Pseudo-Manifold Geometries with Applications

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Abstract: A Smarandache geometry is a geometry which has at least one Smarandachely denied axiom(1969), i.e., an axiom behaves in at least two different ways within the same space, i.e., validated and invalided, or only invalided but in multiple distinct ways and a Smarandache n-manifold is a n-manifold that support a Smarandache geometry. Iseri provided a construction for Smarandache 2-manifolds by equilateral triangular disks on a plane and a more general way for Smarandache 2-manifolds on surfaces, called *map geometries* was presented by the author in $[9]-[10]$ and [12]. However, few observations for cases of $n \geq 3$ are found on the journals. As a kind of Smarandache geometries, a general way for constructing dimensional n pseudo-manifolds are presented for any integer $n \geq 2$ in this paper. Connection and principal fiber bundles are also defined on these manifolds. Following these constructions, nearly all existent geometries, such as those of *Euclid geometry, Lobachevshy-Bolyai geometry, Riemann geometry, Weyl geometry, Kähler geometry* and *Finsler geometry*, ...,etc., are their sub-geometries.

Key Words: Smarandache geometry, Smarandache manifold, pseudo-manifold, pseudo-manifold geometry, multi-manifold geometry, connection, curvature, Finsler geometry, Riemann geometry, Weyl geometry and Kähler geometry.

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§1. Introduction

Various geometries are encountered in update mathematics, such as those of *Euclid geometry, Lobachevshy-Bolyai geometry, Riemann geometry, Weyl geometry, Kähler geometry* and *Finsler geometry*, ..., etc.. As a branch of geometry, each of them has been a kind of spacetimes in physics once and contributes successively to increase human's cognitive ability on the natural world. Motivated by a combinatorial notion for sciences: *combining different fields into a unifying field*, Smarandache introduced *neutrosophy and neutrosophic logic* in references $[14]-[15]$ and Smarandache geometries in [16].

Definition 1.1([8][16]) *An axiom is said to be Smarandachely denied if the axiom behaves in at least two different ways within the same space, i.e., validated and invalided, or only invalided but in multiple distinct ways.*

A Smarandache geometry is a geometry which has at least one Smarandachely *denied axiom(1969).*

Definition 1.2 *For an integer $n, n \geq 2$, a Smarandache n-manifold is a n-manifold that support a Smarandache geometry.*

Smarandache geometries were applied to construct many world from conservation laws as a mathematical tool([2]). For Smarandache n-manifolds, Iseri constructed Smarandache manifolds for $n = 2$ by equilateral triangular disks on a plane in [6] and [7] (see also [11] in details). For generalizing Iseri's Smarandache manifolds, *map geometries* were introduced in $[9] - [10]$ and [12], particularly in [12] convinced us that these map geometries are really Smarandache 2-manifolds. Kuciuk and Antholy gave a popular and easily understanding example on an Euclid plane in [8]. Notice that in [13], these *multi-metric space* were defined, which can be also seen as Smarandache geometries. However, few observations for cases of $n \geq 3$ and their relations with existent manifolds in differential geometry are found on the journals. The main purpose of this paper is to give general ways for constructing dimensional n pseudo-manifolds for any integer $n \geq 2$. Differential structure, connection and principal fiber bundles are also introduced on these manifolds. Following these constructions, nearly all existent geometries, such as those of *Euclid geometry, Lobachevshy-Bolyai geometry, Riemann geometry, Weyl geometry, Kähler geometry* and *Finsler geometry*, ...,etc., are their sub-geometries.

Terminology and notations are standard used in this paper. Other terminology and notations not defined here can be found in these references $[1], [3] - [5]$.

For any integer $n, n \geq 1$, an *n-manifold* is a Hausdorff space M^n, i.e., a space that satisfies the T_2 separation axiom, such that for $\forall p \in M^n$, there is an open neighborhood $U_p, p \in U_p \subset M^n$ and a homeomorphism $\varphi_p : U_p \to \mathbf{R}^n$ or $\mathbf{C}^n$, respectively.

Considering the differentiability of the homeomorphism $\varphi : U \to \mathbf{R}^n$ enables us to get the conception of differential manifolds, introduced in the following.

An *differential n-manifold* $(M^n, \mathcal{A})$ is an n-manifold $M^n, M^n = \bigcup\limits_{i \in I} U_i$, endowed with a C^r differential structure $\mathcal{A} = \{(U_\alpha, \varphi_\alpha) | \alpha \in I\}$ on M^n for an integer r with following conditions hold.

(1) $\{U_\alpha; \alpha \in I\}$ is an open covering of M^n;

(2) For $\forall \alpha, \beta \in I$, atlases $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are *equivalent*, i.e., $U_\alpha \bigcap U_\beta = \emptyset$ or $U_\alpha \bigcap U_\beta \neq \emptyset$ but the *overlap maps*

$$\varphi_\alpha \varphi_\beta^{-1} : \varphi_\beta(U_{\alpha \bigcap U_\beta}) \to \varphi_\beta(U_\beta) \ \text{ and } \ \varphi_\beta \varphi_\alpha^{-1} : \varphi_\beta(U_{\alpha \bigcap U_\beta}) \to \varphi_\alpha(U_\alpha)$$

are C^r;

(3) $\mathcal{A}$ is maximal, i.e., if (U, φ) is an atlas of M^n equivalent with one atlas in $\mathcal{A}$, then $(U, \varphi) \in \mathcal{A}$.

An n-manifold is *smooth* if it is endowed with a C^∞ differential structure. It is well-known that a complex manifold M_c^n is equal to a smooth real manifold M_r^{2n} with a natural base

$$\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i} |\ 1 \leq i \leq n\}$$

for $T_pM_c^n$, where $T_pM_c^n$ denotes the tangent vector space of M_c^n at each point $p \in M_c^n$.

§2. Pseudo-Manifolds

These Smarandache manifolds are non-homogenous spaces, i.e., there are singular or inflection points in these spaces and hence can be used to characterize warped spaces in physics. A generalization of ideas in map geometries can be applied for constructing dimensional n pseudo-manifolds.

Construction 2.1 *Let M^n be an n-manifold with an atlas $\mathcal{A} = \{(U_p, \varphi_p)|p \in M^n\}$. For $\forall p \in M^n$ with a local coordinates $(x_1, x_2, \cdots, x_n)$, define a spatially directional mapping $\omega : p \rightarrow \mathbf{R}^n$ action on φ_p by*

$$\omega : p \rightarrow \varphi_p^\omega(p) = \omega(\varphi_p(p)) = (\omega_1, \omega_2, \cdots, \omega_n),$$

i.e., if a line L passes through $\varphi(p)$ with direction angles $\theta_1, \theta_2, \cdots, \theta_n$ with axes $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n$ in $\mathbf{R}^n$, then its direction becomes

$$\theta_1 - \frac{\vartheta_1}{2} + \sigma_1, \theta_2 - \frac{\vartheta_2}{2} + \sigma_2, \cdots, \theta_n - \frac{\vartheta_n}{2} + \sigma_n$$

after passing through $\varphi_p(p)$, where for any integer $1 \leq i \leq n$, $\omega_i \equiv \vartheta_i (mod 4\pi)$, $\vartheta_i \geq 0$ and

$$\sigma_i = \begin{cases} \pi, & if \ \ 0 \leq \omega_i < 2\pi, \\ 0, & if \ \ 2\pi < \omega_i < 4\pi. \end{cases}$$

A manifold M^n endowed with such a spatially directional mapping $\omega : M^n \rightarrow \mathbf{R}^n$ is called an n-dimensional pseudo-manifold, denoted by $(M^n, \mathcal{A}^\omega)$.

Theorem 2.1 *For a point $p \in M^n$ with local chart (U_p, φ_p), $\varphi_p^\omega = \varphi_p$ if and only if $\omega(p) = (2\pi k_1, 2\pi k_2, \cdots, 2\pi k_n)$ with $k_i \equiv 1(mod 2)$ for $1 \leq i \leq n$.*

Proof By definition, for any point $p \in M^n$, if $\varphi_p^\omega(p) = \varphi_p(p)$, then $\omega(\varphi_p(p)) = \varphi_p(p)$. According to Construction 2.1, this can only happens while $\omega(p) = (2\pi k_1, 2\pi k_2, \cdots, 2\pi k_n)$ with $k_i \equiv 1(mod 2)$ for $1 \leq i \leq n$. $\natural$

Definition 2.1 *A spatially directional mapping $\omega : M^n \rightarrow \mathbf{R}^n$ is euclidean if for any point $p \in M^n$ with a local coordinates $(x_1, x_2, \cdots, x_n)$, $\omega(p) = (2\pi k_1, 2\pi k_2, \cdots, 2\pi k_n)$ with $k_i \equiv 1(mod 2)$ for $1 \leq i \leq n$, otherwise, non-euclidean.*

Definition 2.2 *Let $\omega : M^n \rightarrow \mathbf{R}^n$ be a spatially directional mapping and $p \in (M^n, \mathcal{A}^\omega)$, $\omega(p)(mod 4\pi) = (\omega_1, \omega_2, \cdots, \omega_n)$. Call a point p elliptic, euclidean or hyperbolic in direction $\mathbf{e}_i$, $1 \leq i \leq n$ if $o \leq \omega_i < 2\pi$, $\omega_i = 2\pi$ or $2\pi < \omega_i < 4\pi$.*

Then we get a consequence by Theorem 2.1.

Corollary 2.1 *Let $(M^n, \mathcal{A}^\omega)$ be a pseudo-manifold. Then $\varphi_p^\omega = \varphi_p$ if and only if every point in M^n is euclidean.*

Theorem 2.2 *Let $(M^n, \mathcal{A}^\omega)$ be an n-dimensional pseudo-manifold and $p \in M^n$. If there are euclidean and non-euclidean points simultaneously or two elliptic or hyperbolic points in a same direction in (U_p, φ_p), then $(M^n, \mathcal{A}^\omega)$ is a Smarandache n-manifold.*

Proof On the first, we introduce a conception for locally parallel lines in an n-manifold. Two lines C_1, C_2 are said *locally parallel* in a neighborhood (U_p, φ_p) of a point $p \in M^n$ if $\varphi_p(C_1)$ and $\varphi_p(C_2)$ are parallel straight lines in $\mathbf{R}^n$.

In $(M^n, \mathcal{A}^\omega)$, the axiom that *there are lines pass through a point locally parallel a given line* is Smarandachely denied since it behaves in at least two different ways, i.e., *one parallel, none parallel*, or *one parallel, infinite parallels*, or *none parallel, infinite parallels*.

If there are euclidean and non-euclidean points in (U_p, φ_p) simultaneously, not loss of generality, we assume that u is euclidean but v non-euclidean, $\omega(v)(mod 4\pi) = (\omega_1, \omega_2, \cdots, \omega_n)$ and $\omega_1 \neq 2\pi$. Now let L be a straight line parallel the axis $\mathbf{e}_1$ in $\mathbf{R}^n$. There is only one line C_u locally parallel to $\varphi_p^{-1}(L)$ passing through the point u since there is only one line $\varphi_p(C_q)$ parallel to L in $\mathbf{R}^n$ by these axioms for Euclid spaces. However, if $0 < \omega_1 < 2\pi$, then there are infinite many lines passing through u locally parallel to $\varphi_p^{-1}(L)$ in (U_p, φ_p) since there are infinite many straight lines parallel L in $\mathbf{R}^n$, such as those shown in Fig.2.1(a) in where each straight line passing through the point $\overline{u} = \varphi_p(u)$ from the shade field is parallel to L.

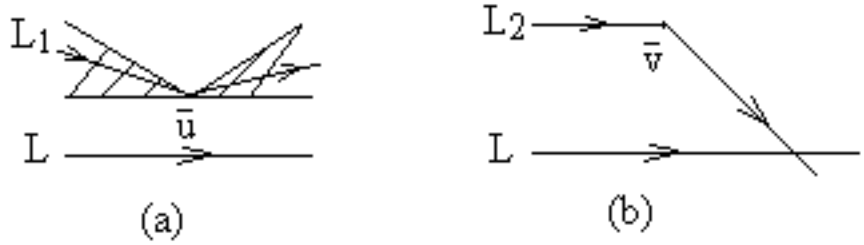

Fig.2.1

But if $2\pi < \omega_1 < 4\pi$, then there are no lines locally parallel to $\varphi_p^{-1}(L)$ in (U_p, φ_p) since there are no straight lines passing through the point $\overline{v} = \varphi_p(v)$ parallel to L in $\mathbf{R}^n$, such as those shown in Fig.2.1(b).

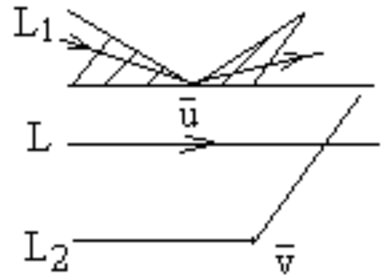

Fig.2.2

If there are two elliptic points u, v along a direction $\overrightarrow{O}$, consider the plane $\mathcal{P}$ determined by $\omega(u), \omega(v)$ with $\overrightarrow{O}$ in $\mathbf{R}^n$. Let L be a straight line intersecting with the line uv in $\mathcal{P}$. Then there are infinite lines passing through u locally parallel to $\varphi_p(L)$ but none line passing through v locally parallel to $\varphi_p^{-1}(L)$ in (U_p, φ_p) since there are infinite many lines or none lines passing through $\overline{u} = \omega(u)$ or $\overline{v} = \omega(v)$ parallel to L in $\mathbf{R}^n$, such as those shown in Fig.2.2.

Similarly, we can also get the conclusion for the case of hyperbolic points. Since there exists a Smarandachely denied axiom in $(M^n, \mathcal{A}^\omega)$, it is a Smarandache manifold. This completes the proof. $\natural$

For an Euclid space $\mathbf{R}^n$, the homeomorphism φ_p is trivial for $\forall p \in \mathbf{R}^n$. In this case, we abbreviate $(\mathbf{R}^n, \mathcal{A}^\omega)$ to $(\mathbf{R}^n, \omega)$.

Corollary 2.2 *For any integer $n \geq 2$, if there are euclidean and non-euclidean points simultaneously or two elliptic or hyperbolic points in a same direction in $(\mathbf{R}^n, \omega)$, then $(\mathbf{R}^n, \omega)$ is an n-dimensional Smarandache geometry.*

Particularly, Corollary 2.2 partially answers an open problem in [12] for establishing Smarandache geometries in $\mathbf{R}^3$.

Corollary 2.3 *If there are points $p, q \in \mathbf{R}^3$ such that $\omega(p)(mod 4\pi) \neq (2\pi, 2\pi, 2\pi)$ but $\omega(q)(mod 4\pi) = (2\pi k_1, 2\pi k_2, 2\pi k_3)$, where $k_i \equiv 1(mod 2), 1 \leq i \leq 3$ or p, q are simultaneously elliptic or hyperbolic in a same direction of $\mathbf{R}^3$, then $(\mathbf{R}^3, \omega)$ is a Smarandache space geometry.*

Definition 2.3 *For any integer $r \geq 1$, a C^r differential Smarandache n-manifold $(M^n, \mathcal{A}^\omega)$ is a Smarandache n-manifold $(M^n, \mathcal{A}^\omega)$ endowed with a differential structure $\mathcal{A}$ and a C^r spatially directional mapping ω. A C^∞ Smarandache n-manifold $(M^n, \mathcal{A}^\omega)$ is also said to be a smooth Smarandache n-manifold.*

According to Theorem 2.2, we get the next result by definitions.

Theorem 2.3 *Let $(M^n, \mathcal{A})$ be a manifold and $\omega : M^n \to \mathbf{R}^n$ a spatially directional mapping action on $\mathcal{A}$. Then $(M^n, \mathcal{A}^\omega)$ is a C^r differential Smarandache n-manifold for an integer $r \geq 1$ if the following conditions hold:*

(1) *there is a C^r differential structure $\mathcal{A} = \{(U_\alpha, \varphi_\alpha) | \alpha \in I\}$ on M^n;*
(2) *ω is C^r;*
(3) *there are euclidean and non-euclidean points simultaneously or two elliptic or hyperbolic points in a same direction in (U_p, φ_p) for a point $p \in M^n$.*

Proof The condition (1) implies that $(M^n, \mathcal{A})$ is a C^r differential n-manifold and conditions (2), (3) ensure $(M^n, \mathcal{A}^\omega)$ is a differential Smarandache manifold by definitions and Theorem 2.2. ♮

For a smooth differential Smarandache n-manifold $(M^n, \mathcal{A}^\omega)$, a function $f : M^n \to \mathbf{R}$ is said smooth if for $\forall p \in M^n$ with an chart (U_p, φ_p),

$$f \circ (\varphi_p^\omega)^{-1} : (\varphi_p^\omega)(U_p) \to \mathbf{R}^n$$

is smooth. Denote by $\Im_p$ all these C^∞ functions at a point $p \in M^n$.

Definition 2.4 *Let $(M^n, \mathcal{A}^\omega)$ be a smooth differential Smarandache n-manifold and $p \in M^n$. A tangent vector v at p is a mapping $v : \Im_p \to \mathbf{R}$* with these following conditions hold.
(1) $\forall g, h \in \Im_p, \forall \lambda \in \mathbf{R},\ v(h + \lambda h) = v(g) + \lambda v(h)$;
(2) $\forall g, h \in \Im_p, v(gh) = v(g)h(p) + g(p)v(h)$.

Denote all tangent vectors at a point $p \in (M^n, \mathcal{A}^\omega)$ by T_pM^n and define addition+and scalar multiplication·for $\forall u, v \in T_pM^n, \lambda \in \mathbf{R}$ and $f \in \Im_p$ by

$$(u + v)(f) = u(f) + v(f), \quad (\lambda u)(f) = \lambda \cdot u(f).$$

Then it can be shown immediately that T_pM^n is a vector space under these two operations+and·.

Let $p \in (M^n, \mathcal{A}^\omega)$ and $\gamma : (-\varepsilon, \varepsilon) \to \mathbf{R}^n$ be a smooth curve in $\mathbf{R}^n$ with $\gamma(0) = p$. In $(M^n, \mathcal{A}^\omega)$, there are four possible cases for tangent lines on γ at the point p, such as those shown in Fig.2.3, in where these bold lines represent tangent lines.

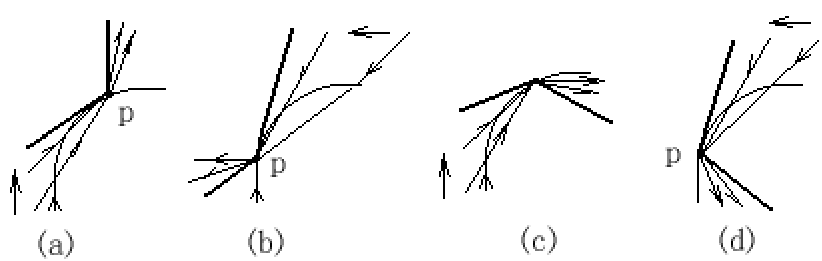

Fig.2.3

By these positions of tangent lines at a point p on γ, we conclude that *there is one tangent line at a point p on a smooth curve if and only if p is euclidean in* $(M^n, \mathcal{A}^\omega)$. This result enables us to get the dimensional number of a tangent vector space T_pM^n at a point $p \in (M^n, \mathcal{A}^\omega)$.

Theorem 2.4 *For any point $p \in (M^n, \mathcal{A}^\omega)$ with a local chart (U_p, φ_p), $\varphi_p(p) = (x_1^0 x_2^0, \cdots, x_n^0)$, if there are just s euclidean directions along $\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \cdots, \mathbf{e}_{i_s}$ for a point , then the dimension of T_pM^n is*

$$\dim T_pM^n \;=\; 2n - s$$

with a basis

$$\{\frac{\partial}{\partial x^{i_j}}|_p \mid 1 \leq j \leq s\} \bigcup \{\frac{\partial^-}{\partial x^l}|_p, \frac{\partial^+}{\partial x^l}|_p \mid 1 \leq l \leq n \text{ and } l \neq i_j, 1 \leq j \leq s\}.$$

Proof We only need to prove that

$$\{\frac{\partial}{\partial x^{i_j}}|_p \mid 1 \leq j \leq s\} \bigcup \{\frac{\partial^-}{\partial x^l}, \frac{\partial^+}{\partial x^l}|_p \mid 1 \leq l \leq n \text{ and } l \neq i_j, 1 \leq j \leq s\} \quad (2.1)$$

is a basis of T_pM^n. For $\forall f \in \Im_p$, since f is smooth, we know that

$$\begin{aligned} f(x) &= f(p) + \sum_{i=1}^{n}(x_i - x_i^0)\frac{\partial^{\epsilon_i} f}{\partial x_i}(p) \\ &+ \sum_{i,j=1}^{n}(x_i - x_i^0)(x_j - x_j^0)\frac{\partial^{\epsilon_i} f}{\partial x_i}\frac{\partial^{\epsilon_j} f}{\partial x_j} + R_{i,j,\cdots,k} \end{aligned}$$

for $\forall x = (x_1, x_2, \cdots, x_n) \in \varphi_p(U_p)$ by the Taylor formula in $\mathbf{R}^n$, where each term in $R_{i,j,\cdots,k}$ contains $(x_i - x_i^0)(x_j - x_j^0)\cdots(x_k - x_k^0)$, $\epsilon_l \in \{+, -\}$ for $1 \leq l \leq n$ but $l \neq i_j$ for $1 \leq j \leq s$ and ϵ_l should be deleted for $l = i_j, 1 \leq j \leq s$.

Now let $v \in T_pM^n$. By Definition 2.4(1), we get that

$$\begin{aligned} v(f(x)) &= v(f(p)) + v(\sum_{i=1}^{n}(x_i - x_i^0)\frac{\partial^{\epsilon_i} f}{\partial x_i}(p)) \\ &+ v(\sum_{i,j=1}^{n}(x_i - x_i^0)(x_j - x_j^0)\frac{\partial^{\epsilon_i} f}{\partial x_i}\frac{\partial^{\epsilon_j} f}{\partial x_j}) + v(R_{i,j,\cdots,k}). \end{aligned}$$

Application of the condition (2) in Definition 2.4 shows that

$$v(f(p)) = 0, \;\; \sum_{i=1}^{n} v(x_i^0)\frac{\partial^{\epsilon_i} f}{\partial x_i}(p) = 0,$$

$$v(\sum_{i,j=1}^{n}(x_i - x_i^0)(x_j - x_j^0)\frac{\partial^{\epsilon_i} f}{\partial x_i}\frac{\partial^{\epsilon_j} f}{\partial x_j}) = 0$$

and

$$v(R_{i,j,\cdots,k}) = 0.$$

Whence, we get that

$$v(f(x)) = \sum_{i=1}^{n} v(x_i)\frac{\partial^{\epsilon_i} f}{\partial x_i}(p) = \sum_{i=1}^{n} v(x_i)\frac{\partial^{\epsilon_i}}{\partial x_i}|_p(f). \quad (2.2)$$

The formula (2.2) shows that any tangent vector v in T_pM^n can be spanned by elements in (2.1).

All elements in (2.1) are linearly independent. Otherwise, if there are numbers $a^1, a^2, \cdots, a^s, a_1^+, a_1^-, a_2^+, a_2^-, \cdots, a_{n-s}^+, a_{n-s}^-$ such that

$$\sum_{j=1}^{s} a_{i_j}\frac{\partial}{\partial x_{i_j}} + \sum_{i\neq i_1,i_2,\cdots,i_s,1\leq i\leq n} a_i^{\epsilon_i}\frac{\partial^{\epsilon_i}}{\partial x_i}|_p = 0,$$

where $\epsilon_i \in \{+,-\}$, then we get that

$$a_{i_j} = (\sum_{j=1}^{s} a_{i_j}\frac{\partial}{\partial x_{i_j}} + \sum_{i\neq i_1,i_2,\cdots,i_s,1\leq i\leq n} a_i^{\epsilon_i}\frac{\partial^{\epsilon_i}}{\partial x_i})(x_{i_j}) = 0$$

for $1 \leq j \leq s$ and

$$a_i^{\epsilon_i} = (\sum_{j=1}^{s} a_{i_j}\frac{\partial}{\partial x_{i_j}} + \sum_{i\neq i_1,i_2,\cdots,i_s,1\leq i\leq n} a_i^{\epsilon_i}\frac{\partial^{\epsilon_i}}{\partial x_i})(x_i) = 0$$

for $i \neq i_1, i_2, \cdots, i_s, 1 \leq i \leq n$. Therefore, (2.1) is a basis of the tangent vector space T_pM^n at the point $p \in (M^n, \mathcal{A}^\omega)$. ♮

Notice that $\dim T_pM^n = n$ in Theorem 2.4 if and only if all these directions are euclidean along $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n$. We get a consequence by Theorem 2.4.

Corollary 2.4([4]-[5]) *Let $(M^n, \mathcal{A})$ be a smooth manifold and $p \in M^n$. Then*

$$\dim T_pM^n = n$$

with a basis

$$\{\frac{\partial}{\partial x^i}|_p \mid 1 \leq i \leq n\}.$$

Definition 2.5 *For $\forall p \in (M^n, \mathcal{A}^\omega)$, the dual space $T_p^*M^n$ is called a co-tangent vector space at p.*

Definition 2.6 *For $f \in \Im_p, d \in T_p^*M^n$ and $v \in T_pM^n$, the action of d on f, called a differential operator $d : \Im_p \to \mathbf{R}$, is defined by*

$$df = v(f).$$

Then we immediately get the following result.

Theorem 2.5 *For any point $p \in (M^n, \mathcal{A}^\omega)$ with a local chart (U_p, φ_p), $\varphi_p(p) = (x_1^{\cdot} x_2^0, \cdots, x_n^0)$, if there are just s euclidean directions along $\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \cdots, \mathbf{e}_{i_s}$ for a point , then the dimension of $T_p^* M^n$ is*

$$\dim T_p^* M^n \;=\; 2n - s$$

with a basis

$$\{dx^{i_j}|_p \mid 1 \leq j \leq s\} \bigcup \{d^- x_p^l, d^+ x^l|_p \mid 1 \leq l \leq n \text{ and } l \neq i_j, 1 \leq j \leq s\},$$

where

$$dx^i|_p(\frac{\partial}{\partial x^j}|_p) = \delta_j^i \text{ and } d^{\epsilon_i} x^i|_p(\frac{\partial^{\epsilon_i}}{\partial x^j}|_p) = \delta_j^i$$

for $\epsilon_i \in \{+, -\}, 1 \leq i \leq n$.

§3. Pseudo-Manifold Geometries

Here we introduce *Minkowski norms* on these pseudo-manifolds $(M^n, \mathcal{A}^\omega)$.

Definition 3.1 *A Minkowski norm on a vector space V is a function $F : V \to \mathbf{R}$ such that*

(1) *F is smooth on $V \setminus \{0\}$ and $F(v) \geq 0$ for $\forall v \in V$;*
(2) *F is 1-homogenous, i.e., $F(\lambda v) = \lambda F(v)$ for $\forall \lambda > 0$;*
(3) *for all $y \in V \setminus \{0\}$, the symmetric bilinear form $g_y : V \times V \to \mathbf{R}$ with*

$$g_y(u, v) = \sum_{i,j} \frac{\partial^2 F(y)}{\partial y^i \partial y^j}$$

is positive definite for $u, v \in V$.

Denote by $TM^n = \bigcup\limits_{p \in (M^n, \mathcal{A}^\omega)} T_p M^n$.

Definition 3.2 *A pseudo-manifold geometry is a pseudo-manifold $(M^n, \mathcal{A}^\omega)$ endowed with a Minkowski norm F on TM^n.*

Then we get the following result.

Theorem 3.1 *There are pseudo-manifold geometries.*

Proof Consider an eucildean $2n$-dimensional space $\mathbf{R}^{2n}$. Then there exists a Minkowski norm $F(\overline{x}) = |\overline{x}|$ at least. According to Theorem 2.4, $T_p M^n$ is $\mathbf{R}^{s+2(n-s)}$

if $\omega(p)$ has s euclidean directions along $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n$. Whence there are Minkowski norms on each chart of a point in $(M^n, \mathcal{A}^\omega)$.

Since $(M^n, \mathcal{A})$ has finite cover $\{(U_\alpha, \varphi_\alpha) | \alpha \in I\}$, where I is a finite index set, by the decomposition theorem for unit, we know that there are smooth functions $h_\alpha, \alpha \in I$ such that

$$\sum_{\alpha \in I} h_\alpha = 1 \text{ with } 0 \leq h_\alpha \leq 1.$$

Choose a Minkowski norm F^α on each chart $(U_\alpha, \varphi_\alpha)$. Define

$$F_\alpha = \begin{cases} h^\alpha F^\alpha, & \text{if} \quad p \in U_\alpha, \\ 0, & \text{if} \quad p \notin U_\alpha \end{cases}$$

for $\forall p \in (M^n, \varphi^\omega)$. Now let

$$F = \sum_{\alpha \in I} F_\alpha.$$

Then F is a Minkowski norm on TM^n since it satisfies all of these conditions $(1)-(3)$ in Definition 3.1. $\natural$

Although the dimension of each tangent vector space maybe different, we can also introduce *principal fiber bundles* and *connections* on pseudo-manifolds.

Definition 3.3 *A principal fiber bundle (PFB) consists of a pseudo-manifold $(P, \mathcal{A}_1^\omega)$, a projection $\pi : (P, \mathcal{A}_1^\omega) \rightarrow (M, \mathcal{A}_0^{\pi(\omega)})$, a base pseudo-manifold $(M, \mathcal{A}_0^{\pi(\omega)})$ and a Lie group G, denoted by (P, M, ω^π, G) such that (1), (2) and (3) following hold.*

(1) There is a right freely action of G on $(P, \mathcal{A}_1^\omega)$, i.e., for $\forall g \in G$, there is a diffeomorphism $R_g : (P, \mathcal{A}_1^\omega) \rightarrow (P, \mathcal{A}_1^\omega)$ with $R_g(p^\omega) = p^\omega g$ for $\forall p \in (P, \mathcal{A}_1^\omega)$ such that $p^\omega(g_1 g_2) = (p^\omega g_1) g_2$ for $\forall p \in (P, \mathcal{A}_1^\omega)$, $\forall g_1, g_2 \in G$ and $p^\omega e = p^\omega$ for some $p \in (P^n, \mathcal{A}_1^\omega)$, $e \in G$ if and only if e is the identity element of G.

(2) The map $\pi : (P, \mathcal{A}_1^\omega) \rightarrow (M, \mathcal{A}_0^{\pi(\omega)})$ is onto with $\pi^{-1}(\pi(p)) = \{pg | g \in G\}$, $\pi\omega_1 = \omega_0\pi$, and regular on spatial directions of p, i.e., if the spatial directions of p are $(\omega_1, \omega_2, \cdots, \omega_n)$, then ω_i and $\pi(\omega_i)$ are both elliptic, or euclidean, or hyperbolic and $|\pi^{-1}(\pi(\omega_i))|$ is a constant number independent of p for any integer $i, 1 \leq i \leq n$.

(3) For $\forall x \in (M, \mathcal{A}_0^{\pi(\omega)})$ there is an open set U with $x \in U$ and a diffeomorphism $T_u^{\pi(\omega)} : (\pi)^{-1}(U^{\pi(\omega)}) \rightarrow U^{\pi(\omega)} \times G$ of the form $T_u(p) = (\pi(p^\omega), s_u(p^\omega))$, where $s_u : \pi^{-1}(U^{\pi(\omega)}) \rightarrow G$ has the property $s_u(p^\omega g) = s_u(p^\omega) g$ for $\forall g \in G, p \in \pi^{-1}(U)$.

We know the following result for principal fiber bundles of pseudo-manifolds.

Theorem 3.2 *Let (P, M, ω^π, G) be a PFB. Then*

$$(P, M, \omega^\pi, G) = (P, M, \pi, G)$$

if and only if all points in pseudo-manifolds $(P, \mathcal{A}_1^\omega)$ are euclidean.

Proof For $\forall p \in (P, \mathcal{A}_1^\omega)$, let (U_p, φ_p) be a chart at p. Notice that $\omega^\pi = \pi$ if and only if $\varphi_p^\omega = \varphi_p$ for $\forall p \in (P, \mathcal{A}_1^\omega)$. According to Theorem 2.1, by definition this is equivalent to that all points in $(P, \mathcal{A}_1^\omega)$ are euclidean. ♮

Definition 3.4 *Let (P, M, ω^π, G) be a PFB with* $\mathrm{dim}G = r$. *A subspace family $H = \{H_p | p \in (P, \mathcal{A}_1^\omega), \mathrm{dim}H_p = \mathrm{dim}T_{\pi(p)}M\}$ of TP is called a connection if conditions (1) and (2) following hold.*

(1) *For $\forall p \in (P, \mathcal{A}_1^\omega)$, there is a decomposition*

$$T_pP = H_p \bigoplus V_p$$

and the restriction $\pi_*|_{H_p} : H_p \to T_{\pi(p)}M$ is a linear isomorphism.

(2) *H is invariant under the right action of G, i.e., for $p \in (P, \mathcal{A}_1^\omega)$, $\forall g \in G$,*

$$(R_g)_{*p}(H_p) = H_{pg}.$$

Similar to Theorem 3.2, the conception of connection introduced in Definition 3.4 is more general than the popular connection on principal fiber bundles.

Theorem 3.3(dimensional formula) *Let (P, M, ω^π, G) be a PFB with a connection H. For $\forall p \in (P, \mathcal{A}_1^\omega)$, if the number of euclidean directions of p is $\lambda_P(p)$, then*

$$\mathrm{dim}V_p = \frac{(\mathrm{dim}P - \mathrm{dim}M)(2\mathrm{dim}P - \lambda_P(p))}{\mathrm{dim}P}.$$

Proof Assume these euclidean directions of the point p being $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_{\lambda_P(p)}$. By definition π is regular, we know that $\pi(\mathbf{e}_1), \pi(\mathbf{e}_2), \cdots, \pi(\mathbf{e}_{\lambda_P(p)})$ are also euclidean in $(M, \mathcal{A}_1^{\pi(\omega)})$. Now since

$$\pi^{-1}(\pi(\mathbf{e}_1)) = \pi^{-1}(\pi(\mathbf{e}_2)) = \cdots = \pi^{-1}(\pi(\mathbf{e}_{\lambda_P(p)})) = \mu = \text{ constant},$$

we get that $\lambda_P(p) = \mu\lambda_M$, where λ_M denotes the correspondent euclidean directions in $(M, \mathcal{A}_1^{\pi(\omega)})$. Similarly, consider all directions of the point p, we also get that $\mathrm{dim}P = \mu\mathrm{dim}M$. Thereafter

$$\lambda_M = \frac{\mathrm{dim}M}{\mathrm{dim}P}\lambda_P(p). \qquad (3.1)$$

Now by Definition 3.4, $T_pP = H_p \bigoplus V_p$, i.e.,

$$\mathrm{dim}T_pP = \mathrm{dim}H_p + \mathrm{dim}V_p. \quad (3.2)$$

Since $\pi_*|_{H_p} : H_p \to T_{\pi(p)}M$ is a linear isomorphism, we know that $\mathrm{dim}H_p = \mathrm{dim}T_{\pi(p)}M$. According to Theorem 2.4, we have formulae

$$\mathrm{dim}T_pP = 2\mathrm{dim}P - \lambda_P(p)$$

and

$$\dim T_{\pi(p)}M = 2\dim M - \lambda_M = 2\dim M - \frac{\dim M}{\dim P}\lambda_P(p).$$

Now replacing all these formulae into (3.2), we get that

$$2\dim P - \lambda_P(p) = 2\dim M - \frac{\dim M}{\dim P}\lambda_P(p) + \dim V_p.$$

That is,

$$\dim V_p = \frac{(\dim P - \dim M)(2\dim P - \lambda_P(p))}{\dim P}. \quad \natural$$

We immediately get the following consequence by Theorem 3.3.

Corollary 3.1 *Let (P, M, ω^π, G) be a PFB with a connection H. Then for $\forall p \in (P, \mathcal{A}_1^\omega)$,*

$$\dim V_p = \dim P - \dim M$$

if and only if the point p is euclidean.

Now we consider conclusions included in Smarandache geometries, particularly in pseudo-manifold geometries.

Theorem 3.4 *A pseudo-manifold geometry (M^n, φ^ω) with a Minkowski norm on TM^n is a Finsler geometry if and only if all points of (M^n, φ^ω) are euclidean.*

Proof According to Theorem 2.1, $\varphi_p^\omega = \varphi_p$ for $\forall p \in (M^n, \varphi^\omega)$ if and only if p is eucildean. Whence, by definition (M^n, φ^ω) is a Finsler geometry if and only if all points of (M^n, φ^ω) are euclidean. $\natural$

Corollary 3.1 *There are inclusions among Smarandache geometries, Finsler geometry, Riemann geometry and Weyl geometry:*

$$\{Smarandache\ geometries\} \supset \{pseudo-manifold\ geometries\}$$
$$\supset \{Finsler\ geometry\} \supset \{Riemann\ geometry\} \supset \{Weyl\ geometry\}.$$

Proof The first and second inclusions are implied in Theorems 2.1 and 3.3. Other inclusions are known in a textbook, such as [4] – [5]. $\natural$

Now we consider complex manifolds. Let $z^i = x^i + \sqrt{-1}y^i$. In fact, any complex manifold M_c^n is equal to a smooth real manifold M^{2n} with a natural base $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\}$ for $T_pM_c^n$ at each point $p \in M_c^n$. Define a *Hermite* manifold M_c^n to be a manifold M_c^n endowed with a Hermite inner product $h(p)$ on the tangent space $(T_pM_c^n, J)$ for $\forall p \in M_c^n$, where J is a mapping defined by

$$J(\frac{\partial}{\partial x^i}|_p) = \frac{\partial}{\partial y^i}|_p, \quad J(\frac{\partial}{\partial y^i}|_p) = -\frac{\partial}{\partial x^i}|_p$$

at each point $p \in M_c^n$ for any integer $i, 1 \leq i \leq n$. Now let

$$h(p) = g(p) + \sqrt{-1}\kappa(p), \quad p \in M_c^m.$$

Then a *Kähler manifold* is defined to be a Hermite manifold (M_c^n, h) with a closed κ satisfying

$$\kappa(X, Y) = g(X, JY), \ \forall X, Y \in T_p M_c^n, \forall p \in M_c^n.$$

Similar to Theorem 3.3 for real manifolds, we know the next result.

Theorem 3.5 *A pseudo-manifold geometry (M_c^n, φ^ω) with a Minkowski norm on TM^n is a Kähler geometry if and only if F is a Hermite inner product on M_c^n with all points of (M^n, φ^ω) being euclidean.*

Proof Notice that a complex manifold M_c^n is equal to a real manifold M^{2n}. Similar to the proof of Theorem 3.3, we get the claim. $\natural$

As a immediately consequence, we get the following inclusions in Smarandache geometries.

Corollary 3.2 *There are inclusions among Smarandache geometries, pseudo-manifold geometry and Kähler geometry:*

$$\begin{aligned} \{Smarandache\ geometries\} &\supset \{pseudo-manifold\ geometries\} \\ &\supset \{K\ddot{a}hler\ geometry\}. \end{aligned}$$

§4. Further Discussions

Undoubtedly, there are many and many open problems and research trends in pseudo-manifold geometries. Further research these new trends and solving these open problems will enrich one's knowledge in sciences.

Firstly, we need to get these counterpart in pseudo-manifold geometries for some important results in Finsler geometry or Riemann geometry.

4.1. **Storkes Theorem** *Let $(M^n, \mathcal{A})$ be a smoothly oriented manifold with the T_2 axiom hold. Then for $\forall \varpi \in A_0^{n-1}(M^n)$,*

$$\int_{M^n} d\varpi = \int_{\partial M^n} \varpi.$$

This is the well-known Storkes formula in Riemann geometry. If we replace $(M^n, \mathcal{A})$ by $(M^n, \mathcal{A}^\omega)$, *what will happens?* Answer this question needs to solve problems following.

(1) *Establish an integral theory on pseudo-manifolds.*

(2) *Find conditions such that the Storkes formula hold for pseudo-manifolds.*

4.2. **Gauss-Bonnet Theorem** *Let S be an orientable compact surface. Then*

$$\int\int_S K d\sigma = 2\pi\chi(S),$$

where K and $\chi(S)$ are the Gauss curvature and Euler characteristic of S This formula is the well-known Gauss-Bonnet formula in differential geometry on surfaces. Then *what is its counterpart in pseudo-manifold geometries?* This need us to solve problems following.

(1) *Find a suitable definition for curvatures in pseudo-manifold geometries.*

(2) *Find generalizations of the Gauss-Bonnect formula for pseudo-manifold geometries, particularly, for pseudo-surfaces.*

For a oriently compact Riemann manifold (M^{2p}, g), let

$$\Omega = \frac{(-1)^p}{2^{2p}\pi^p p!} \sum_{i_1, i_2, \cdots, i_{2p}} \delta^{i_1, \cdots, i_{2p}}_{1, \cdots, 2p} \Omega_{i_1 i_2} \wedge \cdots \wedge \Omega_{i_{2p-1} i_{2p}},$$

where Ω_{ij} is the curvature form under the natural chart $\{e_i\}$ of M^{2p} and

$$\delta^{i_1, \cdots, i_{2p}}_{1, \cdots, 2p} = \begin{cases} 1, & \text{if permutation } i_1 \cdots i_{2p} \text{ is even,} \\ -1, & \text{if permutation } i_1 \cdots i_{2p} \text{ is odd,} \\ 0, & \text{otherwise.} \end{cases}$$

Chern proved that[4]−[5]

$$\int_{M^{2p}} \Omega = \chi(M^{2p}).$$

Certainly, these new kind of global formulae for pseudo-manifold geometries are valuable to find.

4.3. **Gauge Fields** Physicists have established a gauge theory on principal fiber bundles of Riemann manifolds, which can be used to unite gauge fields with gravitation. Similar consideration for pseudo-manifold geometries will induce new gauge theory, which enables us to asking problems following.

Establish a gauge theory on those of pseudo-manifold geometries with some additional conditions.

(1) *Find these conditions such that we can establish a gauge theory on a pseudo-manifold geometry.*

(2) *Find the Yang-Mills equation in a gauge theory on a pseudo-manifold geometry.*

(2) *Unify these gauge fields and gravitation.*

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