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One problem related to the Smarandache quotients

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Abstract For any positive integer $n \geq 1$, the Smarandache quotients $Q(n)$ is defined as the smallest positive integer k such that $n \cdot k$ is a factorial number. That is, $Q(n) = \min\{k : n \cdot k = m!\}$. The main purpose of this paper is using the elementary method to study the properties of the Smarandache quotients sequence, and give an identity involving the Smarandache quotients sequence.
Keywords The Smarandache quotients, identity, infinite series, elementary method.

§1. Introduction and results

In his book "Only problems, not solutions", professor F.Smarandache introduced many functions, sequences and unsolved problems, many authors had studied it, see references [1], [2] and [3]. One of the unsolved problems is the Smarandache quotients sequence $\{Q(n)\}$, it is defined as the smallest positive integer k such that $n \cdot k$ is a factorial number. That is, $Q(n) = \min\{k : n \cdot k = m!\}$, where m is a positive integer. For example, from the definition of $Q(n)$ we can find that the first few values of $Q(n)$ are $Q(1) = 1$, $Q(2) = 1$, $Q(3) = 2$, $Q(4) = 6$, $Q(5) = 24$, $Q(6) = 1$, $Q(7) = 720$, $Q(8) = 3$, $Q(9) = 80$, $Q(10) = 12$, $Q(11) = 3628800$, $Q(12) = 2$, $Q(13) = 479001600$, $Q(14) = 360$, $Q(15) = 8$, $Q(16) = 45$, $\cdots\cdots$.

In reference [4], professor F.Smarandache asked us to study the properties of the sequence $\{Q(n)\}$. About this problem, some authors had studied it, and obtained several simple results. For example, Kenichiro Kashihara [5] proved that for any prime p, we have $Q(p) = (p-1)!$.

In this paper, we use the elementary method to study the properties of an infinite series involving the Smarandache quotients sequence, and give an interesting identity. That is, we shall prove the following conclusion:

Theorem 1. Let $d(n)$ denotes the Dirichlet divisor function, then we have the identity

$$\sum_{n=1}^{+\infty} \frac{1}{Q(n) \cdot n} = \sum_{m=1}^{+\infty} \frac{m \cdot d(m!)}{(m+1)!}.$$

Let $Q_1(2n-1)$ denotes the smallest positive odd number such that $Q_1(2n-1) \cdot (2n-1)$ is a two factorial number. That is, $Q_1(2n-1) \cdot (2n-1) = (2m-1)!!$, where $(2m-1)!! = 1 \times 3 \times 5 \times \cdots \times (2m-1)$. Then for the sequence $\{Q_1(2n-1)\}$, we can also get the following:

Theorem 2. Let $d(n)$ denote the Dirichlet divisor function, then we also have the identity

$$\sum_{n=1}^{+\infty} \frac{1}{Q_1(2n-1)\cdot(2n-1)} = 2\cdot\sum_{m=1}^{+\infty} \frac{m\cdot d((2m-1)!!)}{(2m+1)!!}.$$

§2. Proof of the theorems

In this section, we shall use the elementary method to complete the proof of our theorems directly. For any positive integer n, let $Q(n)=k$, from the definition of $Q(n)$ we know that k is the smallest positive integer such that $n\cdot k=m!$. So k must be a divisor of $m!$. This implies that $k\mid m!$, and the number of all k (such that $k\mid m!$) is $d(m!)$. On the other hand, if $k\mid(m-1)!$, then $Q(n)\neq k$, and the number of all k (such that $k\mid(m-1)!$) is $d((m-1)!)$. So the number of all k (such that $k\mid m!$ and $k\dagger(m-1)!$) is $d(m!)-d((m-1)!)$. That is means, the number of all k (such that $Q(n)=k$ and $Q(n)\cdot n=m!$) is $d(m!)-d((m-1)!)$. From this we may immediately get

$$\begin{aligned}\sum_{n=1}^{+\infty}\frac{1}{Q(n)\cdot n} &= \sum_{m=1}^{+\infty}\sum_{\substack{n=1\\ Q(n)\cdot n=m!}}^{+\infty}\frac{1}{m!}=\sum_{m=1}^{+\infty}\frac{\#\{Q(n)\cdot n=m!\}}{m!}\\ &= 1+\sum_{m=2}^{+\infty}\frac{d(m!)-d((m-1)!)}{m!}=\sum_{m=1}^{+\infty}\frac{d(m!)}{m!}-\sum_{m=1}^{+\infty}\frac{d(m!)}{(m+1)!}\\ &= \sum_{m=1}^{+\infty}\frac{(m+1)\cdot d(m!)-d(m!)}{(m+1)!}=\sum_{m=1}^{+\infty}\frac{m\cdot d(m!)}{(m+1)!},\end{aligned}$$

where $d(m)$ is the Dirichlet divisor function, and $\#\{Q(n)\cdot n=m!\}$ denotes the number of all solutions of the equation $Q(n)\cdot n=m!$. This proves Theorem 1.

Similarly, we can also deduce Theorem 2. This completes the proof of Theorems.

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The Smarandache bisymmetric determinant natural sequence

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Abstract Murthy [1] introduced the Smarandache Bisymmetric Determinant Natural Sequence. In this paper, we derive the sum of the first n terms of the sequence.

Keywords The Smarandache bisymmetric determinant natural sequence, the n-th term, the sum of the first n terms.

§1. Introduction

The Smarandache bisymmetric determinant natural sequence (SBDNS), introduced by Murthy [1], is defined as follows.

Definition 1.1. The Smarandache bisymmetric determinant natural sequence, {SBDNS(n)}, is

$$\left\{ |1|, \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 2 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 3 \\ 3 & 4 & 3 & 2 \\ 4 & 3 & 2 & 1 \end{vmatrix}, \ldots \right\}.$$

A first few terms of the sequence are 1, -3, -8, 20, 48, -112, -256, 576, $\cdots$.

The following result is due to Majumdar [2].

Theorem 1.1. Let a_n be the n-th term of the Smarandache bisymmetric determinant natural sequence. Then,

$$a_n = \begin{vmatrix} 1 & 2 & 3 & \cdots & n-2 & n-1 & n \\ 2 & 3 & 4 & \cdots & n-1 & n & n-1 \\ 3 & 4 & 5 & \cdots & n & n-1 & n-2 \\ \vdots & & & & & & \\ n-2 & n-1 & n & & 5 & 4 & 3 \\ n-1 & n & n-1 & \cdots & 4 & 3 & 2 \\ n & n-1 & n-2 & \cdots & 3 & 2 & 1 \end{vmatrix} = (-1)^{\left[\frac{n}{2}\right]} (n+1)\, 2^{n-2}.$$

Let $\{S_n\}$ be the sequence of $n-th$ partial sums of the sequence $\{a_n\}$, so that

$$S_n = \sum_{k=1}^{n} a_k, \ n \geqslant 1.$$

This paper gives explicit expressions for the sequence $\{S_n\}$. This is given in Theorem 3.1 in Section 3. In Section 2, we give some preliminary results that would be necessary for the proof of the theorem. We conclude this paper with some remarks in the final section, Section 4.

§2. Some preliminary results

In this section, we derive some preliminary results that would be necessary in deriving the expressions of S_n in the next section. These are given in the following two lemmas.

Lemma 2.1. For any integer $m \geq 1$,

$$\sum_{k=1,3,\ldots,(2m-1)} 2^{2(k-1)} = \frac{1}{15}(2^{4m}-1).$$

Proof. Since

$$\sum_{k=1,3,\ldots,(2m-1)} 2^{2(k-1)} = 1 + 2^4 + 2^8 + ... + 2^{4(m-1)}$$

is a geometric series with common ratio 2^4, the result follows.

Lemma 2.2. For any integer $m \geq 1$,

$$\sum_{k=1,3,\ldots,(2m-1)} k\ 2^{2(k-1)} = \frac{1}{15}(2m-1)2^{4m} - \frac{1}{225}(2^{4m+1}-17).$$

Proof. Denoting by s the series on the left above, we see that

$$s = 1 + 3\cdot 2^4 + 5\cdot 2^8 + \cdots + (2m-1)\cdot 2^{4(m-1)}, \qquad (*)$$

so that, multiplying throughout by 2^4, we get

$$2^4 s = 1\cdot 2^4 + 3\cdot 2^8 + \cdots + (2m-3)\cdot 2^{4(m-1)} + (2m-1)\cdot 2^{4m}\ \cdot \qquad (**)$$

Now, subtracting (**) from (*), we have

$$\begin{aligned}(1-2^4)s &= 1 + 2\cdot 2^4\left[1 + 2^4 + \cdots + 2^{4(m-2)}\right] - (2m-1)\cdot 2^{4m}\\ &= 1 + 2^5\cdot\left\{\frac{2^{4(m-1)}-1}{2^4-1}\right\} - (2m-1)\cdot 2^{4m}\\ &= \frac{1}{15}\left(2^{4m+1}-17\right) - (2m-1)\cdot 2^{4m}\end{aligned}$$

which now gives the desired result.

§3. Main results

In this section, we derive the explicit expressions of the n-th partial sums, S_n, of the Smarandache bisymmetric determinant natural sequence.

From Theorem 1.1, we see that, for any integer $k \geq 1$,

$$a_{2k} + a_{2k+1} = (-1)^k (6k+5) 2^{2(k-1)},$$

$$a_{2k+2} + a_{2k+3} = (-1)^{k+1} (6k+11) 2^{2k},$$

so that

$$a_{2k} + a_{2k+1} + a_{2k+2} + a_{2k+3} = 3(-1)^{k+1} (6k+13) 2^{2(k-1)}. \tag{1}$$

Letting

$$S_n = a_1 + a_2 + \cdots + a_n,$$

we can prove the following result.

Theorem 3.1. For any integer $m \geq 0$,

1). $S_{4m+1} = \frac{3}{5} \cdot m \cdot 2^{2(2m+1)} + \frac{31}{25} \cdot 2^{4m} - \frac{6}{25} = \frac{2}{25} \left\{(60m+31) 2^{4m-1} - 3\right\};$

2). $S_{4m+2} = -\frac{1}{5} \cdot m \cdot 2^{4m+3} - \frac{11}{25} \cdot 2^{2(2m+1)} - \frac{6}{25} = -\frac{2}{25} \left\{(10m+11) 2^{4m+1} + 3\right\};$

3). $S_{4m+3} = -\frac{3}{5} \cdot m \cdot 2^{4(m+1)} - \frac{61}{25} \cdot 2^{2(2m+1)} - \frac{6}{25} = -\frac{2}{25} \left\{(60m+61) 2^{4m+1} + 3\right\};$

4). $S_{4m+4} = \frac{1}{5} \cdot m \cdot 2^{4m+5} + \frac{1}{25} \cdot 2^{4(2m+2)} - \frac{6}{25} = \frac{2}{25} \left\{(5m+8) 2^{4(m+1)} - 3\right\}.$

Proof. To prove the theorem, we make use of Lemma 2.1 and Lemma 2.2, as well as Theorem 1.1.

1). Since S_{4m+1} can be written as

$$S_{4m+1} = a_1 + a_2 + \cdots + a_{4m+1} = a_1 + \sum_{k=1,3,\ldots,(2m-1)} (a_{2k} + a_{2k+1} + a_{2k+2} + a_{2k+3}),$$

by virtue of (1),

$$\begin{aligned} S_{4m+1} &= a_1 + 3 \sum_{k=1,3,\ldots,(2m-1)} (-1)^{k+1} (6k+13) 2^{2(k-1)} \\ &= a_1 + 3 \left\{ 6 \sum_{k=1,3,\ldots,(2m-1)} k 2^{2(k-1)} + 13 \sum_{k=1,3,\ldots,(2m-1)} 2^{2(k-1)} \right\}. \end{aligned}$$

Now, appealing to Lemma 2.1 and Lemma 2.2, we get

$$S_{4m+1} = 1 + \left[\frac{6}{5}(2m-1) 2^{4m} - \frac{2}{25}(2^{4m+1} - 17)\right] + \frac{13}{5}(2^{4m} - 1)$$

which now gives the desired result after some algebraic manipulations.

2). Since

$$S_{4m+2} = S_{4m+1} + a_{4m+2},$$

from part (1) above, together with Theorem 1.1, we get

$$S_{4m+2} = \left[\frac{3}{5} \cdot m \cdot 2^{2(2m+1)} + \frac{31}{25} \cdot 2^{4m} - \frac{6}{25}\right] - (4m+3) \cdot 2^{4m},$$

which gives the desired expression for S_{4m+2} after algebraic simplifications.

3). Since

$$S_{4m+3} = S_{4m+2} + a_{4m+3} = \left[-\frac{1}{5} \cdot m \cdot 2^{4m+3} - \frac{11}{25} \cdot 2^{2(2m+1)} - \frac{6}{25}\right] - (4m+4) \cdot 2^{4m+1},$$

we get the desired expression for S_{4m+3} after simplifications.

4). Since

$$S_{4m+4} = S_{4m+3} + a_{4m+4} = \left[-\frac{6}{5} \cdot m \cdot 2^{4m+3} - \frac{61}{25} \cdot 2^{2(2m+1)} - \frac{6}{25}\right] - (4m+5) \cdot 2^{4m+2},$$

the result follows after some algebraic simplifications.

The case when $m = 0$ can easily be verified.

Hence, the proof is complete.

§4. Remarks

Theorem 3.1 in the previous section gives the n-th term of the sequence of partial sums, $\{S_n\}$, in all the possible four cases. The following lemmas prove that, in each case, S_n is indeed an integer.

Lemma 4.1. For any integer $m \geq 0$, $2^{4m-1}(60m+31) - 3$ is divisible by 25.

Proof. The result is true for $m = 0, 1$. So, we assume its validity for some positive integer m. Now, since

$$[2^{4m+3}\{60(m+1)+31\} - 3] - \{2^{4m-1}(60m+31) - 3\} = 25(36m+57)2^{4m-1},$$

it follows, by virtue of the induction hypothesis, that $2^{4m+3}\{60(m+1)+31\} - 3$ is also divisible by 25. Thus, the result is true for $m+1$ as well, completing induction.

Lemma 4.2. For any integer $m \geq 0$, $2^{4m+1}(10m+11) + 3$ is divisible by 25.

Proof. is by induction on m. The result is clearly true for $m = 0, 1$. Now, assuming its validity for some positive integer m, since

$$[2^{4m+5}\{10(m+1)+11\} + 3] - \{2^{4m+1}(10m+11) + 3\} = 25(6m+13)2^{4m+1},$$

is divisible by 25, it follows that the result is true for $m+1$ as well. This completes the proof.

Lemma 4.3. For any integer $m \geq 0$, $2^{4m+1}(60m+61) + 3$ is divisible by 25.

Proof. is by induction on m. The result is clearly true for $m = 0, 1$. We now assume that the result is true for some positive integer m. Then, since

$$[2^{4m+5}\{60(m+1)+61\}+3] - \{2^{4m+1}(60m+61)+3\} = 25(36m+75)2^{4m+1},$$

this, together with the induction hypothesis, shows that the result is true for m + 1 as well. This completes the proof by induction.

Lemma 4.4. For any integer $m \geq 0$, $2^{4(m+1)}(5m+32) - 3$ is divisible by 25.

Proof. We first assume that the result is true for some positive integer m. Now, since

$$[2^{4(m+2)}\{5(m+1)+13\}-3] - \{2^{4(m+1)}(5m+13)-3\} = 25(3m+8)2^{4(m+1)},$$

this together with the induction hypothesis, shows that $2^{4(m+2)}\{5(m+1)+13\} - 3$ is also divisible by 25. This, in turn, shows that the result is true for $m+1$ as well. To complete, we have to prove the validity of the result for $m = 0, 1$, which can easily be checked.

The Smarandache bisymmetric arithmetic determinant sequence, introduced by Murthy [1], is

$$\left\{ a, \begin{vmatrix} a & a+d \\ a+d & a \end{vmatrix}, \begin{vmatrix} a & a+d & a+2d \\ a+d & a+2d & a+d \\ a+2d & a+d & a \end{vmatrix}, \cdots \right\}.$$

The n-th term of the above sequence has been found by Majumdar [2] to be

$$(-1)^{\left[\frac{n}{2}\right]} \left(a + \frac{n-1}{2}d\right)(2d)^{n-1}.$$

Note that the Smarandache bisymmetric determinant natural sequence is a particular case of the Smarandache bisymmetric arithmetic determinant sequence when $a = 1$ and $d = 1$.

Open Problem: To find a formula for the sum of the first n terms of the Smarandache bisymmetric arithmetic determinant sequence.

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