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Remarks on some of the Smarandache's problem. Part 2

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To Dr. Florentin Smarandache
for his 50th birthday

0. In 1999, the second author of this remarks published a book over 30 of Smarandache's problems in area of elementary number theory (see [1, 2]). After this, we worked over new 20 problems that we collected in our book [28]. These books contain Smarandache's problems, described in [10, 16]. The present paper contains some of the results from [28].

In [16] Florentin Smarandache formulated 105 unsolved problems, while in [10] C.Dumitresu and V. Seleacu formulated 140 unsolved problems of his. The second book contains almost all the problems from [16], but now each problem has unique number and by this reason in [1, 28] and here the authors use the numeration of the problems from [10].

In the text below the following notations are used.

$\mathcal{N}$ - the set of all natural numbers (i.e., the set of all positive integers);

$[x]$ - "floor function" (or also so called "bracket function") - the greatest integer which is not greater than the real non-negative number x;

ζ - Riemann's Zeta-function;

Γ - Euler's Gamma-function;

π - the prime counting function, i.e., $\pi(n)$ denotes the number of prime p such that $p \leq n$;

$]x[$ - the largest natural number strongly smaller than the real (positive) number x;

$\lceil x \rceil$ - the inferior integer part of x, i.e, the smallest integer greater than or equal to x.

For an arbitrary increasing sequence of natural number $C \equiv \{c_n\}_{n=1}^{\infty}$ we denote by $\pi_C(n)$ the number of terms of C, which are not greater than n. When $n < c_1$ we put $\pi_C(n) = 0$.

1. The results in this section are taken from [8].

The second problem from [10] (see also 16-th problem from [16]) is the following:

Smarandache circular sequence:

$$\underbrace{1}_{1}, \underbrace{12, 21}_{2}, \underbrace{123, 231, 312}_{3}, \underbrace{1234, 2341, 3412, 4123}_{4},$$

$$\underbrace{12345, 23451, 34512, 45123, 51234}_{5},$$

$$\underbrace{123456, 234561, 345612, 456123, 561234, 612345}_{6}, \cdots$$

Let $f(n)$ be the n-th member of the above sequence. We shall prove the following

Theorem 1.1. For each natural number n:

$$f(n) = \overline{s(s+1)\dots k12\dots(s-1)},$$

where

$$k \equiv k(n) =]\frac{\sqrt{8n+1}-1}{2}[$$

and

$$s \equiv s(n) = n - \frac{k(k+1)}{2}.$$

2. The results in this section are taken from [25].

The eight problem from [10] (see also 16-th problem from [16]) is the following:

Smarandache mobile periodicals (I):

$$\begin{array}{ccccccccccccc}
\dots 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\dots \\
\dots 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0\dots \\
\dots 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0\dots \\
\dots 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0\dots \\
\dots 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\dots \\
\dots 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0\dots \\
\dots 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0\dots \\
\dots 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0\dots \\
\dots 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0\dots \\
\dots 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0\dots \\
\dots 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\dots \\
\dots 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0\dots \\
\dots 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0\dots \\
\dots 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0\dots \\
\dots 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0\dots \\
\dots 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0\dots \\
\dots 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0\dots \\
\dots 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0\dots \\
\dots 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\dots
\end{array}$$

$$
\begin{array}{ccccccccccccc}
\ldots 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\ldots \\
\ldots 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0\ldots \\
\ldots 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0\ldots \\
\ldots 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0\ldots \\
\ldots 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0\ldots \\
\ldots 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0\ldots \\
\ldots 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0\ldots \\
 & & & . & & & . & & & . & & & \\
 & & & . & & & . & & & . & & &
\end{array}
$$

This sequence has the form

$$
\underbrace{1, 111, 11011, 111, 1}_{3}, \underbrace{1, 111, 11011, 1100011, 11011, 111, 1}_{7},
$$
$$
\underbrace{1, 111, 11011, 1100011, 110000011, 1100011, 11011, 111, 1}_{9}, \ldots
$$

All digits from the above table generate an infinite matrix A. We described the elements of A.

Let us take a Cartesian coordinate system C with origin in the point containing element "1" in the topmost (i.e., the first) row of A. We assume that this row belongs to the ordinate axis of C (see Fig. 1) and that the points to the right of the origin have positive ordinates.

The above digits generate an infinite sequence of squares, located in the half-plane (determined by C) where the abscissa of the points are nonnegative. Their diameters have the form

$$"110\ldots011".$$

Exactly one of the diameters of each of considered square lies on the abscissa of C. It can be seen (and proved, e.g.,by induction) that the s-th square, denoted by $G_s (s = 0, 1, 2, \ldots)$ has a diameter with length $2s + 4$ and the same square has a highest vertex with coordinates $\langle s^2 + 3s, 0\rangle$ in C and a lowest vertex with coordinates $\langle s^2 + 5s + 4, 0\rangle$ in C.

Let us denote by $a_{k,i}$ an element of A with coordinates $\langle k, i\rangle$ in C.

First, we determine the minimal nonnegative s for which the inequality

$$s^2 + 5s + 4 \geq k$$

holds. We denote it by $s(k)$. Directly it is seen the following

Lemma 2.1 The number $s(k)$ admits the explicit representation:

$$
s(k) = \begin{cases}
0, & \text{if} \quad 0 \leq k \leq 4 \\
\left[\frac{\sqrt{4k+9}-5}{2}\right], & \text{if } k \geq 5 \text{ and } 4k+9 \text{ is} \\
 & \text{a square of an integer} \\
\left[\frac{\sqrt{4k+9}-5}{2}\right] + 1, & \text{if } k \geq 5 \text{ and } 4k+9 \text{ is} \\
 & \text{not a square of an integer}
\end{cases}
\tag{2.1}
$$

and the inequality

$$(s(k))^2 + 3s(k) \leq k \leq (s(k))^2 + 5s(k) + 4 \tag{2.2}$$

hold.

Second, we introduce the integer $\delta(k)$ and $\varepsilon(k)$ by

$$\delta(k) \equiv k - (s(k))^2 - 3s(k), \tag{2.3}$$

$$\varepsilon(k) \equiv (s(k))^2 + 5s(k) + 4 - k. \tag{2.4}$$

From (2.2) we have $\delta(k) \geq 0$ and $\varepsilon(k) \geq 0$. Let P_k be the infinite strip orthogonal to the abscissa of C and lying between the straight lines passing through those vertices of the square $G_{s(k)}$ lying on the abscissa of C. Then $\delta(k)$ and $\varepsilon(k)$ characterize the location of point with coordinate $\langle k, i\rangle$ in C in strip P_k. Namely, the following assertion is true.

Proposition 2.1. The elements $a_{k,i}$ of the infinite matrix A are described as follows:
if $k \leq (s(k))^2 + 4s(k) + 2$, then

$$a_{k,i} = \begin{cases} 0, & \text{if } \delta(k) < |i| \text{ or } \delta(k) \geq |i| + 2, \\ 1, & \text{if } |i| \leq \delta(k) \leq |i| + 1 \end{cases}$$

if $k \geq (s(k))^2 + 4s(k) + 2$, then

$$a_{k,i} = \begin{cases} 0, & \text{if } \varepsilon(k) < |i| \text{ or } \varepsilon(k) \geq |i| + 2, \\ 1, & \text{if } |i| \leq \varepsilon(k) \leq |i| + 1 \end{cases}$$

where here and below $s(k)$ is given by (2.1), $\delta(k)$ and $\varepsilon(k)$ are given by (2.3) and (2.4), respectively.

Below, we propose another description of elements of A, which can be proved (e.g., by induction) using the same considerations.

$$a_{k,i} = \begin{cases} 1, & \text{if } \langle k, i\rangle \in \\ & \{\langle (s(k))^2 + 3s(k), 0\rangle, \langle (s(k))^2 + 5s(k) + 4, 0\rangle\} \\ & \bigcup\{\langle (s(k))^2 + 3s(k) + j, -j\rangle, \\ & \langle (s(k))^2 + 3s(k) + j, -j + 1\rangle, \\ & \langle (s(k))^2 + 3s(k) + j, j - 1\rangle, \\ & \langle (s(k))^2 + 3s(k) + j, j\rangle : 1 \leq j \leq s(k) + 2\} \\ & \langle (s(k))^2 + 5s(k) + 4 - j, -j\rangle, \\ & \langle (s(k))^2 + 5s(k) + 4 - j, -j + 1\rangle, \\ & \langle (s(k))^2 + 5s(k) + 4 - j, j - 1\rangle, \\ & \langle (s(k))^2 + 5s(k) + 4 - j, j\rangle : \\ & 1 \leq j \leq s(k) + 1\} \\ 0, & \text{otherwise} \end{cases}$$

Similar representations are possible for the ninth, tenth and eleventh problems. In [28] we introduce eight modifications of these problems, giving formulae for their (k, i)-th members $a_{k,i}$.

Essentially more interesting is Problem 103 from [10]:
Smarandache numerical carpet:
has the general form

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & . & & & & & & & \\
 & & & & & & & . & & & & & & & \\
 & & & & & & & . & & & & & & & \\
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & a & 1 & & & & & & \\
 & & & & & 1 & a & b & a & 1 & & & & & \\
 & & & & 1 & a & b & c & b & a & 1 & & & & \\
 & & & 1 & a & b & c & d & c & b & a & 1 & & & \\
 & & 1 & a & b & c & d & e & d & c & b & a & 1 & & \\
 & 1 & a & b & c & d & e & f & e & d & c & b & a & 1 & \\
1 & a & b & c & d & e & f & g & f & e & d & c & b & a & 1 \\
 & 1 & a & b & c & d & e & f & e & d & c & b & a & 1 & \\
 & & 1 & a & b & c & d & e & d & c & b & a & 1 & & \\
 & & & 1 & a & b & c & d & c & b & a & 1 & & & \\
 & & & & 1 & a & b & c & b & a & 1 & & & & \\
 & & & & & 1 & a & b & a & 1 & & & & & \\
 & & & & & & 1 & a & 1 & & & & & & \\
 & & & & & & & 1 & & & & & & & \\
 & & & & & & & . & & & & & & & \\
 & & & & & & & . & & & & & & & \\
 & & & & & & & . & & & & & & &
\end{array}
$$

On the border of level 0, the elements are equal to "1";
they form a rhomb.
Next, on the border of level 1, the elements are equal to "a";
where "a" is the sum of all elements of the previous border;
the "a"s form a rhomb too inside the previous one.
Next again, on the border of level 2, the elements are equal to "b";

where "b" is the sum of all elements of the previous border;

the "b"s form a rhomb too inside the previous one.

And so on ...

The above square, that Smarandache named *"rhomb"*, corresponds to the square from our construction for the case of $s = 6$, if we begin to count from $s = 1$, instead of $s = 0$. In [10] a particular solution of the Problem 103 is given, but there a complete solution is not introduced. We will give a solution below firstly for the case of Problem 103 and then for a more general case.

It can be easily seen that the number of the elements of the s-th square side is $s + 2$ and therefore the number of the elements from the contour of this square is just equal to $4s + 4$.

The s-th square can be represented as a set of sub-squares, each one included in the next. Let us number them inwards, so that the outmost (boundary) square is the first one. As it is written in Problem 103, all of its elements are equal to 1. Hence, the value of the elements of the subsequent (second) square will be (using also the notation from problem 103):

$$a_1 = a = (s + 2) + (s + 1) + (s + 1) + s = 4(s + 1);$$

the value of the elements of the third square will be

$$a_2 = b = a(4(s - 1) + 4 + 1) = 4(s + 1)(4s + 1);$$

the value of the elements of the fourth square will be

$$a_3 = c = b(4(s - 2) + 4 + 1) = 4(s + 1)(4s + 1)(4s - 3);$$

the value of the elements of the fifth square will be

$$a_4 = d = c(4(s - 3) + 4 + 1) = 4(s + 1)(4s + 1)(4s - 3)(4s - 7);$$

etc.,where the square, corresponding to the initial square *(rhomb)*, from Problem 103 has the form

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & \cdot & \cdot & \cdot & & & & \\
 & & 1 & a_1 & \cdot & \cdot & \cdot & a_1 & 1 & & \\
 & 1 & a_1 & a_2 & \cdot & \cdot & \cdot & a_2 & a_1 & 1 & \\
1 & a_1 & a_2 & a_3 & \cdot & \cdot & \cdot & a_3 & a_2 & a_1 & 1 \\
 & 1 & a_1 & a_2 & \cdot & \cdot & \cdot & a_2 & a_1 & 1 & \\
 & & 1 & a_1 & \cdot & \cdot & \cdot & a_1 & 1 & & \\
 & & & & \cdot & \cdot & \cdot & & & & \\
 & & & & & 1 & & & & &
\end{array}$$

It can be proved by induction that the elements of this square that stay on t-th place are given by the formula

$$a_t = 4(s+1)\prod_{i=0}^{t-2}(4s+1-4i).$$

If we would like to generalize the above problem, we can construct, e.g., the following extension:

$$\begin{array}{ccccccccccc}
 & & & & & x & & & & & \\
 & & & & \cdot & \cdot & \cdot & & & & \\
 & & & x & a_1 & \cdot\ \cdot\ \cdot & a_1 & x & & & \\
 & & x & a_1 & a_2 & \cdot\ \cdot\ \cdot & a_2 & a_1 & x & & \\
 & x & a_1 & a_2 & a_3 & \cdot\ \cdot\ \cdot & a_3 & a_2 & a_1 & x & \\
 & & x & a_1 & a_2 & \cdot\ \cdot\ \cdot & a_2 & a_1 & x & & \\
 & & & x & a_1 & \cdot\ \cdot\ \cdot & a_1 & x & & & \\
 & & & & \cdot & \cdot & \cdot & & & & \\
 & & & & & x & & & & &
\end{array}$$

where x is given number. Then we obtain

$$a_1 = 4(s+1)x$$

$$a_2 = 4(s+1)(4s+1)x$$

$$a_3 = 4(s+1)(4s+1)(4s-3)x$$

$$a_4 = 4(s+1)(4s+1)(4s-3)(4s-7)x$$

etc. and for $t \geq 1$

$$a_t = 4(s+1)\prod_{i=0}^{t-2}(4s+1-4i)x.$$

where it assumed that

$$\prod_{i=0}^{-1} \cdot = 1.$$

3. The results in this section are taken from [21].

The 15-th Smarandache's problem from [10] is the following: *"Smarandache's simple numbers"*:

$$2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 17, 19, 21, 22, 23, 25, 26, 27,$$

$$29, 31, 33, \ldots$$

A number n is called "Smarandache's simple number" if the product of its proper divisors is less than or equal to n. Generally speaking, n has the form $n = p$, or $n = p^2$, or $n = p^3$, or $n = pq$, where p and q are distinct primes".

Let us denote: by S - the sequence of all Smarandache's simple numbers and by s_n - the n-th term of S; by $\mathcal{P}$ - the sequence of all primes and by p_n - the n-th term of $\mathcal{P}$; by $\mathcal{P}^2$ - the sequence $\{p_n^2\}_{n=1}^{\infty}$; by $\mathcal{P}^3$ - the sequence $\{p_n^3\}_{n=1}^{\infty}$; by $\mathcal{PQ}$ - the sequence $\{p.q\}_{p,q} \in \mathcal{P}$, where $p < q$.

In the present section we find $\pi_S(n)$ in an explicit form and using this, we find the n-th term of S in explicit form, too.

First, we note that instead of $\pi_P(n)$ we use the notation $\pi(n)$.

Hence

$$\pi_{\mathcal{P}^2}(n) = \pi(\sqrt{n}), \pi_{\mathcal{P}^3}(n) = \pi(\sqrt[3]{n}),$$

Thus, using the definition of S, we get

$$\pi_S(n) = \pi(n) + \pi(\sqrt{n}) + \pi(\sqrt[3]{n}) + \pi_{\mathcal{PQ}}(n) \tag{4.1}$$

Our first aim is to express $\pi_S(n)$ in an explicit form. For $\pi(n)$ some explicit formulae are proposed in [18]. Other explicit formulae for $\pi(n)$ are given in [14]. One of them is known as Mináč's formula. It is given below

$$\pi(n) = \sum_{k=2}^{n} [\frac{(k-1)!+1}{k} - [\frac{(k-1)!}{k}]]. \tag{4.2}$$

Therefore, the problem of finding of explicit formulae for functions $\pi(n), \pi(\sqrt{n}), \pi(\sqrt[3]{n})$ is solved successfully. It remains only to express $\pi_{\mathcal{PQ}}(n)$ in an explicit form.

Let $k \in \{1, 2, \ldots, \pi(\sqrt{n})\}$ be fixed. We consider all numbers of the kind $p_k q$, which $p \in \mathcal{P}$, $q > p_k$ for which $p_k.q \le n$. The quality of these numbers is $\pi(\frac{n}{p_k}) - \pi(p_k)$, or which is the same

$$\pi(\frac{n}{p_k}) - k. \tag{4.3}$$

When $k = 1, 2, \ldots, \pi(\sqrt{n})$, the number $p_k.q$, as defined above, describe all numbers of the kind $p.q$, with $p, q \in \mathcal{P}$, $p < q, p.q < n$. But the quantity of the last numbers is equal to $\pi_{\mathcal{PQ}}(n)$. Hence

$$\pi_{\mathcal{PQ}}(n) = \sum_{k=1}^{\pi(\sqrt{n})} (\pi(\frac{n}{p_k}) - k), \tag{4.4}$$

because of (4.3). The equality (4.4), after a simple computation yields the formula

$$\pi_{\mathcal{PQ}}(n) = \sum_{k=1}^{\pi(\sqrt{n})} \pi(\frac{n}{p_k}) - \frac{\pi(\sqrt{n})(\pi(\sqrt{n})+1)}{2}. \tag{4.5}$$

In [20] the identity

$$\sum_{k=1}^{\pi(b)} \pi(\frac{n}{p_k}) = \pi(\frac{n}{b}).\pi(b) + \sum_{k=1}^{\pi(\frac{n}{2})-\pi(\frac{n}{b})} \pi(\frac{n}{p_{\pi(\frac{n}{b})+k}}) \tag{4.6}$$

is proved, under the condition $b > 2$ (b is a real number). When $\pi(\frac{n}{2}) = \pi(\frac{n}{b})$, the right hand-side of (4.6) is reduced to $\pi(\frac{n}{b}).\pi(b)$. In the case $b = \sqrt{n}$ and $n \ge 4$ equality (4.6) yields

$$\sum_{k=1}^{\pi(\sqrt{n})} \pi(\frac{n}{p_k}) = (\pi(\sqrt{n}))^2 + \sum_{k=1}^{\pi(\frac{n}{2})-\pi(\sqrt{n})} \pi(\frac{n}{p_{\pi(\sqrt{n})+k}}). \tag{4.7}$$

If we compare (4.5) with (4.7) we obtain for $n \geq 4$

$$\pi_{\mathcal{PQ}}(n) = \frac{\pi(\sqrt{n})(\pi(\sqrt{n})-1)}{2} + \sum_{k=1}^{\pi(\frac{n}{2})-\pi(\sqrt{n})} \pi(\frac{n}{p_{\pi(\sqrt{n})+k}}). \tag{4.8}$$

Thus, we have two different explicit representations for $\pi_{\mathcal{PQ}}(n)$. These are formulae (4.5) and (4.8). We note that the right hand side of (4.8) reduces to $\frac{\pi(\sqrt{n})(\pi(\sqrt{n})-1)}{2}$, when $\pi(\frac{n}{2}) = \pi(\sqrt{n})$.

Finally, we observe that (4.1) gives an explicit representation for $\pi_S(n)$, since we may use formula (4.2) for $\pi(n)$ (or other explicit formulae for $\pi(n)$) and (4.5), or (4.8) for $\pi_{\mathcal{PQ}}(n)$.

The following assertion solves the problem for finding of the explicit representation of s_n.

Theorem 4.1. The n-th term s_n of S admits the following three different explicit representations:

$$s_n = \sum_{k=0}^{\theta(n)} [\frac{1}{1+[\frac{\pi_S(n)}{n}]}]; \tag{4.9}$$

$$s_n = -2\sum_{k=0}^{\theta(n)} \zeta(-2[\frac{\pi_S(n)}{n}]); \tag{4.10}$$

$$s_n = \sum_{k=0}^{\theta(n)} \frac{1}{\Gamma(1-[\frac{\pi_S(n)}{n}])}, \tag{4.11}$$

where

$$\theta(n) \equiv [\frac{n^2+3n+4}{4}],\ n = 1, 2, \dots$$

We note that (4.9)-(4.11) are representations using, respectively, "floor function", Riemann's Zeta-function and Euler's Gamma-function. Also, we note that in (4.9)-(4.11) $\pi_S(k)$ is given by (4.1), $\pi(k)$ is given by (4.2) (or by others formulae like (4.2)) and $\pi_{\mathcal{PQ}}(n)$ is given by (4.5), or by (4.8). Therefore, formulae (4.9)-(4.11) are explicit.

4. The results in this section are taken from [6].

The 17-th problem from [10] (see also the 22-nd problem from [16]) is the following:

Smarandache's digital products:

$$\underbrace{0,1,2,3,4,5,6,7,8,9}, \underbrace{0,1,2,3,4,5,6,7,8,9},$$

$$\underbrace{0,2,4,6,8,10,12,14,16,18}, \underbrace{0,3,6,9,12,15,18,21,24,27},$$

$$\underbrace{0,4,8,12,16,20,24,28,32,36}, \underbrace{0,5,10,15,20,25\dots}$$

($d_p(n)$is the product of digits.)

Let the fixed natural number n have the form $n = \overline{a_1 a_2 \dots a_k}$, where $a_1, a_2, \dots, a_k \in \{0, 1, \dots, 9\}$ and $a_1 \geq 1$. Therefore,

$$n = \sum_{i=1}^{k} a_i 10^{i-1}.$$

Hence, $k = [\log_{10} n] + 1$ and

$$a_1(n) \equiv a_1 = [\frac{n}{10^{k-1}}],$$

$$a_2(n) \equiv a_2 = [\frac{n - a_1 10^{k-1}}{10^{k-2}}],$$

$$a_3(n) \equiv a_3 = [\frac{n - a_1 10^{k-1} - a_2 10^{k-2}}{10^{k-3}}],$$

$$\dots$$

$$a_{[\log_{10}(n)]}(n) \equiv a_{k-1} = [\frac{n - a_1 10^{k-1} - \dots - a_{k-2} 10^2}{10}],$$

$$a_{[\log_{10}(n)]+1}(n) \equiv a_k = n - a_1 10^{k-1} - \dots - a_{k-1} 10.$$

Obviously, $k, a_1, a_2, \dots, a_k$ are functions only of n. Therefore,

$$d_p(n) = \prod_{i=1}^{[\log_{10}(n)]+1} a_i(n).$$

5. The results in this section are taken from [4, 27].

The 20-th problem from [10] is the following (see also Problem 25 from [16]):
Smarandache devisor products:

$$1, 2, 3, 8, 5, 36, 7, 64, 27, 100, 11, 1728, 13, 196, 225, 1024, 17, 5832, 19,$$

$$8000, 441, 484, 23, 331776, 125, 676, 729, 21952, 29, 810000, 31, 32768,$$

$$1089, 1156, 1225, 10077696, 37, 1444, 1521, 2560000, 41, \dots$$

($P_d(n)$ is the product of all positive divisors of n.)

The 21-st problem from [10] is the following (see also Problem 26 from [16]):
Smarandache proper devisor products:

$$1, 1, 1, 2, 1, 6, 1, 8, 3, 10, 1, 144, 1, 14, 15, 64, 1, 324, 1, 400, 21, 22, 1,$$

$$13824, 5, 26, 27, 784, 1, 27000, 1, 1024, 33, 34, 35, 279936, 1, 38, 39,$$

$$64000, 1, \dots$$

($p_d(n)$ is the product of all positive divisors of n but n.)

Let us denote by $\tau(n)$ the number of all devisors of n. It is well-known (see, e.g., [13]) that

$$P_d(n) = \sqrt{n^{\tau(n)}} \tag{6.1}$$

and of course, we have

$$p_d(n) = \frac{P_d(n)}{n}. \tag{6.2}$$

But (6.1) is not a good formula for $P_d(n)$, because it depends on function τ and to express $\tau(n)$ we need the prime number factorization of n.

Below, we give other representations of $P_d(n)$ and $p_d(n)$, which do not use the prime number factorization of n.

Proposition 6.1. For $n \geq 1$ representation

$$P_d(n) = \prod_{k=1}^{n} k^{[\frac{n}{k}]-[\frac{n-1}{k}]} \tag{6.3}$$

holds.

Here and further the symbols

$$\prod_{k/n} \cdot \quad \text{and} \quad \sum_{k/n} \cdot$$

mean the product and the sum, respectively, of all divisors of n.

Let

$$\theta(n,k) \equiv [\frac{n}{k}] - [\frac{n-1}{k}]$$
$$= \begin{cases} 1, & \text{if} \quad k \quad \text{is a divisor of} \quad n \\ 0, & \text{otherwise} \end{cases}$$

The following assertion is obtained as a corollary of (6.2) and (6.3).

Proposition 6.2. For $n \geq 1$ representation

$$p_d(n) = \prod_{k=1}^{n-1} k^{[\frac{n}{k}]-[\frac{n-1}{k}]}$$

holds.

For $n = 1$ we have

$$p_d(1) = 1.$$

Proposition 6.3. For $n \geq 1$ representation

$$P_d(n) = \prod_{k=1}^{n} \frac{[\frac{n}{k}]!}{[\frac{n-1}{k}]!} \tag{6.5}$$

holds, where here and further we assume that $0! = 1$.

Now (6.2) and (6.5) yield.

Proposition 6.4. For $n \geq 2$ representation

$$p_d(n) = \prod_{k=2}^{n} \frac{[\frac{n}{k}]!}{[\frac{n-1}{k}]!}$$

holds.

Another type of representation of $p_d(n)$ is the following

Proposition 6.5. For $n \geq 3$ representation

$$p_d(n) = \prod_{k=1}^{n-2} (k!)^{\theta(n,k)-\theta(n,k+1)},$$

where $\theta(n,k)$ is given by (6.4).

Further, we need the following

Theorem 6.1. [22] For $n \geq 2$ the identity

$$\prod_{k=2}^{n}[\frac{n}{k}]! = \prod_{k=1}^{n-1}(k!)^{[\frac{n}{k}]-[\frac{n}{k+1}]} \tag{6.6}$$

holds.

Now, we shall deduce some formulae for

$$\prod_{k=1}^{n} P_d(k) \quad \text{and} \quad \prod_{k=1}^{n} p_d(k).$$

Proposition 6.6. Let f be an arbitrary arithmetic function. then the identity

$$\prod_{k=1}^{n}(P_d(k))^{f(k)} = \prod_{k=1}^{n} k^{\rho(n,k)} \tag{6.7}$$

holds, where

$$\rho(n,k) = \sum_{s=1}^{[\frac{n}{k}]} f(ks).$$

Now we need the following

Lemma 6.1. For $n \geq 1$ the identity

$$\prod_{k=1}^{n}[\frac{n}{k}]! = \prod_{k=1}^{n} k^{[\frac{n}{k}]}$$

holds.

Proposition 6.7. For $n \geq 1$ the identity

$$\prod_{k=1}^{n} P_d(k) = \prod_{k=1}^{n}[\frac{n}{k}]! \tag{6.8}$$

holds. As a corollary from (6.2) and (6.8), we also obtain

Proposition 6.8. For $n \geq 2$ the identity

$$\prod_{k=1}^{n} p_d(k) = \prod_{k=2}^{n}[\frac{n}{k}]! \tag{6.9}$$

holds.

From (6.6) and (6.9), we obtain

Proposition 6.9. For $n \geq 2$ the identity

$$\prod_{k=1}^{n} p_d(k) = \prod_{k=1}^{n-1}(k!)^{[\frac{n}{k}]-[\frac{n}{k+1}]} \tag{6.10}$$

holds.

As a corollary from (6.10) we obtain, because of (6.2)

Proposition 6.10. For $n \geq 1$ the identity

$$\prod_{k=1}^{n} P_d(k) = \prod_{k=1}^{n} (k!)^{[\frac{n}{k}]-[\frac{n}{k+1}]} \tag{6.10}$$

holds.

Now, we return to (6.7) and suppose that

$$f(k) > 0 \ (k = 1, 2, \ldots).$$

Then after some simple computations we obtain

Proposition 6.11. For $n \geq 1$ representation

$$P_d(k) = \prod_{k=1}^{n} k^{\sigma(n,k)} \tag{6.11}$$

holds, where

$$\sigma(n,k) = \frac{\sum_{s=1}^{[\frac{n}{k}]} f(ks) - \sum_{s=1}^{[\frac{n-1}{k}]} f(ks)}{f(n)}.$$

For $n \geq 2$ representation

$$p_d(k) = \prod_{k=1}^{n-1} k^{\sigma(n,k)} \tag{6.12}$$

holds.

Note that although f is an arbitrary arithmetic function, the situation with (6.11) and (6.12) is like the case $f(x) \equiv 1$, because

$$\frac{\sum_{s=1}^{[\frac{n}{k}]} f(ks) - \sum_{s=1}^{[\frac{n-1}{k}]} f(ks)}{f(n)} = \begin{cases} 1, & \text{if } k \text{ is a divisor of } n \\ 0, & \text{otherwise} \end{cases}$$

Finally, we use (6.7) to obtain some new inequalities, involving $P_d(k)$ and $p_d(k)$ for $k = 1, 2, \ldots, n$.

Putting

$$F(n) = \sum_{k=1}^{n} f(k)$$

we rewrite (6.7) as

$$\prod_{k=1}^{n} (P_d(k))^{\frac{f(k)}{F(n)}} = \prod_{k=1}^{n} k^{(\sum_{s=1}^{[\frac{n}{k}]} f(ks))/(F(n))}. \tag{6.13}$$

Then we use the well-known Jensen's inequality

$$\sum_{k=1}^{n} \alpha_k x_k \geq \prod_{k=1}^{n} x_k^{\alpha_k},$$

that is valid for arbitrary positive numbers x_k, $\alpha_k (k = 1, 2, \ldots, n)$ such that

$$\sum_{k=1}^{n} \alpha_k = 1,$$

for the case:

$$x_k = P_d(k),$$

$$\alpha_k = \frac{f(k)}{F(n)}.$$

Thus we obtain from (6.13) inequality

$$\sum_{k=1}^{n} f(k).P_d(k) \geq (\sum_{k=1}^{n} f(k)).\prod_{k=1}^{n} k^{(\sum_{s=1}^{[\frac{n}{k}]} f(ks))/(\sum_{s=1}^{n} f(s))}. \tag{6.14}$$

If $f(x) \equiv 1$, then (6.14) yields the inequality

$$\frac{1}{n}\sum_{k=1}^{n} P_d(k) \geq \prod_{k=1}^{n} (\sqrt[n]{k})^{[\frac{n}{k}]}.$$

If we put in (6.14)

$$f(k) = \frac{g(k)}{k}$$

for $k = 1, 2, \ldots, n$, then we obtain

$$\sum_{k=1}^{n} g(k).p_d(k) \geq (\sum_{k=1}^{n} \frac{g(k)}{k}).\prod_{k=1}^{n} (\sqrt[k]{k})^{(\sum_{s=1}^{[\frac{n}{k}]} \frac{g(ks)}{s})/(\sum_{s=1}^{n} \frac{g(s)}{s})}. \tag{6.15}$$

because of (6.2).

Let $g(x) \equiv 1$. Then (6.15) yields the very interesting inequality

$$(\frac{1}{H_n}\sum_{k=1}^{n} p_d(k))^{H_n} \geq \prod_{k=1}^{n} (\sqrt[k]{k})^{H_{[\frac{n}{k}]}},$$

where H_m denotes the m-th partial sum of the harmonic series, i.e.,

$$H_m = \frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{m}.$$

All of the above inequalities become equalities if and only if $n = 1$.

6. The results in this section are taken from [29].

The 25-th and the 26-th problems from [10] (see also the 30-th and the 31-st problems from [16]) are the following:

Smarandache's cube free sieve:

$$2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14, 15, 17, 18, 19, 20, 21, 22, 23, 25, 26,$$

$$28, 29, 30, 31, 33, 34, 35, 36, 37, 38, 39, 41, 42, 43, 44, 45, 46, 47, 49, 50,$$

$$51, 52, 53, 55, 57, 58, 59, 60, 61, 62, 63, 65, 66, 67, 68, 69, 70, 71, 73, \ldots$$

Definition: from the set of natural numbers (except 0 and 1):
- take off all multiples of 2^3 *(i.e. 8,16,24,32,40,...)*

- *take off all multiples of* 3^3
- *take off all multiples of* 5^3

... and so on (take off all multiples of all cubic primes).

Smarandache's m-power free sieve:
Definition: from the set of natural numbers (except 0 and 1) take off all multiples of 2^m*, afterwards all multiples of* 3^m *... and so on(take off all multiples of all m-power primes,* $m \geq 2$*). (One obtains all m-power free numbers.)*

Here we introduce the solution for both of these problems.

For every natural number m we denote the increasing sequence $a_1^{(m)}, a_2^{(m)}, a_3^{(m)}, \ldots$ of all m-power free numbers by $\overline{m}$. Then we have

$$\emptyset \equiv \overline{1} \subset \overline{2} \subset \ldots \subset \overline{(m-1)} \subset \overline{m} \subset \overline{(m+1)} \subset \ldots$$

Also, for $m \geq 2$ we have

$$\overline{m} = \bigcup_{k=1}^{m-1} (\overline{2})^k$$

where

$$(\overline{2})^k = \{x | (\exists x_1, \ldots, x_k \in \overline{2})(x = x_1.x_2 \ldots x_k)\}$$

for each natural number $k \geq 1$.

Let us consider $\overline{m}$ as an infinite sequence for $m = 2, 3, \ldots$. Then $\overline{2}$ is a subsequence of $\overline{m}$. Therefore, the inequality

$$a_n^{(m)} \leq a_n^{(2)}$$

holds for $n = 1, 2, 3, \ldots$.

Let $p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7, \ldots$ be the sequence of all primes. It is obvious that this sequence is a subsequence of $\overline{2}$. Hence the inequality

$$a_n^{(2)} \leq p_n$$

holds for $n = 1, 2, 3, \ldots$. But it is well-known that

$$p_n \leq \theta(n) \equiv [\frac{n^2 + 3n + 4}{4}]$$

(see [12]). Therefore, for any $m \geq 2$ and $n = 1, 2, 3, \ldots$ we have

$$a_n^{(m)} \leq a_n^{(2)} \leq \theta(n).$$

Hence, there exits $\lambda(n)$ such that $\lambda(n) \leq \theta(n)$ and inequality:

$$a_n^{(m)} \leq a_n^{(2)} \leq \lambda(n).$$

holds. In particular, it is possible to use $\theta(n)$ instead of $\lambda(n)$.

In [28] we find the following explicit formulae for $a_n^{(m)}$ when $m \geq 2$ is fixed:

$$a_n^{(m)} = \sum_{k=0}^{\lambda(n)} [\frac{1}{1 + [\frac{\pi_{\overline{m}}(k)}{n}]}]; \tag{7.1}$$

$$a_n^{(m)} = -2\sum_{k=0}^{\lambda(n)} \zeta(-2[\frac{\pi_{\overline{m}}(k)}{n}]); \tag{7.2}$$

$$a_n^{(m)} = \sum_{k=0}^{\lambda(n)} \frac{1}{\Gamma(1-[\frac{\pi_{\overline{m}}(k)}{n}])}. \tag{7.3}$$

Thus, the 26-th Smarandache's problem is solved and for $m = 3$ the 25-th Smarandache's problem is solved, too.

The following problems are interesting.

Problem 7.1. Does there exist a constant $C > 1$, such that $\lambda(n) \leq C.n$?

Problem 7.2. Is $C \leq 2$?

Below we give the main explicit representation of function $\pi_{\overline{m}}(n)$, that takes part in formulae (7.1) - (7.3). In this way we find the main explicit representation for $a_n^{(m)}$, that is based on formulae (7.1) - (7.3), too.

Theorem 7.1. Function $\pi_{\overline{m}}(n)$ allows representation

$$\pi_{\overline{m}}(n) = n - 1 + \sum_{s\in\overline{2}\bigcap\{2,3,\ldots,[\sqrt[m]{n}]\}} (-1)^{\omega(s)}.[\frac{n}{s^m}],$$

where $\omega(s)$ denotes the number of all different prime divisors of s.

7. The results in this section are taken from [24].

The 28-th problem from [10] (see also the 94-th problem from [16]) is the following:

Smarandache odd sieve:

$$7, 13, 19, 23, 25, 31, 33, 37, 43, 47, 49, 53, 55, 61, 63, 67, 73, 75, 83,$$

$$85, 91, 93, 97, \ldots$$

(All odd numbers that are not equal to the difference of two primes).

A sieve is to get this sequence:

- subtract 2 from all prime numbers and obtain a temporary sequence;

- choose all odd numbers that do not belong to the temporary one.

We find an explicit form of the n-th term of the above sequence, that will be denoted by $C = \{C_n\}_{n=1}^{\infty}$ below.

Firstly, we shall note that the above definition of C can be interpreted to the following equivalent form as follows, having in mind that every odd number is a difference of two prime numbers if and only if it is a difference of a prime number and 2:

Smarandache's odd sieve contains exactly these odd numbers that cannot be represented as a difference of a prime and 2.

We rewrite the last definition to the following equivalent form, too:

Smarandache's odd sieve contains exactly these odd numbers that are represented as a difference of a composite odd number and 2.

We find an explicit form of the n-th term of the above sequence, using the third definition of it. Initially, we use the following two assertions.

Lemma 8.1. For every natural number $n \geq 1$, C_{n+1} is exactly one of the numbers: $u \equiv C_n + 2, v \equiv C_n + 4$ or $w \equiv C_n + 6$.
Corollary 8.1. For every natural number $n \geq 1$:

$$C_{n+1} \leq C_n + 6.$$

Corollary 8.2. For every natural number $n \geq 1$:

$$C_n \leq 6n + 1. \tag{8.1}$$

Now, we return to the Smarandache's problem.

In [18] the following three universal explicit formulae are introduced, using numbers $\pi_C(k)$ ($k = 0, 1, 2, \ldots$), that can be used to represent numbers C_n:

$$C_n = \sum_{k=0}^{\infty} [\frac{1}{1 + [\frac{\pi_C(k)}{n}]}],$$

$$C_n = -2\sum_{k=0}^{\infty} \zeta(-2[\frac{\pi_C(k)}{n}]),$$

$$C_n = \sum_{k=0}^{\infty} \frac{1}{\Gamma(1 - [\frac{\pi_C(k)}{n}])}.$$

For the present case, having in mind (8.1), we substitute symbol ∞ with $6n + 1$ in sum $\sum_{k=0}^{\infty}$ for C_n and we obtain the following sums:

$$C_n = \sum_{k=0}^{6n+1} [\frac{1}{1 + [\frac{\pi_C(k)}{n}]}], \tag{8.2}$$

$$C_n = -2\sum_{k=0}^{6n+1} \zeta(-2[\frac{\pi_C(k)}{n}]), \tag{8.3}$$

$$C_n = \sum_{k=0}^{6n+1} \frac{1}{\Gamma(1 - [\frac{\pi_C(k)}{n}])}. \tag{8.4}$$

We must explain why $\pi_C(n)$ ($n = 1, 2, 3, \ldots$) is represented in an explicit form. It can be directly seen that the number of the odd numbers, that are not bigger than n, is exactly equal to

$$\alpha(n) = n - [\frac{n}{2}], \tag{8.5}$$

because the number of the even numbers that are not greater than n is exactly equal to $[\frac{n}{2}]$.

Let us denote by $\beta(n)$ the number of all odd numbers not bigger than n, that can be represented as a difference of two primes. According to the second form of the above given definition, $\beta(n)$ coincides with the number of all odd numbers m such that $m \leq n$ and m has the form $m = p - 2$, where p is an odd prime number. Therefore, we must study all odd prime numbers, because of the inequality $m \leq n$. The number of these prime numbers is exactly $\pi(n + 2) - 1$. therefore,

$$\beta(n) = \pi(n + 2) - 1. \tag{8.6}$$

Omitting from the number of all odd numbers that are not greater than n the quantity of those numbers that are a difference of two primes, we find exactly the quantity of these odd numbers that are not greater than n and that are not a difference of two prime numbers, i.e., $\pi_C(n)$. Therefore, the equality

$$\pi_C(n) = \alpha(n) - \beta(n)$$

holds and from (8.5) and (8.6) we obtain:

$$\pi_C(n) = (n - [\frac{n}{2}]) - (\pi(n+2) - 1) = n + 1 - [\frac{n}{2}] - \pi(n+2).$$

But $\pi(n+2)$ can be represented in an explicit form, e.g., by Mináč's formula and therefore, we obtain that the explicit form of $\pi_C(N)$ is

$$\pi_C(N) = N + 1 - [\frac{N}{2}] - \sum_{k=2}^{N+2} [\frac{(k-1)! + 1}{k} - [\frac{(k-1)!}{k}]], \tag{8.7}$$

where $N \geq 1$ is a fixed natural number.

It is possible to put $[\frac{N+3}{2}]$ instead of $N + 1 - [\frac{N}{2}]$ into (8.7).

Now, using each of the formulae (8.2) - (8.4), we obtain C_n in an explicit form, using (8.7).

It can be checked directly that

$$C_1 = 7, C_2 = 13, C_3 = 19, C_4 = 23, C_5 = 25, C_6 = 31,$$

$$C_7 = 33, \ldots$$

and

$$\pi_C(0) = \pi_C(1) = \pi_C(2) = \pi_C(3) = \pi_C(4) = \pi_C(5) = \pi_C(6) = 0.$$

Therefore from (8.2) - (8.4) we have the following explicit formulae for C_n

$$C_n = 7 + \sum_{k=7}^{6n+1} [\frac{1}{1 + [\frac{\pi_C(k)}{n}]}],$$

$$C_n = 7 + -2 \sum_{k=7}^{6n+1} \zeta(-2.[\frac{\pi_C(k)}{n}]),$$

$$C_n = 7 + \sum_{k=7}^{6n+1} \frac{1}{\Gamma(1 - [\frac{\pi_C(k)}{n}])},$$

where $\pi_C(k)$ is given by (8.7).

8. The results in this section are taken from [7, 26].

The 46-th Smarandache's problem from [10] is the following:

Smarandache's prime additive complements;

$$1, 0, 0, 1, 0, 1, 0, 3, 2, 1, 0, 1, 0, 3, 2, 1, 0, 1, 0, 3, 2, 1, 0, 1, 0, 5, 4, 3, 2, 1,$$

$$0, 1, 0, 5, 4, 3, 2, 1, 0, 3, 2, 1, 0, 5, 4, 3, 2, 1, 0 \ldots$$

(For each n to find the smallest k such that $n + k$ is prime.)

Remarks: Smarandache asked if it is possible to get as large as we want but finite decreasing $k, k-1, k-2, \ldots, 2, 1, 0$ (odd k) sequence included in the previous sequence - i.e., for any even integer are there two primes those difference is equal to it? He conjectured the answer is negative.

Obviously, the members of the above sequence are differences between first prime number that is greater or equal to the current natural number n and the same n. It is well-known that the number of primes smaller than or equal to n is $\pi(n)$. Therefore, the prime number smaller than or equal to n is $p_{\pi(n)}$. Hence, the prime number that is greater than or equal to n is the next prime number, i.e., $p_{\pi(n)+1}$. Finally, the n-th member of the above sequence will be equal to

$$\begin{cases} p_{\pi(n)+1} - n, & \text{if } n \text{ is not a prime number} \\ 0, & \text{otherwise} \end{cases}$$

We shall note that in [3] the following new formula p_n for every natural number n is given:

$$p_n = \sum_{i=0}^{\theta(n)} sg(n - \pi(i)),$$

where $\theta(n) = [\frac{n^2+3n+4}{4}]$ and

$$sg(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ 1, & \text{if } x > 0. \end{cases}$$

Let us denote by a_n the n-th term of the above sequence. Next, we propose a way for obtaining an explicit formula for a_n ($n = 1, 2, 3, \ldots$). Extending the below results, we give an answer to the Smarandache's question from his own remark in [10]. At the end, we propose a generalization of Problem 46 and present a proof of an assertion related to Smarandache's conjecture for Problem 46.

Proposition 9.1. a_n admits the representation

$$a_n = p_{\pi(n-1)+1} - n, \tag{9.1}$$

where $n = 1, 2, 3, \ldots$, π is the prime counting function and p_k is the k-th term of prime number sequence.

It is clear that (9.1) gives an explicit representation for a_n since several explicit formulae for $\pi(k)$ and p_k are known (see, e.g. [14]).

Let us define

$$n(m) = m! + 2.$$

Then all numbers

$$n(m), n(m) + 1, n(m) + 2, \ldots, n(m) + m - 2$$

are composite. Hence

$$a_{n(m)} \geq m - 1.$$

This proves the Smarandache's conjecture, since m may grow up to infinity. Therefore $\{a_n\}_{n=1}^{\infty}$ is unbounded sequence.

Now, we shall generalize Problem 46.

Let

$$\mathbf{c} \equiv c_1, c_2, c_3, \ldots$$

be a strictly increasing sequence of positive integers.

Definition. Sequence

$$\mathbf{b} \equiv b_1, b_2, b_3, \ldots$$

is called c-additive complement of $\mathbf{c}$ if and only if b_n is the smallest non-negative integer, such that $n + b_n$ is a term of $\mathbf{c}$.

The following assertion generalizes Proposition 1.

Proposition 9.2. b_n admits the representation

$$b_n = c_{\pi_c(n-1)+1} - n,$$

where $n = 1, 2, 3, \ldots$, $\pi_c(n)$ is the counting function of $\mathbf{c}$, i.e., $\pi_c(n)$ equals to the quantity of c_m, $m = 1, 2, 3, \ldots$, such that $c_m \leq n$.

Let

$$d_n \equiv c_{n+1} - c_n \ (n = 1, 2, 3, \ldots).$$

The following assertion is related to the Smarandache's conjecture from Problem 46.

Proposition 9.3. If $\{d_n\}_{n=1}^{\infty}$ is unbounded sequence, then $\{b_n\}_{n=1}^{\infty}$ is unbounded sequence, too.

Open Problem. Formulate necessary conditions for the sequence $\{b_n\}_{n=1}^{\infty}$ to be unbounded.

9.

The results in this section are taken from [23].

Solving of the Diophantine equation

$$2x^2 - 3y^2 = 5 \tag{10.1}$$

i.e.,

$$2x^2 - 3y^2 - 5 = 0$$

was put as an open Problem 78 by F. Smarandache in [16]. In [28] this problem is solved completely. Also, we consider here the Diophantine equation

$$l^2 - 6m^2 = -5,$$

i.e.,

$$l^2 - 6m^2 + 5 = 0$$

and the Pellian equation

$$u^2 - 6v^2 = 1,$$

i.e.,

$$u^2 - 6v^2 - 1 = 0.$$

In [28] we introduce a generalization of the Smarandache's problem 78 from [16].

If we consider the Diophantine equation

$$2x^2 - 3y^2 = p, \tag{10.2}$$

where $p \neq 2$ is a prime number, then using [13], Chapter VII, exercise 2 and the same method as in the case of (10.1), we obtain the following result.

Theorem 10.1. (1) The necessary and sufficient condition for solvability of (10.2) is:

$$p \equiv 5(\mathrm{mod} 24) \text{ or } p \equiv 23(\mathrm{mod} 24) \tag{10.3}$$

(2) if (10.3) is valid, then there exist exactly one solution $< x, y > \in \mathcal{N}^2$ of (10.2) such that the inequalities

$$x < \sqrt{\frac{3}{2}.p}$$

and

$$y < \sqrt{\frac{3}{2}.p}$$

holds. Every other solution $< x, y > \in \mathcal{N}^2$ of (10.2) has the form:

$$x = l + 3m$$

$$y = l + 2m,$$

where $< l, m > \in \mathcal{N}^2$ is a solution of the Diophantine equation

$$l^2 - 6m^2 = -p.$$

The problem how to solve the Diophantine equation, a special case of which is the above one, is considered in Theorem 110 from [13].

10. The results in this section are taken from [9]. In [15, 17] F. Smarandache formulates the following four problems:

Problem 1. *Let p be an integer ≥ 3. Then:*

p is a prime if and only if

$$(p-3)! \textit{ is congruent to } \frac{p-1}{2}(\mathrm{mod} p).$$

Problem 2. *Let p be an integer ≥ 4. Then:*

p is a prime if and only if

$$(p-4)! \textit{ is congruent to } (-1)^{\lceil \frac{p}{3} \rceil + 1} \lceil \frac{p+1}{6} \rceil (\mathrm{mod} p). \tag{11.1}$$

Problem 3. *Let p be an integer ≥ 5. Then:*

p is a prime if and only if

$$(p-5)! \textit{ is congruent to } \ rh + \frac{r^2-1}{24}(\text{mod}p), \tag{11.2}$$

with $h = \lceil \frac{p}{24} \rceil$ *and* $r = p - 24$.

Problem 4. *Let* $p = (k-1)!h + 1$ *be a positive integer* $k > 5$, h *natural number. Then:*

$$p \textit{ is a prime if and only if}$$

$$(p-k)! \textit{ is congruent to } \ (-1)^t h(\text{mod}p). \tag{11.3}$$

with $t = h + \lceil \frac{p}{h} \rceil + 1$.

Everywhere above $\lceil x \rceil$ means the inferior integer part of x, i.e., the smallest integer greater than or equal to x.

In [28] we discussed these four problems.

Problem 1. Admits the following representation:
Let $p \geq 3$ *be an odd number. Then:*

$$p \textit{ is a prime if and only if } (p-3)! \equiv \frac{p-1}{2}(\textit{mod}p).$$

Different than Smarandache's proof of this assertion is given in [28].

Problem 2. Is false, because, for example, if $p = 7$, then (11.1) obtains the form

$$6 \equiv (-1)^4 2(\text{mod}7),$$

where

$$6 = (7-4)!$$

and

$$(-1)^4 2 = (-1)^{\lceil \frac{7}{3} \rceil + 1} \lceil \frac{8}{6} \rceil,$$

i.e.,

$$6 \equiv 2(\text{mod}7),$$

which is impossible.

Problem 3. Can be modified, having in mind that from $r = p - 24h$ it follows:

$$\begin{aligned} rh + \frac{r^2-1}{24} &= (p-24h).h + \frac{p^2 - 48ph + 24^2h^2 - 1}{24} \\ = \quad ph - 24h^2 + \frac{p^2-1}{24} &- 2ph + 24h^2 = \frac{p^2-1}{24} - ph, \end{aligned}$$

i.e., (11.2) has the form

$$p \textit{ is a prime if and only if}$$

$$(p-5)! \textit{ is congruent to } \ \frac{p^2-1}{24}(\text{mod}p),$$

Different than the Smarandache's proof of this assertion is given in [28].

Problem 4. Also can be simplified, because

$$t = h + \lceil \frac{p}{h} \rceil + 1$$

$$= h + \lceil \frac{(k-1)!h+1}{h} \rceil + 1$$

$$= h + (k-1)! + 1 + 1 = h + (k-1)! + 2,$$

i.e.,

$$(-1)^t = (-1)^h,$$

because for $k > 2$: $(k-1)! + 2$ is an even number. Therefore, (11.3) obtains the form

p is a prime if and only if

$$(p-k)! \textit{ is congruent to } (-1)^h h (\mathrm{mod} p),$$

Let us assume that (11.4) is valid. We use again the congruences

$$(p-1) \equiv -1 (\mathrm{mod} p)$$

$$(p-2) \equiv -2 (\mathrm{mod} p)$$

$$\dots$$

$$(p-(k-1)) \equiv -(k-1)(\mathrm{mod} p)$$

and obtain the next form of (11.4)

p is a prime if and only if

$$(p-1)! \equiv (-1)^h.(-1)^{k-1}.(k-1)!.h(\mathrm{mod} p)$$

or

p is a prime if and only if

$$(p-1)! \equiv (-1)^{h+k-1}.(p-1)(\mathrm{mod} p).$$

But the last congruence is not valid, because, e.g., for $k = 5, h = 3, p = 73 = (5-1)! + 1$[1] holds

$$72! \equiv (-1)^9.72(\mathrm{mod} 73),$$ [2]

i.e.,

$$72! \equiv 1(\mathrm{mod} 73),$$

[1] In [28] there is a misprint: 3! instead of 3.

[2] In [28] there is a misprint: $(-1)^9$ instead of $(-1)^7$.

while from Wilson's Theorem follows that

$$72! \equiv -1(\text{mod}73).$$

11. The results in this section are taken from [5].

In [17] F. Smarandache discussed the following particular cases of the well-known characteristic functions (see, e.g., [11, 30]).

12.1) Prime function: $P : N \to \{0, 1\}$, with

$$P(n) = \begin{cases} 0, & \text{if } n \text{ is a prime} \\ 1, & \text{otherwise} \end{cases}$$

More generally: $P_k : N^k \to \{0, 1\}$, where $k \geq 2$ is an integer, and

$$P_k(n_1, n_2, \ldots, n_k) = \begin{cases} 0, & \text{if } n_1, n_2, \ldots, n_k \text{ are all prime numbers} \\ 1, & \text{otherwise} \end{cases}$$

12.2) Coprime function is defined similarly: $C_k : N^k \to \{0, 1\}$, where $k \geq 2$ is an integer, and

$$C_k(n_1, n_2, \ldots, n_k) = \begin{cases} 0, & \text{if } n_1, n_2, \ldots, n_k \text{ are coprime numbers} \\ 1, & \text{otherwise} \end{cases}$$

In [28] we formulate and prove four assertions related to these functions.

Proposition 12.1. For each $k, n_1, n_2, \ldots, n_k$ natural numbers:

$$P_k(n_1, n_2, \ldots, n_k) = 1 - \prod_{i=1}^{k}(1 - P(n_i)).$$

Proposition 12.2. For each $k, n_1, n_2, \ldots, n_k$ natural numbers:

$$C_k(n_1, n_2, \ldots, n_k) = 1 - \prod_{i=1}^{k} \prod_{j=i+1}^{k} (1 - C_2(n_i, n_j)).$$

Proposition 12.3. For each natural number n:

$$C_{\pi(n)+P(n)}(p_1, p_2, \ldots, p_{\pi(n)+P(n)-1}, n) = P(n).$$

Proposition 12.4. For each natural number n:

$$P(n) = 1 - \prod_{i=1}^{\pi(n)+P(n)-1} (1 - C_2(p_i, n)).$$

Corollary 12.1. For each natural number $k, n_1, n_2, \ldots, n_k$:

$$P_k(n_1, n_2, \ldots, n_k) = 1 - \prod_{i=1}^{k} \prod_{j=1}^{\pi(n_i)+P(n_i)-1} (1 - C_2(p_j, n_i)).$$

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