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On the primitive numbers of power P and its mean value properties[1]

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Abstract Let p be a prime, n be any fixed positive integer. $S_p(n)$ denotes the smallest positive integer such that $S_p(n)!$ is divisible by p^n. In this paper, we study the mean value properties of $S_p(n)$ for p, and give a sharp asymptotic formula for it.
Keywords Primitive numbers; Mean value; Asymptotic formula.

§1. Introduction

Let p be a prime, n be any fixed positive integer, $S_p(n)$ denotes the smallest positive integer such that $S_p(n)!$ is divisible by p^n. For example, $S_3(1) = 3$, $S_3(2) = 6$, $S_3(3) = 9$, $S_3(4) = 9$, $S_3(5) = 12$, $\cdots$. In problem 49 of book [1], Professor F. Smarandache asks us to study the properties of the sequence $S_p(n)$. About this problem, some asymptotic properties of this sequence have been studied by Zhang Wenpeng and Liu Duansen [2], they proved that

$$S_p(n) = (p-1)n + O\left(\frac{p}{\log p}\log n\right).$$

The problem is interesting because it can help us to calculate the Smarandache function. In this paper, we use the elementary methods to study the mean value properties of $S_p(n)$ for p, and give a sharp asymptotic formula for it. That is, we shall prove the following:

Theorem Let $x \geq 2$, for any fixed positive integer n, we have the asymptotic formula

$$\sum_{p\leq x} S_p(n) = \frac{nx^2}{2\log x} + \sum_{m=1}^{k-1}\frac{na_m x^2}{\log^{m+1} x} + O\left(\frac{nx^2}{\log^{k+1} x}\right),$$

where $a_m (m = 1, 2, \cdots, k-1)$ are computable constants.

§2. Some Lemmas

To complete the proof of the theorem, we need the following:

Lemma Let p be a prime, n be any fixed positive integer. Then we have the estimate

$$(p-1)n \leq S_p(n) \leq np.$$

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Proof. Let $S_p(n) = k = a_1p^{\alpha_1} + a_2p^{\alpha_2} + \cdots + a_sp^{\alpha_s}$ with $\alpha_s > \alpha_{s-1} > \cdots > \alpha_1 \geq 0$ under the base p. Then from the definition of $S_p(n)$ we know that $p \mid S_p(n)!$ and the $S_p(n)$ denotes the smallest integer satisfy the condition. However, let

$$(np)! = 1 \cdot 2 \cdot 3 \cdots p \cdot (p+1) \cdots 2p \cdot (2p+1) \cdots np = up^l.$$

where $l \geq n$, $p \dagger u$.

So combining this and $p \mid S_p(n)!$ we can easily obtain

$$S_p(n) \leq np. \tag{1}$$

On the other hand, from the definition of $S_p(n)$ we know that $p \mid S_p(n)!$ and $p^n \dagger (S_p(n)-1)!$, so that $\alpha_1 \geq 1$, note that the factorization of $S_p(n)!$ into prime powr is

$$k! = \prod_{q \leq k} q^{\alpha_q(k)}.$$

where $\prod_{q \leq k}$ denotes the product over all prime , and

$$\alpha_q(k) = \sum_{i=1}^{\infty} [\frac{k}{q^i}]$$

because $p \mid S_p(n)!$, so we have

$$n \leq \alpha_p(k) = \sum_{i=1}^{\infty} [\frac{k}{p^i}] = \frac{k}{p-1}$$

or

$$(p-1)n \leq k \tag{2}$$

combining (1) and (2) we immediately get the estimate

$$(p-1)n \leq S_p(n) \leq np.$$

This completes the proof of the lemma.

§3. Proof of the theorem

In this section, we complete the proof of Theorem. Based on the result of lemma

$$(p-1)n \leq S_p(n) \leq np$$

we can easily get

$$\sum_{p \leq x} (p-1)n \leq \sum_{p \leq x} S_p(n) \leq \sum_{p \leq x} np.$$

Let

$$a(n) = \begin{cases} 1, & \text{if } n \text{ is prime;} \\ 0, & \text{otherwise,} \end{cases}$$

Then from [3] we know that for any positive integer k,

$$\sum_{n\leq x} a(n) = \pi(x) = \frac{x}{logx}(1+\sum_{m=1}^{k-1}\frac{m!}{\log^m x}) + O\left(\frac{x}{\log^{k+1} x}\right).$$

By Abel's identity we have

$$\begin{aligned} & \sum_{p\leq x} p = \sum_{m\leq x} a(m)m \\ = & \ \pi(x)x - \int_2^x \pi(t)dt \\ = & \ \frac{x^2}{logx} + \frac{x^2}{logx}\sum_{m=1}^{k-1}\frac{m!}{\log^m x} - \int_2^x \frac{t}{logt}(1+\sum_{m=1}^{k-1}\frac{m!}{\log^m t})dt + O\left(\frac{x^2}{\log^{k+1} x}\right) \\ = & \ \frac{x^2}{2\log x} + \sum_{m=1}^{k-1}\frac{a_m x^2}{\log^(m+1)x} + O\left(\frac{x^2}{\log^{k+1} x}\right) \end{aligned}$$

where $a_m (m = 1, 2, \cdots, k-1)$ are computable constants. From above we have

$$\sum_{p\leq x}(p-1) = \sum_{p\leq x} p - \pi(x) = \frac{x^2}{2\log x} + \sum_{m=1}^{k-1}\frac{a_m x^2}{\log^(m+1)x} + O\left(\frac{x^2}{\log^{k+1} x}\right).$$

Therefore

$$\sum_{p\leq x} S_p(n) = \sum_{p\leq x} k == \frac{x^2}{2\log x} + \sum_{m=1}^{k-1}\frac{a_m x^2}{\log^(m+1)x} + O\left(\frac{x^2}{\log^{k+1} x}\right).$$

This completes the proof of the theorem.

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