j=12p+4C-18+4,$$

$$\sum d_i^2+2\sum C_j=4p+4C-6,$$

$$\sum d_i^2+2\sum C_j-4p-4C+6=0.$$

Since every cut-point is of degree two in a path, we have,

$$\sum C_j=2C.$$

Therefore

$$\sum d_i^2+6-4p=4C-2x2C=0.$$

Hence $\sum d_i^2+6-4p=0$. By Theorem 1.4, it follows that T is a non-empty path.

Conversely, Suppose T is a non-empty path. We now prove that $P_n[S(T)]$ is maximal outerplanar by induction on the number of points (≥ 2). Suppose T is K_2. Then $P_n[S(T)]=K_4-x$. Hence it is maximal outerplanar. As the inductive hypothesis, let the pathos lict subdivision of a non-empty path P with n points be maximal outerplanar. We now show that $P_n[S(T)]$ of a path P with $n+1$ points is maximal outerplanar. First we prove that it is outerplanar. Let the point and line sequence of the path P' be $v_1,e_1,v_2,e_2,v_3,e_3,\ldots,v_n,e_n,v_{n+1}$. P', $S(P')$ and $P_n[S(P')]$ are shown in Figure 2. Without loss of generality, $P'-v_{n+1}=P$. By inductive hypothesis $P_n[S(P)]$ is maximal outerplanar. Now the point v_{n+1} is one point more in $P_n[S(P')]$ than in $P_n[S(P)]$. Also there are only eight lines (e'_{n-1},e_n), (e'_{n-1},e_{n-1}), (e_{n-1},e_n), (e_n,R), (e_n,e'_n), (e_n,C'_n), (C'_n,e'_n), (e'_n,R) more in $P_n[S(P')]$. Clearly, the induced subgraph on the points e'_{n-1}, C_{n-1}, e_n, e'_n, C'_n, R is not K_4. Hence $P_n[S(P')]$ is outerplanar. We now prove $P_n[S(P')]$ is maximal outerplanar. Since $P_n[S(P)]$ is maximal outerplanar, it has $2(3q+C+1)-3$ lines. The outerplanar graph $P_n[S(P')]$ has $2(3q+C+1)-3+8$ lines $=2[3(q+1)+(C+)+1]-3$ lines. By Theorem 1.3, $P_n[S(P')]$ is maximal outerplanar. □

Theorem 3.4 *For any tree T, $P_n[S(T)]$ is minimally nonouterplanar if and only if $\Delta(T)\leq 3$ and T has a unique point of degree* 3.

Proof Suppose $P_n[S(T)]$ is minimally non-outerplanar. Assume $\Delta(T)>3$. By Theorem 3.1, $P_n[S(T)]$ is nonplanar, a contradiction. Hence $\Delta(T)\leq 3$.

Assume $\Delta(T)<3$. By Theorem 3.2, $P_n[S(T)]$ is outerplanar, a contradiction. Thus $\Delta(T)=3$.

Assume there exist two points of degree 3 in T. Then $n[S(T)]$ has at least two blocks as K_4. Any pathos point of $S(T)$ is adjacent to atmost two points of each block in $n[S(T)]$ which gives $i(P_n[S(T)])>1$, a contradiction. Hence T has exactly one point point of degree 3.

Conversely, suppose every point of T has degree ≤ 3 and has a unique point of degree 3, then $n[S(T)]$ has exactly one block as K_4 and remaining blocks are K_3's. Each pathos point is adjacent to atmost two points of each block. Hence $i(P_n[S(T)]) = 1$. □

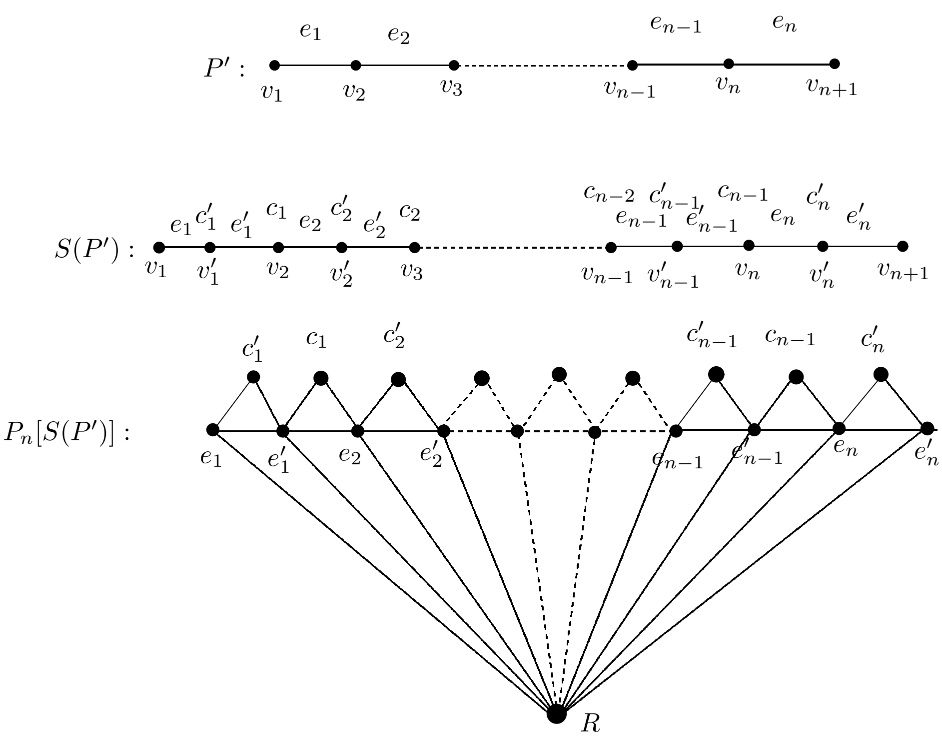

Figure 2

§4. Traversability in Pathos Lict Subdivision of a Tree

In this section, we characterize the trees whose $P_n[S(T)]$ is eulerian and hamiltonian.

Theorem 4.1 *For any non-trivial tree T, the pathos lict subdivision of a tree is non-eulerian.*

Proof Let T be a non-trivial tree. Remark 1.4 implies $P_n[S(T)]$ always contains a point of odd degree. Hence by Theorem 1.5, the result follows. □

Theorem 4.2 *The pathos lict subdivision $P_n[S(T)]$ of a tree T is hamiltonian if and only if every cut-point of T is even of degree.*

Proof If $T = P_2$, then $P_n[S(T)]$ is $K_4 - x$. If T is a tree with $p \geq 3$ points. Suppose $P_n[S(T)]$ is hamiltonian. Assume that T has at least one cut-point v of odd degree m. Then $G = K_{1,m}$ is a subgraph of T. Clearly, $n(S(K_{1,m})) = K_{m+1}$, together with each point of K_m incident to a line of K_3. In number of path of pathos of $S(T)$ there exist at least one path of pathos P_i such that it begins with the cut-point v of $S(T)$. In $P_n[S(T)]$ each pathos point is adjacent to exactly two points of K_m. Further the pathos beginning with the cut-point v of $S(T)$ is adjacent to exactly one point of K_m in $n(S(T))$. Hence this creates a cut-point in $P_n[S(T)]$, a contradiction.

Conversely, suppose every cut-point of T is even. Then every path of pathos starts and ends at pendant points of T.

We consider the following cases.

Case 1 If T has only cut-points of degree two. Clearly, T is a path. Further $S(T)$ is also a path with $p + q$ points and has exactly one path of pathos. Let $T = P_l, v_1, v_2, \cdots, v_l$ is a path. Now $S(T)$:$v_1, v_1', v_2, v_2', \cdots, v_{l-1}', v_l$ for all $v_i \in V[S(P_l)]$ such that $v_i v_i' = e_i$, $v_i' v_{i+1} = e_i'$ are consecutive lines and for all $e_i, e_i' \in E[S(P_n)]$. Further $V[n(S(T))]$=$\{e_1, e_1', e_2, e_2', \cdots, e_i, e_i'\} \cup \{C_1', C_1, C_2', C_2, \cdots, C_i'\}$ where,$(C_1', C_1, C_2', C_2, \cdots, C_i')$ are cut-points of $S(T)$. Since each block is a triangle in $n(S(T))$ and each block consist of points as $B_1 = (e_1 C_1' e_1')$, $B_2 = (e_2 C_2' e_2')$, $\cdots$, $B_m = (e_i C_i' e_i')$. In $P_n[S(T)]$, the pathos point w is adjacent to $e_1, e_1', e_2, e_2', \cdots, e_i, e_i'$. Hence, $P_n[S(T)]$= $e_1, e_1', e_2, e_2', \cdots, e_i, e_i' \cup (C_1', C_1, C_2', C_2, \cdots, C_i') \cup w$ form a cycle as $w e_1 C_1' e_1' C_1 e_2 C_2' e_2' \cdots e_i' w$ containing all the points of $P_n[S(T)]$.Hence $P_n[S(T)]$ is hamiltonian.

Case 2 If T has all cut-points of even degree and is not a path.

we consider the following subcases of this case.

Subcase 2.1. If T has exactly one cut-point v of even degree m, $v = \Delta(T)$ and is $K_{1,m}$. Clearly, $S(K_{1,m}) = F$, such that $E(F)$=$\{e_1, e_1', e_2, e_2', \cdots, e_q, e_q'\}$. Now $n(F)$ contains point set as $\{e_1, e_1', e_2, e_2', \cdots, e_q, e_q'\} \cup \{v, C_1', C_2', C_3', \cdots, C_q'\}$. For $S[K_{1,m}]$, it has $\frac{m}{2}$ paths of pathos with pathos point as $P_1, P_2, \cdots, P_{\frac{m}{2}}$. By definition of $P_n[S(T)]$, each pathos point is adjacent to exactly two points of $n(S(T))$. Also, $V[P_n[S(T)]]$=$\{e_1, e_1', e_2, e_2', \cdots, e_q, e_q'\} \cup \{v, C_1', C_2', C_3', \cdots, C_q'\} \cup \{P_1, P_2, \cdots, P_{\frac{m}{2}}\}$. Then there exist a cycle containing all the points of $P_n[S(T)]$ as $P_1, e_1', C_1', e_1, v, e_2, C_2', e_2', P_2, \cdots, P_{\frac{m}{2}}, e_{q-1}', C_{q-1}', e_{q-1}, e_q, C_q', e_q', P_1$.

Subcase 2.2. Assume T has more than one cut-point of even degree. Then in $n(S(T))$ each block is complete and every cut-point lies on exactly two blocks of $n(S(T))$. Let $V[n(S(T))]$=$\{e_1, e_1', e_2, e_2', \cdots, e_q, e_q'\} \cup \{C_1, C_2, \cdots, C_i\} \cup \{C_1', C_2', C_3', \cdots, C_q'\} \cup \{P_1, P_2, \cdots, P_j\}$. But each P_j is adjacent to exactly two point of the block B_j except $\{C_1, C_2, \cdots, C_i\} \cup \{C_1', C_2', C_3', \cdots, C_q'\}$ and all these points together form a hamiltonian cycle of the type, $\{P_1, e_1', C_1', e_1, v, e_2, C_2', e_2', P_2, \cdots, P_r, e_k', C_k', e_k, e_{k+1}, C_{k+1}', e_{k+1}', P_{r+1}, \cdots, P_j, e_{q-1}', C_{q-1}', e_{q-1}, e_q, C_q', e_q', P_1\}$.

Hence $P_n[S(T)]$ is hamiltonian. □

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