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On the Roman Edge Domination Number of a Graph

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Abstract: For an integer $n \geq 2$, let $I \subset \{0, 1, 2, \cdots, n\}$. A *Smarandachely Roman s-dominating function* for an integer s, $2 \leq s \leq n$ on a graph G = (V,E) is a function $f : V \rightarrow \{0, 1, 2, \cdots, n\}$ satisfying the condition that $|f(u) - f(v)| \geq s$ for each edge $uv \in E$ with $f(u)$ or $f(v) \in I$. Similarly, a *Smarandachely Roman edge s-dominating function* for an integer s, $2 \leq s \leq n$ on a graph $G = (V, E)$ is a function $f : E \rightarrow \{0, 1, 2, \cdots, n\}$ satisfying the condition that $|f(e) - f(h)| \geq s$ for adjacent edges $e, h \in E$ with $f(e)$ or $f(h) \in I$. Particularly, if we choose $n = s = 2$ and $I = \{0\}$, such a Smarandachely Roman s-dominating function or Smarandachely Roman edge s-dominating function is called *Roman dominating function* or *Roman edge dominating function*. The Roman edge domination number $\gamma_{re}(G)$ of G is the minimum of $f(E) = \sum_{e \in E} f(e)$ over such functions. In this paper we first show that for any connected graph G of $q \geq 3$, $\gamma_{re}(G) + \gamma_e(G)/2 \leq q$ and $\gamma_{re}(G) \leq 4q/5$, where $\gamma_e(G)$ is the edge domination number of G. Also we prove that for any $\gamma_{re}(G)$-function $f = \{E_0, E_1, E_2\}$ of a connected graph G of $q \geq 3$, $|E_0| \geq q/5 + 1$, $|E_1| \leq 4q/5 - 2$ and $|E_2| \leq 2q/5$.

Key Words: Smarandachely Roman s-dominating function, Smarandachely Roman edge s-dominating function.

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§1. Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. As usual $|V| = p$ and $|E| = q$ denote the number of vertices and edges of the graph G, respectively. The open neighborhood $N(e)$ of the edge e is the set of all edges adjacent to e in G. And its closed neighborhood is $N[e] = N(e) \cup \{e\}$. Similarly, the open neighborhood of a set $S \subseteq E$ is the set $N(S) = \bigcup_{e \in S} N(e)$, and its closed neighborhood is $N[S] = N(S) \cup S$.

The degree of an edge $e = uv$ of G is defined by $deg\, e = deg\, u + deg\, v - 2$ and $\delta'(G)$ ($\Delta'(G)$) is the minimum (maximum) degree among the edges of G (the degree of an edge is the number of edges adjacent to it). A vertex of degree one is called a pendant vertex or a leaf and its neighbor is called a support vertex.

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Let $e \in S \subseteq E$. Edge h is called a private neighbor of e with respect to S (denoted by h is an S-pn of e) if $h \in N[e] - N[S - \{e\}]$. An S-pn of e is external if it is an edge of $E - S$. The set $pn(e, S) = N[e] - N[S - \{e\}]$ of all S-pn's of e is called the private neighborhood set of e with respect to S. The set S is said to be irredundant if for every $e \in S$, $pn(e, S) \neq \varnothing$. And a set S of edges is called independent if no two edges in S are adjacent.

A set $D \subseteq V$ is said to be a dominating set of G, if every vertex in $V - D$ is adjacent to some vertex in D. The minimum cardinality of such a set is called the domination number of G and is denoted by $\gamma(G)$. For a complete review on the topic of domination and its related parameters, see [5].

Mitchell and Hedetniemi in [6] introduced the notion of edge domination as follows. A set F of edges in a graph G is an edge dominating set if every edge in $E - F$ is adjacent to at least one edge in F. The minimum number of edges in such a set is called the edge domination number of G and is denoted by $\gamma_e(G)$. This concept is also studied in [1].

The concept of Roman dominating function (RDF) was introduced by E. J. Cockayne, P. A. Dreyer, S. M. Hedetniemi and S. T. Hedetniemi in [3]. (See also [2,4,8]). A Roman dominating function on a graph $G = (V, E)$ is a function $f : V \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of a Roman dominating function is the value $f(V) = \sum_{u \in V} f(u)$. The Roman domination number of a graph G, denoted by $\gamma_R(G)$, equals the minimum weight of a Roman dominating function on G.

A Roman edge dominating function (REDF) on a graph $G = (V, E)$ is a function $f : E \to \{0, 1, 2\}$ satisfying the condition that every edge e for which $f(e) = 0$ is adjacent to at least one edge h for which $f(h) = 2$. The weight of a Roman edge dominating function is the value $f(E) = \sum_{e \in E} f(e)$. The Roman edge domination number of a graph G, denoted by $\gamma_{re}(G)$, equals the minimum weight of a Roman edge dominating function on G. A Roman edge dominating function $f : E \to \{0, 1, 2\}$ can be represented by the ordered partition (E_0, E_1, E_2) of E, where $E_i = \{e \in E \mid f(e) = i\}$ and $|E_i| = q_i$ for $i = 0, 1, 2$. This concept is studied in Soner et al. in [9] (see also [7]). A $\gamma - set$, $\gamma_r - set$ and γ_{re}-set, can be defined as a minimum dominating set (MDS), a minimum Roman dominating set (MRDS) and a minimum Roman edge dominating set (MREDS), respectively.

Theorem A. *For a graph G of order p,*

$$\gamma_e(G) \leq \gamma_{re}(G) \leq 2\gamma_e(G).$$

It is clear that if G has at least one edge then $1 \leq \gamma_{re}(G) \leq q$, where q is the number of edges in G. However if a graph is totally disconnected or trivial, we define $\gamma_{re}(G) = 0$. We note that $E(G)$ is the unique maximum REDS of G. Since every edge dominating set in G is a dominating set in the line graph of G and an independent set of edges of G is an independent set of vertices in the line graph of G, the following results can easily be proved from the well-known analogous results for dominating sets of vertices and independent sets.

Proposition 1. *A Roman edge dominating set S is minimal if and only if for each $e \in S$, one of the following two conditions holds.*

(i) $N(e) \cap S = \varnothing$.

(ii) *There exists an edge* $h \in E - S$, *such that* $N(h) \cap S = \{e\}$.

Proposition 2. *Let* $S = E_1 \cup E_2$ *be a REDS such that* $|E_1| + 2|E_2| = \gamma_{re}(G)$. *Then*

$$|E(G) - S| \leq \sum_{e \in S} deg(e),$$

and the equality holds if and only if S *is independent and for every* $e \in E - S$ *there exists only one edge* $h \in S$ *such that* $N(e) \cap S = \{h\}$.

Proof Since every edge in $E(G) - S$ is adjacent to at least one edge of S, each edge in $E(G) - S$ contributes at least one to the sum of the degrees of the edges of S, hence

$$|E(G) - S| \leq \textstyle\sum_{e \in S} deg(e)$$

Let $|E(G) - S| = \sum_{e \in S} deg(e)$. Suppose S is not independent. Since S is a REDS, every edge in $E - S$ is counted in the sum $\sum_{e \in S} deg(e)$. Hence if e_1 and e_2 have a common point in S, then e_1 is counted in $deg(e_2)$ and vice versa. Then the sum exceeds $|E - S|$ by at least two, contrary to the hypothesis. Hence S must be independent.

Now suppose $N(e) \cap S = \varnothing$ or $|N(e) \cap S| \geq 2$ for $e \in E - S$. Since S is a REDS the former case does not occur. Let e_1 and e_2 belong to $N(e) \cap S$. In this case $\sum_{e \in S} deg(e)$ exceeds $|E(G) - S|$ by at least one since e_1 is counted twice: once in $deg(e_1)$ and once in $deg(e_2)$, a contradiction. Hence equality holds if S is independent and for every $e \in E - S$ there exists only one edge $h \in S$ such that $N(e) \cap S = \{h\}$. Conversely, if S is independent and for every $e \in E - S$ there exists only one edge $h \in S$ such that $N(e) \cap S = \{h\}$, then equality holds. □

Proposition 3. *Let* G *be a graph and* $S = E_1 \cup E_2$ *be a minimum REDS of* G *such that* $|S| = 1$, *then the following condition hold.*

(i) S *is independent.*

(ii) $|E - S| = \sum_{e \in S} deg(e)$.

(iii) $\Delta'(G) = q - 1$.

(iv) $q/(\Delta' + 1) = 1$.

An immediate consequence of the above result is.

Corollary 1 *For any* (p, q) *graph,* $\gamma_{re}(G) = p - q + 1$ *if and only if* G *has* γ_{re} *components each of which is isomorphic to a star.*

Proposition 4. *Let* G *be a graph of* q *edges which contains a edge of degree* $q - 1$, *then* $\gamma_e(G) = 1$ *and* $\gamma_{re}(G) = 2$.

Proposition 5.([9]) *Let* $f = (E_0, E_1, E_2)$ *be any REDF. Then*

(i) $\langle E_1 \rangle$ *has maximum degree one.*

(ii) *Each edge of* E_0 *is adjacent to at most two edges of* E_1.

(iii) E_2 *is an* γ_e*-set of* $H = G[E_0 \cup E_2]$.

Proposition 6. *Let $f = (E_0, E_1, E_2)$ be any γ_{re}-function. Then*

(i) *No any edge of E_1 is adjacent to any edge of E_2.*

(ii) *Let $H = G[E_0 \cup E_2]$. Then each edge $e \in E_2$ has at least two H-pn's (i.e private neighbors relative to E_2 in the graph H).*

(iii) *If e is isolated in $G[E_2]$ and has precisely one external H-pn, say $h \in E_0$, then $N(h) \cap E_1 = \varnothing$.*

Proof (i) Let $e_1, e_2 \in E$, where e_1 adjacent to e_2, $f(e_1) = 1$ and $f(e_2) = 2$. Form f' by changing $f(e_1)$ to 0. Then f' is a REDF with $f'(E) < f(E)$, a contradiction.

(ii) By Proposition 5(iii), E_2 is an γ_e-set of H and hence is a maximal irredundant set in H. Therefore, each $e \in E_2$ has at least one E_2-pn in H.

Let e be isolated in $G[E_2]$. Then e is a E_2-pn of e. Suppose that e has no external E_2-pn. Then the function produced by changing $f(e)$ from 2 to 1 is an REDF of smaller weight, a contradiction. Hence, e has at least two E_2-pns in H.

Suppose that e is not isolated in $G[E_2]$ and has precisely one E_2-pn (in H), say w. Consider the function produced by changing $f(e)$ to 0 and $f(h)$ to 1. The edge e is still dominated because it has a neighbor in E_2. All of e's neighbors in E_0 are also obtained, since every edge in E_0 has another neighbor in E_2 except for h, which is now in E_1. Therefore, this new function is an REDF of smaller weight, which is a contradiction. Again, we can conclude that e has at least two E_2-pns in H.

(iii) Suppose the contrary. Define a new function f' with $f'(e) = 0$, $f'(e') = 0$ for $e' \in N(h) \cap E_1$, $f'(h) = 2$, and $f'(x) = f(x)$ for all other edges x. $f'(E) = f(E) - |N(h) \cap E_1| < f(E)$, contradicting the minimality of f. □

Proposition 7. *Let $f = (E_0, E_1, E_2)$ be a γ_{re}-function of an isolate-free graph G, such that $|E_2| = q_2$ is a maximum. Then*

(i) *E_1 is independent.*

(ii) *The set E_0 dominates the set E_1.*

(iii) *Each edge of E_0 is adjacent to at most one edge of E_1.*

(iv) *Let $e \in G[E_2]$ have exactly two external H-pn's e_1 and e_2 in E_0. Then there do not exist edges $h_1, h_2 \in E_1$ such that (h_1, e_1, e, e_2, h_2) is the edge sequence of a path P_6.*

Proof (i) By Proposition 5(i), $G[E_1]$ consists of disjoint K_2's and P_3's. If there exists a P_3, then we can change the function values of its edges to 0 and 2. The resulting function $g = (W_0, W_1, W_2)$ is a γ_{re}-function with $|W_2| > |E_2|$, which is a contradiction. Therefore, E_1 is an independent set.

(ii) By (i) and Proposition 6(i), no edge $e \in E_1$ is adjacent to an edge in $E_1 \cup E_2$. Since G is isolate-free, e is adjacent to some edge in E_0. Hence the set E_0 dominates the set E_1.

(iii) Let $e \in E_0$ and $B = N(e) \cap E_1$, where $|B| = 2$. Note that $|B| \leq 2$, by Proposition 5(ii). Let

$$W_0 = (E_0 \cup B) - \{e\},$$
$$W_1 = E_1 - B,$$

$W_2 = E_2 \cup \{e\}$.

We know that E_2 dominates E_0, so that $g = (W_0, W_1, W_2)$ is an REDF.

$g(E) = |W_1| + 2|W_2| = |E_1| - B + 2|E_2| - 2 = f(E)$. Hence, g is a γ_{re}-function with $|W_2| > |E_2|$, which is a contradiction.

iv) Suppose the contrary. Form a new function by changing the function values of (h_1, e_1, e, e_2, h_2) from $(1, 0, 2, 0, 1)$ to $(0, 2, 0, 0, 2)$. Then the new function is a γ_{re}-function with bigger value of q_2, which is a contradiction. □

§2. Graph for Which $\gamma_{re}(G) = 2\gamma_e(G)$

From Theorem A we know that for any graph G, $\gamma_{re}(G) \leq 2\gamma_e(G)$. We will say that a graph G is a Roman edge graph if $\gamma_{re}(G) = 2\gamma_e(G)$.

Proposition 8. *A graph G is Roman edge graph if and only if it has a γ_{re}-function $f = (E_0, E_1, E_2)$ with $q_1 = |E_1| = 0$.*

Proof Let G be a Roman edge graph and let $f = (E_0, E_1, E_2)$ be a γ_{re}-function of G. Proposition 5(iii) we know that E_2 dominates E_0, and $E_1 \cup E_2$ dominates E, and hence

$$\gamma_e(G) \leq |E_1 \cup E_2| = |E_1| + |E_2| \leq |E_1| + 2|E_2| = \gamma_{re}(G).$$

But since G is Roman edge, we know that

$$2\gamma_e(G) = 2|E_1| + 2|E_2| = \gamma_{re}(G) = |E_1| + 2|E_2|.$$

Hence, $q_1 = |E_1| = 0$.

Conversely, let $f = (E_0, E_1, E_2)$ be a γ_{re}-function of G with $q_1 = |E_1| = 0$. Then, $\gamma_{re}(G) = 2|E_2|$, and since by definition $E_1 \cup E_2$ dominates E, it follows that E_2 is a dominating set of G. But by Proposition 5(iii), we know that E_2 is a γ_e-set of $G[E_0 \cup E_2]$, i.e. $\gamma_e(G) = |E_2|$ and $\gamma_{re}(G) = 2\gamma_e(G)$, i.e. G is a Roman edge graph. □

§3. Bound on the Sum $\gamma_{re}(G) + \gamma_e(G)/2$

For q-edge graphs, always $\gamma_{re}(G) \leq q$, with equality when G is isomorphic with mK_2 or mP_3. In this section we prove that $\gamma_{re}(G) + \gamma_e(G)/2 \leq q$ and $\gamma_{re}(G) \leq 4q/5$ when G is a connected q-edge graph.

Theorem 9. *For any connected graph G of $q \geq 3$,*

(*i*) $\gamma_{re}(G) + \gamma_e(G)/2 \leq q$.
(*ii*) $\gamma_{re}(G) \leq 4q/5$.

Proof Let $f = (E_0, E_1, E_2)$ be a $\gamma_{re}(G)$-function such that $|E_2|$ is maximum. It is proved in Proposition 6(i) that for such a function no edge of E_1 is adjacent to any edge of E_2 and every edge e of E_2 has at least two E_2-private neighbors, one of them can be e itself if it is isolated in

E_2 (true for every $\gamma_{re}(G)$-function). The set E_1 is independent and every edge of E_0 has at most one neighbor in E_1. Moreover we add the condition the number $\mu(f)$ of edges of E_2 with only one neighbor in E_0 is minimum. Suppose that $N_{E_0}(e) = \{h\}$ for some $e \in E_2$. Then partition $E'_0 = (E_0 \backslash \{h\}) \cup \{e\} \cup N_{E_1}(h)$, $E'_1 = E_1 \backslash N_{E_1}(h)$ and $E'_2 = (E_2 \backslash \{e\}) \cup \{h\}$ is a Roman edge dominating function f' such that $w(f') = w(f) - 1$ if $N_{E_1}(h) \neq \varnothing$, or $w(f') = w(f)$, $|E'_2| = |E_2|$ but $\mu(f') < \mu(f)$ if $N_{E_1}(h) = \varnothing$ since then, G being connected $q \geq 3$, h is not isolated in E_0. Therefore every edge of E_2 has at least two neighbors in E_0. Let A be a largest subset of E_2 such that for each $e \in A$ there exists a subset A_e of $N_{E_0}(e)$ such that the set A_e is disjoint, $|A_e| \geq 2$ and sets $\cup_{e\in A} A_e = \cup_{e \in A} N_{E_0}(e)$. Note that A_e contains all the external E_2-private neighbors of e. $A' = E_2 \backslash A$.

Case 1 $A' = \varnothing$.

In this case $|E_0| \geq 2|E_2|$ and $|E_1| \leq |E_0|$ since every edge of E_0 has at most one neighbor in E_1. Since E_0 is an edge dominating set of G and $|E_0|/2 \geq |E_2|$ we have

(i) $\gamma_{re}(G) + \gamma_e(G)/2 \leq |E_1| + 2|E_2| + |E_0|/2 \leq |E_0| + |E_1| + |E_2| = q$.

(ii) $5\gamma_{re}(G) = 5|E_1| + 10|E_2| = 4q - 4|E_0| + |E_1| + 6|E_2| = 4q - 3(|E_0| - 2|E_2|) - (|E_0| - |E_1|) \leq 4q$. Hence $\gamma_{re}(G) \leq 4q/5$.

Case 2 $A' \neq \varnothing$.

Let $B = \cup_{e\in A} A_e$ and $B' = E_0 \backslash B$. Every edge ε in A' has exactly one E_2-private neighbor ε' in E_0 and $N_{B'}(\varepsilon) = \{\varepsilon'\}$ for otherwise ε could be added to A. This shows that $|A'| = |B'|$. Moreover since $|N_{E_0}(\varepsilon)| \geq 2$, each edge $\varepsilon \in A'$ has at least one neighbor in B. Let $\varepsilon_B \in B \cap N_{E_0}(\varepsilon)$ and let ε_A be the edge of A such that $\varepsilon_B \in A_{\varepsilon_A}$. The edge ε_A is well defined since the sets A_e with $e \in A$ form a partition of B.

Claim 1 $|A_{\varepsilon_A}| = 2$ *for each* $\varepsilon \in A'$ *and each* $\varepsilon_B \in B \cap N_{E_0}(\varepsilon)$.

Proof of Claim 1 If $|A_{\varepsilon_A}| > 2$, then by putting $A'_{\varepsilon_A} = A_{\varepsilon_A} \backslash \{\varepsilon_B\}$ and $A_\varepsilon = \{\varepsilon', \varepsilon_B\}$ we can see that $A_1 = A \cup \{\varepsilon\}$ contradicts the choice of A. Hence $|A_{\varepsilon_A}| = 2$, ε_A has a unique external E_2-private neighbor ε'_A and $A_{\varepsilon_A} = \{\varepsilon_B, \varepsilon'_A\}$. Note that the edges ε_A and ε are isolated in E_2 since they must have a second E_2-private neighbor.

Claim 2 *If* $\varepsilon, y \in A'$ *then* $\varepsilon_B \neq y_B$ *and* $A_{\varepsilon_A} \neq A_{y_A}$.

Proof of Claim 2 Let ε' and y' be respectively the unique external E_2-private neighbors of ε and y. Suppose that $\varepsilon_B = y_B$, and thus $\varepsilon_A = y_A$. The function $g : E(G) \to \{0, 1, 2\}$ defined by $g(\varepsilon_B) = 2$, $g(\varepsilon) = g(y) = g(\varepsilon_A) = 0$, $g(\varepsilon'_A) = g(y') = g(\varepsilon') = 1$ and $g(e) = f(e)$ otherwise, is a REDF of G of weight less than $\gamma_{re}(G)$, a contradiction. Hence $\varepsilon_B \neq y_B$. Since $A_{\varepsilon_A} \supseteq \{\varepsilon_B, \varepsilon'_A\}$ and $|A_{\varepsilon_A}| = 2$, the edge y_B is not in A_{ε_A}. Therefore $A_{\varepsilon_A} \neq A_{y_A}$.

Let $A'' = \{\varepsilon_A \mid \varepsilon \in A' \textit{ and } \varepsilon_B \in B \cap N_{E_0}(\varepsilon)\}$ and $B'' = \cup_{e \in A''} A_e$. By Claims 1 and 2,

$$|B''| + 2|A''| \quad \text{and} \quad |A''| \geq |A'|.$$

Let $A''' = E_2 \backslash (A' \cup A'')$ and $B''' = \cup_{e \in A'''} A_e = E_0 \backslash (B' \cup B'')$. By the definition of the sets A_e,

$$|B'''| \geq |2A'''|.$$

Claim 3 *If $\varepsilon \in A'$ and $\varepsilon_B \in B \cap N_{E_0}(\varepsilon)$, then $\varepsilon', \varepsilon_B$ and ε'_A have no neighbor in E_1. Hence B''' dominates E_1.*

Proof of Claim 3 Let h be a edge of E_1. If h has a neighbor in $B' \cup B''$, Let $g : E(G) \to \{0, 1, 2\}$ be defined by $g(\varepsilon'_A) = 2$, $g(h) = g(\varepsilon_A) = 0$, $g(e) = f(e)$ otherwise if h is adjacent to ε'_A, $g(\varepsilon') = 2$, $g(h) = g(\varepsilon) = 0$, $g(e) = f(e)$ otherwise if h is adjacent to ε',

$g(\varepsilon_B) = 2$, $g(h) = g(\varepsilon_A) = g(\varepsilon) = 0$, $g(\varepsilon'_A) = g(\varepsilon') = 1$, $g(e) = f(e)$ otherwise if h is adjacent to ε_B. In each case, g is a REDF of weight less than $\gamma_{re}(G)$, a contradiction. Therefore $N(h) \subseteq B'''$.

We are now ready to establish the two parts of the Theorem.

(i) By Claim 3, $B''' \cup A' \cup A''$ is an edge dominating set of G. Therefore, since $|A'| = |B'|$ and $|B'''| \geq |2A'''|$ we have,

$$\gamma_e(G) \leq |B'''| + |A'| + |A''| \leq |B'''| + |B''| \leq (2|B'''| - 2|A'''|) + (2|B''| - 2|A''|) + (2|B'| - 2|A'|).$$

Hence $\gamma_e(G) \leq 2|E_0| - 2|E_2|$ and $\gamma_{re}(G) + \gamma_e(G)/2 \leq (|E_1| + 2|E_2|) + (|E_0| - |E_2|) = q$.

(ii) By Claim 3 and since each edge of E_1 has at most one neighbor in E_0 and $|E_1| \leq |B'''|$. Using this inequality and since $|A'| = |B'|$ and $|B'''| \geq |2A'''|$ we get

$$\begin{aligned} 5\gamma_{re}(G) &= 5|E_1| + 10|E_2| = 4q - 4|E_0| + |E_1| + 6|E_2| \leq 4q - 4|B'| - 4|B''| - 4|B'''| \\ &\quad + |B'''| + 6|A'| + 6|A''| + 6|A'''| \leq 4q + 2(|A'| - |A''|) + 3(2|A'''| - |B'''|) \leq 4q. \end{aligned}$$

Hence $\gamma_{re}(G) \leq 4q/5$. □

Corollary 10 *Let $f = (E_0, E_1, E_2)$ be a $\gamma_{re}(G) - function$ of a connected graph G. If $k|E_2| \leq |E_0|$ such that $k \geq 4$, then $\gamma_{re}(G) \leq (k - 1)q/k$.*

§4. Bounds on $|E_0|$, $|E_1|$ and $|E_2|$ for a $\gamma_{re}(G)$-Function (E_0, E_1, E_2)

Theorem 11. *Let $f = (E_0, E_1, E_2)$ be any $\gamma_{re}(G) - function$ of a connected graph G of $q \geq 3$. Then*

(1) $1 \leq |E_2| \leq 2q/5$;
(2) $0 \leq |E_1| \leq 4q/5 - 2$;
(3) $q/5 + 1 \leq |E_0| \leq q - 1$.

Proof By Theorem 9, $|E_1| + 2|E_2| \leq 4q/5$.

(1) If $E_2 = \varnothing$, then $E_1 = q$ and $E_0 = \varnothing$. The REDF $(0, q, 0)$ is not minimum since $|E_1| + 2|E_2| > 4q/5$. Hence $|E_2| \geq 1$. On the other hand, $|E_2| \leq 2q/5 - |E_1|/2 \leq 2q/5$.

(2) Since $|E_2| \geq 1$, then $|E_1| \leq 4q/5 - 2|E_2| \leq 4q/5 - 2$.

(3) The upper bound comes from $|E_0| \leq q - |E_2| \leq q - 1$. For the lower bound, adding on side by side $2|E_0| + 2|E_1| + 2|E_2| = 2q, -|E_1| - 2|E_2| \geq -4q/5$ and $-|E_1| \geq -4q/5 + 2$ gives $2|E_0| \geq 2q/5 + 2$. Therefor, $|E_0| \geq q/5 + 1$. □

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