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Smarandache's ratio theorem

Edited by M. Khoshnevisan

Neuro Intelligence Center, Australia
E-mail: mkhoshnevisan@neurointelligence-center.org

Abstract In this paper we present the Smarandache's ratio theorem in the geometry of the triangle.

Keywords Smarandache's ratio theorem, triangle.

§1. The main result

Smarandache's Ratio Theorem

If the points A_1, B_1, C_1 divide the sides $\| BC \|= a, \| CA \|= b$, respectively $\| AB \|= c$ of a triangle $\triangle ABC$ in the same ratio $k > 0$, then

$$\| AA_1 \|^2 + \| BB_1 \|^2 + \| CC_1 \|^2 \geq \frac{3}{4}(a^2 + b^2 + c^2).$$

Proof. Suppose $k > 0$ because we work with distances.

$$\| BA_1 \|= k \| BC \|, \| CB_1 \|= k \| CA \|, \| AC_1 \|= k \| AB \| .$$

We'll apply tree times Stewart's theorem in the triangle $\triangle ABC$, with the segments AA_1, BB_1, respectively CC_1:

$$\| AB \|^2 \cdot \| BC \| (1-k) + \| AC \|^2 \cdot \| BC \| k - \| AA_1 \|^2 \cdot \| BC \| = \| BC \|^3 (1-k)k,$$

where

$$\| AA_1 \|^2 = (1-k) \| AB \|^2 + k \| AC \|^2 - (1-k)k \| BC \|^2 .$$

Similarly,

$$\| BB_1 \|^2 = (1-k) \| BC \|^2 + k \| BA \|^2 - (1-k)k \| AC \|^2 .$$

$$\| CC_1 \|^2 = (1-k) \| CA \|^2 + k \| CB \|^2 - (1-k)k \| AB \|^2 .$$

By adding these three equalities we obtain:

$$\| AA_1 \|^2 + \| BB_1 \|^2 + \| CC_1 \|^2 = (k^2 - k + 1)(\| AB \|^2 + \| BC \|^2 + \| CA \|^2),$$

which takes the minimum value when $k = \frac{1}{2}$, which is the case when the three lines from the enouncement are the medians of the triangle.

The minimum is $\frac{3}{4}(\| AB \|^2 + \| BC \|^2 + \| CA \|^2)$.

§2. Open problems on Smarandache's ratio theorem

1. If the points $A'_1, A'_2, \ldots, A'_n$ divide the sides $A_1A_2, A_2A_3, \ldots, A_nA_1$ of a polygon in a ratio $k > 0$, determine the minimum of the expression:

$$\| A_1A'_1 \|^2 + \| A_2A'_2 \|^2 + \cdots + \| A_nA'_n \|^2 .$$

2. Similarly question if the points $A'_1, A'_2, \ldots, A'_n$ divide the sides $A_1A_2, A_2A_3, \ldots, A_nA_1$ in the positive ratios $k_1, k_2, \ldots, k_n$ respectively.

3. Generalize this problem for polyhedrons.

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