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The relationship between $S_p(n)$ and $S_p(kn)$

Weiyi Zhu

College of Mathematics, Physics and Information Engineer, Zhejiang Normal University
Jinhua, Zhejiang, P.R.China

Abstract For any positive integer n, let $S_p(n)$ denotes the smallest positive integer such that $S_p(n)!$ is divisible by p^n, where p be a prime. The main purpose of this paper is using the elementary methods to study the relationship between $S_p(n)$ and $S_p(kn)$, and give an interesting identity.

Keywords The primitive numbers of power p, properties, identity

§1. Introduction and Results

Let p be a prime and n be any positive integer. Then we define the primitive numbers of power p (p be a prime) $S_p(n)$ as the smallest positive integer m such that $m!$ is divided by p^n. For example, $S_3(1) = 3,\ S_3(2) = 6,\ S_3(3) = S_3(4) = 9,\ \cdots$. In problem 49 of book [1], Professor F.Smarandache asked us to study the properties of the sequence $\{S_p(n)\}$. About this problem, Zhang Wenpeng and Liu Duansen [3] had studied the asymptotic properties of $S_p(n)$, and obtained an interesting asymptotic formula for it. That is, for any fixed prime p and any positive integer n, they proved that

$$S_p(n) = (p-1)n + O\left(\frac{p}{\ln p}\ln n\right).$$

Yi Yuan [4] had studied the asymptotic property of $S_p(n)$ in the form $\frac{1}{p}\sum\limits_{n\le x}|S_p(n+1) - S_p(n)|$, and obtained the following result: for any real number $x \ge 2$, let p be a prime and n be any positive integer,

$$\frac{1}{p}\sum_{n\le x}|S_p(n+1) - S_p(n)| = x\left(1-\frac{1}{p}\right) + O\left(\frac{\ln x}{\ln p}\right).$$

Xu Zhefeng [5] had studied the relationship between the Riemann zeta-function and an infinite series involving $S_p(n)$, and obtained some interesting identities and asymptotic formulae for $S_p(n)$. That is, for any prime p and complex number s with $\mathrm{Re}s > 1$, we have the identity:

$$\sum_{n=1}^{\infty}\frac{1}{S_p^s(n)} = \frac{\zeta(s)}{p^s - 1},$$

where $\zeta(s)$ is the Riemann zeta-function.

And, let p be a fixed prime, then for any real number $x \geq 1$ he got

$$\sum_{\substack{n=1 \\ S_p(n)\leq x}}^{\infty} \frac{1}{S_p(n)} = \frac{1}{p-1}\left(\ln x + \gamma + \frac{p\ln p}{p-1}\right) + O(x^{-\frac{1}{2}+\varepsilon}),$$

where γ is the Euler constant, ε denotes any fixed positive number.

Chen Guohui [7] had studied the calculation problem of the special value of famous Smarandache function $S(n) = \min\{m :\ m \in N,\ n|m!\}$. That is, let p be a prime and k an integer with $1 \leq k < p$. Then for polynomial $f(x) = x^{n_k} + x^{n_{k-1}} + \cdots + x^{n_1}$ with $n_k > n_{k-1} > \cdots > n_1$, we have:

$$S(p^{f(p)}) = (p-1)f(p) + pf(1).$$

And, let p be a prime and k an integer with $1 \leq k < p$, for any positive integer n, we have:

$$S\left(p^{kp^n}\right) = k\left(\phi(p^n) + \frac{1}{k}\right)p,$$

where $\phi(n)$ is the Euler function. All these two conclusions above also hold for primitive function $S_p(n)$ of power p.

In this paper, we shall use the elementary methods to study the relationships between $S_p(n)$ and $S_p(kn)$, and get some interesting identities. That is, we shall prove the following:

Theorem. Let p be a prime. Then for any positive integers n and k with $1 \leq n \leq p$ and $1 < k < p$, we have the identities:

$S_p(kn) = kS_p(n)$, if $1 < kn < p$;

$S_p(kn) = kS_p(n) - p\left[\frac{kn}{p}\right]$, if $p < kn < p^2$, where $[x]$ denotes the integer part of x.

§2. Two simple Lemmas

To complete the proof of the theorem, we need two simple lemmas which stated as following:

Lemma 1. For any prime p and any positive integer $2 \leq l \leq p-1$, we have:

(1) $S_p(n) = np$, if $1 \leq n \leq p$;

(2) $S_p(n) = (n-l+1)p$, if $(l-1)p + l - 2 < n \leq lp + l - 1$.

Proof. First we prove the case (1) of Lemma 1. From the definition of $S_p(n) = \min\{m : p^n|m!\}$, we know that to prove the case (1) of Lemma 1, we only to prove that $p^n\|(np)!$. That is, $p^n|(np)!$ and $p^{n+1}\dagger(np)!$. According to Theorem 1.7.2 of [6] we only to prove that $\sum_{j=1}^{\infty}\left[\frac{np}{p^j}\right] = n$. In fact, if $1 \leq n < p$, note that $\left[\frac{n}{p^i}\right] = 0,\ i = 1,\ 2, \cdots$, we have

$$\sum_{j=1}^{\infty}\left[\frac{np}{p^j}\right] = \sum_{j=1}^{\infty}\left[\frac{n}{p^{j-1}}\right] = n + \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \cdots = n.$$

This means $S_p(n) = np$. If $n = p$, then $\sum_{j=1}^{\infty}\left[\frac{np}{p^j}\right] = n+1$, but $p^p \dagger (p^2-1)!$ and $p^p|p^2!$. This prove the case (1) of Lemma 1. Now we prove the case (2) of Lemma 1. Using the same method of proving the case (1) of Lemma 1 we can deduce that if $(l-1)p+l-2 < n \le lp+l-1$, then

$$\left[\frac{n-l+1}{p}\right] = l-1,\ \left[\frac{n-l+1}{p^i}\right] = 0,\ i = 2, 3, \cdots.$$

So we have

$$\begin{aligned}\sum_{j=1}^{\infty}\left[\frac{(n-l+1)p}{p^j}\right] &= \sum_{j=1}^{\infty}\left[\frac{n-l+1}{p^{j-1}}\right] \\ &= n-l+1+\left[\frac{n-l+1}{p}\right]+\left[\frac{n-l+1}{p^2}\right]+\cdots \\ &= n-l+1+l-1 = n.\end{aligned}$$

From Theorem 1.7.2 of reference [6] we know that if $(l-1)p+l-2 < n \le lp+l-1$, then $p^n \| ((n-l+1)p)!$. That is, $S_p(n) = (n-l+1)p$. This proves Lemma 1.

Lemma 2. For any prime p, we have the identity $S_p(n) = (n-p+1)p$, if $p^2-2 < n \le p^2$.

Proof. It is similar to Lemma 1, we only need to prove $p^n \| ((n-p+1)p)!$. Note that if $p^2-2 < n \le p^2$, then $\left[\frac{n-p+1}{p}\right] = p-1$, $\left[\frac{n-p+1}{p^i}\right] = 0$, $i = 2,\ 3, \cdots$. So we have

$$\begin{aligned}\sum_{j=1}^{\infty}\left[\frac{(n-p+1)p}{p^j}\right] &= \sum_{j=1}^{\infty}\left[\frac{n-p+1}{p^{j-1}}\right] \\ &= n-p+1+\left[\frac{n-p+1}{p}\right]+\left[\frac{n-p+1}{p^2}\right]+\cdots \\ &= n-p+1+p-1 = n.\end{aligned}$$

From Theorem 1.7.2 of [6] we know that if $p^2-2 < n \le p^2$, then $p^n \| ((n-p+1)p)!$. That is, $S_p(n) = (n-p+1)p$. This completes the proof of Lemma 2.

§3. Proof of Theorem

In this section, we shall use above Lemmas to complete the proof of our theorem.

Since $1 \le n \le p$ and $1 < k < p$, therefore we deduce $1 < kn < p^2$. We can divide $1 < kn < p^2$ into three interval $1 < kn < p$, $(m-1)p+m-2 < kn \le mp+m-1$ $(m = 2,\ 3, \cdots, p-1)$ and $p^2-2 < kn \le p^2$. Here, we discuss above three interval of kn respectively:

i) If $1 < kn < p$, from the case (1) of Lemma 1 we have

$$S_p(kn) = knp = kS_p(n).$$

ii) If $(m-1)p+m-2 < kn \le mp+m-1$ $(m = 2, 3, \cdots, p-1)$, then from the case (2) of Lemma 1 we have

$$S_p(kn) = (kn-m+1)p = knp-(m-1)p = kS_p(n)-(m-1)p.$$

In fact, note that if $(m-1)p+m-2 < kn < mp+m-1$ $(m = 2, 3, \cdots, p-1)$, then $m-1+\left[\frac{m-2}{p}\right] < \left[\frac{kn}{p}\right] < m+\left[\frac{m-1}{p}\right]$. Hence, $\left[\frac{kn}{p}\right] = m-1$. If $kn = mp+m-1$, then $\left[\dfrac{kn}{p}\right] = m$, but $p^{mp+m-1} \dagger ((mp+m-1)p-1)!$ and $p^{mp+m-1} | ((mp+m-1)p)!$. So we immediately get

$$S_p(kn) = kS_p(n) - p\left[\frac{kn}{p}\right].$$

iii) If $p^2 - 2 < kn \leq p^2$, from Lemma 2 we have

$$S_p(kn) = (kn - p + 1)p = knp - (p-1)p.$$

Similarly, note that if $p^2 - 2 < kn \leq p^2$, then $p - \left[\dfrac{2}{p}\right] < \left[\dfrac{kn}{p}\right] \leq p$. That is, $\left[\dfrac{kn}{p}\right] = p - 1$. So we may immediately get

$$S_p(kn) = kS_p(n) - p\left[\frac{kn}{p}\right].$$

This completes the proof of our Theorem.

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