ON THE M-POWER FREE PART OF AN INTEGER

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Abstract The main purpose of this paper is using the elementary method to study the mean value properties of a new arithmetical function involving the m-power free part of an integer, and give an interesting asymptotic formula for it.

Keywords: Arithmetical function; Mean value; Asymptotic formula

§1. Introduction

For any positive integer n, it is clear that we can assume $n = u^m v$, where v is a m-power free number. Let $b_m(n) = v$ be the m-power free part of n. For example, $b_3(8) = 1$, $b_3(24) = 3$, $b_2(12) = 3$, $\cdots\cdots$. Now for any positive integer $k > 1$, we define another function $\delta_k(n)$ as following:

$$\delta_k(n) = \max\{d : d \mid n, (d, k) = 1\}.$$

From the definition of $\delta_k(n)$, we can prove that $\delta_k(n)$ is also a completely multiplicative function. In reference [1], Professor F.Smarandache asked us to study the properties of the sequence $\{b_m(n)\}$. It seems that no one knows the relations between sequence $\{b_m(n)\}$ and the arithmetical function $\delta_k(n)$ before. The main purpose of this paper is to study the mean value properties of $\delta_k(b_m(n))$, and obtain an interesting mean value formula for it. That is, we shall prove the following conclusion:

Theorem. *Let m and k be any fixed positive integer. Then for any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{n\leq x} \delta_k(b_m(n)) = \frac{x^2}{2}\frac{\zeta(2m)}{\zeta(m)}\prod_{p|k}\frac{p^m+1}{p^{m-1}(p+1)} + O\left(x^{\frac{3}{2}+\epsilon}\right),$$

where ϵ denotes any fixed positive number, $\zeta(s)$ is the Riemann zeta-function, and $\prod_{p|k}$ denotes the product over all different prime divisors of k.

Taking $m = 2$ in this Theorem, we may immediately obtain the following:

Corollary. *For any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{n\leq x} \delta_k(b_2(n)) = \frac{\pi^2}{30} x^2 \prod_{p|k} \frac{p^2+1}{p(p+1)} + O(x^{\frac{3}{2}+\varepsilon}).$$

§2. Proof of the Theorem

In this section, we shall use the analytic method to complete the proof of the theorem. In fact, we know that $b_m(n)$ is a completely multiplicative function, so we can use the properties of the Riemann zeta-function to obtain a generating function. For any complex s, if $\text{Re}(s) > 2$, we define the Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} \frac{\delta_k(b_m(n))}{n^s}.$$

If positive integer $n = p^\alpha$, then from the definition of $\delta_k(n)$ and $b_m(n)$ we have:

$$\delta_k(b_m(n)) = \delta_k(b_m(p^\alpha)) = 1, \qquad if \qquad p|k,$$

and

$$\delta_k(b_m(n)) = \delta_k(b_m(p^\alpha)) = p^\beta, \qquad if \alpha \equiv \beta mod m, 0 \leq \beta < m \qquad and \qquad p \dagger k.$$

From the above formula and the Euler product formula (See Theorem 11.6 of [3]) we can get

$$\begin{aligned}
f(s) &= \prod_p \left(1 + \frac{\delta_k(b_m(p))}{p^s} + \frac{\delta_k(b_m(p^2))}{p^{2s}} + \frac{\delta_k(b_m(p^3))}{p^{3s}} + \cdots\right) \\
&= \prod_{p|k} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots + \frac{1}{p^{(m-1)s}} + \frac{1}{p^{ms}} + \frac{1}{p^{(m+1)s}} + \cdots\right) \\
&\quad \times \prod_{p\dagger k} \left(1 + \frac{p}{p^s} + \frac{p^2}{p^{2s}} + \cdots + \frac{p^{m-1}}{p^{(m-1)s}} + \frac{1}{p^{ms}} + \frac{p}{p^{(m+1)s}} + \cdots\right) \\
&= \prod_{p|k} \frac{1}{1-\frac{1}{p^s}} \prod_{p\dagger k} \left[\left(1 + \frac{p}{p^s} + \ldots + \frac{p^{m-1}}{p^{(m-1)s}}\right)\left(1 + \frac{1}{p^{ms}} + \frac{1}{p^{2ms}} + \cdots\right)\right] \\
&= \prod_{p|k} \frac{1}{1-\frac{1}{p^s}} \prod_{p\dagger k} \frac{1}{1-\frac{1}{p^{ms}}} \prod_{p\dagger k} \left(1 + \frac{p}{p^s} + \ldots + \frac{p^{m-1}}{p^{(m-1)s}}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \prod_{p|k} \frac{1}{1-\frac{1}{p^s}} \prod_{p\dagger k} \frac{1-\frac{1}{p^{m(s-1)}}}{1-\frac{1}{p^{s-1}}} \times \frac{1}{1-\frac{1}{p^{ms}}} \\
&= \frac{\zeta(s-1)\zeta(ms)}{\zeta(ms-m)} \prod_{p|k} \frac{(1-\frac{1}{p^{s-1}})(1-\frac{1}{p^{ms}})}{(1-\frac{1}{p^{m(s-1)}})(1-\frac{1}{p^s})}.
\end{aligned}
$$

Because the Riemann zeta-function $\zeta(s)$ have a simple pole point at $s=1$ with the residue 1, we know that $f(s)\frac{x^s}{s}$ also have a simple pole point at $s=2$ with the residue $\dfrac{\zeta(2m)}{\zeta(m)} \prod\limits_{p|k} \dfrac{p^m+1}{p^{m-1}(p+1)} \dfrac{x^2}{2}$. By Perron formula (See [2]), taking $s_0=0$, $b=3$, $T>1$, then we have

$$
\sum_{n\le x} \delta_k(b_m(n)) = \frac{1}{2\pi i}\int_{3-iT}^{3+iT} f(s)\frac{x^s}{s}ds + O\left(\frac{x^{3+\epsilon}}{T}\right).
$$

Now we move the integral line to Re $s=\frac{3}{2}+\epsilon$, then taking $T=x^{\frac{3}{2}}$, we can get

$$
\begin{aligned}
&\sum_{n\le x} \delta_k(b_m(n)) \\
= &\frac{\zeta(2m)}{\zeta(m)} \prod_{p|k} \frac{p^m+1}{p^{m-1}(p+1)} \frac{x^2}{2} + \frac{1}{2\pi i}\int_{\frac{3}{2}+\epsilon-iT}^{\frac{3}{2}+\epsilon+iT} f(s)\frac{x^s}{s}ds + O\left(x^{\frac{3}{2}+\epsilon}\right) \\
= &\frac{\zeta(2m)}{\zeta(m)} \prod_{p|k} \frac{p^m+1}{p^{m-1}(p+1)} \frac{x^2}{2} + O\left(\int_{-T}^{T} \left|f(\frac{3}{2}+\epsilon+it)\right| \frac{x^{\frac{3}{2}+\epsilon}}{1+|t|}dt\right) \\
&+O\left(x^{\frac{3}{2}+\epsilon}\right) \\
= &\frac{\zeta(2m)}{\zeta(m)} \prod_{p|k} \frac{p^m+1}{p^{m-1}(p+1)} \frac{x^2}{2} + O\left(x^{\frac{3}{2}+\epsilon}\right).
\end{aligned}
$$

This completes the proof of Theorem.

Note that $\zeta(2)=\dfrac{\pi^2}{6}$ and $\zeta(4)=\dfrac{\pi^4}{90}$, taking $m=2$ in the theorem, we may immediately obtain the Corollary.

References

[1] F.Smarandache, Only Problems,Not Solutions, Chicago, Xiquan Publishing House, 1993.

[2] Pan Chengdong and Pan Chengbiao, Foundation of Analytic Number Theory, Beijing, Science Press, 1991.

[3] Tom M.Apstol, Introduction to Analytic Number Theory, New York, Springer-Verlag, 1976.