Smarandache Continued Fractions

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Abstract: The theory of general continued fractions is developed to the extent required in order to calculate Smarandache continued fractions to a given number of decimal places. Proof is given for the fact that Smarandache general continued fractions built with positive integer Smarandache sequences having only a finite number of terms equal to 1 is convergent. A few numerical results are given.

Introduction

The definitions of Smarandache continued fractions were given by Jose Castillo in the Smarandache Notions Journal, Vol. 9, No 1-2 [1].

A Smarandache Simple Continued Fraction is a fraction of the form:

$$a(1)+\cfrac{1}{a(2)+\cfrac{1}{a(3)+\cfrac{1}{a(4)+\cfrac{1}{a(5)+\cdots}}}}$$

where a(n), for n≥1, is a Smarandache type Sequence, Sub-Sequence or Function.

Particular attention is given to the Smarandache General Continued Fraction defined as

$$a(1)+\cfrac{b(1)}{a(2)+\cfrac{b(2)}{a(3)+\cfrac{b(3)}{a(4)+\cfrac{b(4)}{a(5)+\cdots}}}}$$

where a(n) and b(n), for n≥1, are Smarandache type Sequences, Sub-Sequences or Functions.

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As a particular case the following example is quoted

$$1+\cfrac{1}{12+\cfrac{21}{123+\cfrac{321}{1234+\cfrac{4321}{12345+\cdots}}}}$$

Here 1, 12, 123, 1234, 12345, ... is the Smarandache Consecutive Sequences and 1, 21, 321, 4321, 54321, ... is the Smarandache Reverse Sequence.

The interest in Castillo's article is focused on the calculation of such fractions and their possible convergens when the number of terms approaches infinity. The theory of simple continued fractions is well known and given in most standard textbooks in Number Theory. A very comprehensive theory of continued fractions, including general continued fractions is found in *Die Lehre von den Kettenbrüchen* [2]. The symbols used to express facts about continued fractions vary a great deal. The symbols which will be used in this article correspond to those used in Hardy and Wright *An Introduction to the Theory of Numbers* [3]. However, only simple continued fractions are treated in the text of Hardy and Wright. Following more or less the same lines the theory of general continued fractions will be developed in the next section as far as needed to provide the necessary tools for calculating Smarandache general continued fractions.

General Continued Fractions

We define a finite general continued fraction through

$$C_n = q_0 + \cfrac{r_1}{q_1 + \cfrac{r_2}{q_2 + \cfrac{r_3}{q_3 + \cfrac{r_4}{q_4 + \ldots}}}} = q_0 + \frac{r_1}{q_1 +}\,\frac{r_2}{q_2 +}\,\frac{r_3}{q_3 +}\,\frac{r_3}{q_4 +}\,\cdots\,\frac{r_n}{q_n} \qquad (1)$$

where $\{q_0, q_1, q_2, \ldots q_n\}$ and $\{r_1, r_2, r_3 \ldots r_n\}$ are integers which we will assume to be positive.

The above definition is an extension of the definition of a simple continued fraction where $r_1=r_2= \ldots =r_n=1$. The theory developed here will apply to simple continued fractions as well by replacing r_k ($k=1, 2, \ldots$) in formulas by 1 and simply ignoring the reference to r_k when not relevant.

The formula (1) will usually be expressed in the form

$$C_n=[q_0,q_1,q_2,q_3, \ldots q_n,r_1,r_2,r_3 \ldots r_n] \qquad (2)$$

For a simple continued fraction we would write

$$C_n=[q_0,q_1,q_2,q_3, \dots q_n] \tag{2'}$$

If we break off the calculation for m≤n we will write

$$C_m=[q_0,q_1,q_2,q_3, \dots q_m,r_1,r_2,r_3 \dots r_m] \tag{3}$$

Equation (3) defines a sequence of finite general continued fractions for m=1, m=2, m=3, … . Each member of this sequence is called a **convergent** to the continued fraction

Working out the general continued fraction in stages, we shall obviously obtain expressions for its convergents as quotients of two sums, each sum comprising various products formed with q_0, q_1, q_2, $\dots q_m$ and r_1, r_2, $\dots r_m$.

If m=1, we obtain the first convergent

$$C_1=[q_0,q_1,r_1]=q_0+\frac{r_1}{q_1}=\frac{q_0q_1+r_1}{q_1} \tag{4}$$

For m=2 we have

$$C_2=[q_0,q_1,q_2,r_1,r_2]=q_0+\frac{q_2r_1}{q_1q_2+r_2}=\frac{q_0q_1q_2+q_0r_2+q_2r_1}{q_1q_2+r_2} \tag{5}$$

In the intermediate step the value of $q_1+\frac{r_2}{q_2}$ from the previous calculation has been quoted, putting q_1, q_2 and r_2 in place of q_0, q_1 and r_1. We can express this by

$$C_2=[q_0,[q_1,q_2,r_2],r_1] \tag{6}$$

Proceeding in the same way we obtain for m=3

$$C_3=[q_0,q_1,q_2,q_3,r_1,r_2,r_3]=q_0+\frac{(q_2q_3+r_3)r_1}{q_1q_2q_3+q_1r_3+q_3r_2}=$$

$$\frac{q_0q_1q_2q_3+q_0q_1r_3+q_0q_3r_2+q_2q_3r_1+r_1r_3}{q_1q_2q_3+q_1r_3+q_3r_2} \tag{7}$$

or generally

$$C_m=[q_0,q_1,\dots q_{m-2},[q_{m-1},q_m,r_m],r_1,r_2,\dots r_{m-1}] \tag{8}$$

which we can extend to

$$C_n=[q_0,q_1,\dots q_{m-2},[q_{m-1},q_m,\dots q_n,r_m,\dots r_n],r_2,r_2,\dots r_{m-1}] \tag{9}$$

Theorem 1:

Let A_m and B_m be defined through

$$A_0=q_0,\quad A_1=q_0q_1+r_1,\qquad A_m=q_mA_{m-1}+r_mA_{m-2}\ \ (2\le m\le n)$$

$$B_0=1,\quad B_1=q_1,\qquad B_m=q_mB_{m-1}+r_mB_{m-2}\ \ (2\le m\le n) \tag{10}$$

then $C_m=[q_0,q_1,\ldots q_m,r_1,\ldots r_m] = \frac{A_m}{B_m}$, i.e. $\frac{A_m}{B_m}$ is the m*th* convergent to the general continued fraction.

Proof: The theorem is true for m=0 and m=1as is seen from $[q_0]=\frac{q_0}{1}=\frac{A_0}{B_0}$ and $[q_0,q_1,r_1]=\frac{q_0q_1+r_1}{q_1}=\frac{A_1}{B_1}$. Let us suppose that it is true for a given $m<n$. We will induce that it is true for m+1

$$\begin{aligned}[q_0,q_1,\ldots q_{m+1},r_1,\ldots r_{m+1}] &= [q_0,q_1,\ldots q_{m-1},[q_m,q_{m+1},r_{m+1}],r_1,\ldots r_m]\\ &=\frac{[q_m,q_{m+1},r_{m+1}]A_{m-1}+r_mA_{m-2}}{[q_m,q_{m+1},r_{m+1}]B_{m-1}+r_mB_{m-2}}\\ &=\frac{(q_m+\frac{r_{m+1}}{q_{m+1}})A_{m-1}+r_mA_{m-2}}{(q_m+\frac{r_{m+1}}{q_{m+1}})B_{m-1}+r_mB_{m-2}}\\ &=\frac{q_{m+1}(q_mA_{m-1}+r_mA_{m-2})+r_{m+1}A_{m-1}}{q_{m+1}(q_mB_{m-1}+r_mB_{m-2})+r_{m+1}B_{m-1}}\\ &=\frac{q_{m+1}A_{m-1}+r_{m+1}A_{m-1}}{q_{m+1}B_{m-1}+r_{m+1}B_{m-1}}=\frac{A_{m+1}}{B_{m+1}}\end{aligned}$$

□

The recurrence relations (10) provide the basis for an effective computer algorithm for successive calculation of the convergents C_m.

Theorem 2:

$$A_mB_{m-1}-B_mA_{m-1}=(-1)^{m-1}\prod_{k=1}^{m} r_k \tag{11}$$

Proof: For m=1 we have $A_1B_0-B_1A_0=q_0q_1+r_1-q_0q_1=r_1$.

$$A_mB_{m-1}-B_mA_{m-1}=(q_mA_{m-1}+r_mA_{m-2})B_{m-1}-(q_mB_{m-1}+r_mB_{m-2})A_{m-1}=$$
$$-r_m(A_{m-1}B_{m-2}-B_{m-1}A_{m-2})$$

By repeating this calculation with m-1, m-2, ..., 2 in place of m, we arrive at

$$A_mB_{m-1}-B_mA_{m-1}= \ldots = (A_1B_0-B_1A_0)(-1)^{m-1}\prod_{k=2}^{m} r_k =(-1)^{m-1}\prod_{k=1}^{m} r_k$$

□

Theorem 3:

$$A_mB_{m-2}-B_mA_{m-2}=(-1)^m q_m \prod_{k=1}^{m-1} r_k \tag{12}$$

Proof: This theorem follows from theorem 3 by inserting expressions for A_m and B_m

$$A_mB_{m-2}-B_mA_{m-2}=(q_mA_{m-1}+r_mA_{m-2})B_{m-2}-(q_mB_{m-1}+r_mB_{m-2})A_{m-2}=$$
$$q_m(A_{m-1}B_{m-2}-B_{m-1}A_{m-2})=(-1)^m q_m \prod_{k=1}^{m-1} r_k$$

□

Using the symbol $C_m=\frac{A_m}{B_m}$ we can now express important properties of the number sequence C_m, m=1, 2, ... , n. In particular we will be interested in what happens to C_n as n approaches infinity.

From (11) we have

$$C_n - C_{n-1} = \frac{A_n}{B_n} - \frac{A_{n-1}}{B_{n-1}} = \frac{(-1)^{n-1}\prod_{k=1}^{n} r_k}{B_{n-1}B_n} \tag{13}$$

while (12) gives

$$C_n - C_{n-2} = \frac{A_n}{B_n} - \frac{A_{n-2}}{B_{n-2}} = \frac{(-1)^{n-1} q_n \prod_{k=1}^{n-1} r_k}{B_{n-2}B_n} \tag{14}$$

We will now consider infinite positive integer sequences $\{q_0, q_1, q_2, \ldots\}$ and $\{r_1, r_2, \ldots\}$ where only a finite number of terms are equal to 1. This is generally the case for Smarandache sequences. We will therefore prove the following important theorem.

Theorem 4:

A general continued fraction for which the sequences $q_0, q_1, q_2, \ldots$ and $r_1, r_2, \ldots$ are positive integer sequences with at most a finite number of terms equal to 1 is convergent.

Proof: We will first show that the product $B_{n-1}B_n$, which is a sum of terms formed by various products of elements from $\{q_1, q_2, \ldots q_n, r_1, r_2, \ldots r_{n-1}\}$, has one term which is a multiple of $\sum_{k=2}^{n} r_k$. Looking at the process by which we calculated C_1, C_2, and C_3, equations

4, 5 and 7, we see how terms with the largest number of factors r_k evolve in numerators and denominators of C_k. This is made explicit in figure 1.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
Num. A_m	r_1	q_0r_2	r_1r_3	$q_0r_2r_4$	$r_1r_3r_5$	$q_0r_2r_4r_6$	$r_1r_3r_5r_7$	$q_0r_2r_4r_6r_8$
Den. B_m	-	r_2	q_1r_3	r_2r_4	$q_1r_3r_5$	$r_2r_4r_6$	$q_1r_3r_5r_7$	$r_2r_4r_6r_8$

Figure 1. The terms with the largest number of r-factors in numerators and denominators.

As is seen from figure 1 two consecutive denominators B_nB_{n-1} will have a term with $r_2r_3 \ldots r_n$ as factor. This means that the numerator of (13) will not cause C_n-C_{n-1} to diverge. On the other hand $B_{n-1}B_n$ contains the term $(q_1q_2 \ldots q_{n-1})^2q_n$ which approaches ∞ as $n\to\infty$. It follows that $\lim_{n\to\infty}(C_n - C_{n-1}) = 0$.

From (14) we see that

1. If n is odd, say n=2k+1, than $C_{2k+1} < C_{2k-1}$ forming a monotonously decreasing number sequence which is bounded below (positive terms). It therefore has limit. $\lim_{k\to\infty} C_{2k+1} = C_1$.
2. If n is even, n=2k, than $C_{2k} > C_{2k-2}$ forming a monotonously increasing number sequence. This sequence has an upper bound because $C_{2k} < C_{2k+1} \to C_1$ as $k \to \infty$. It therefore has limit. $\lim_{k\to\infty} C_{2k} = C_2$.
3. Since $\lim_{n\to\infty}(C_n - C_{n-1}) = 0$ we conclude that $C_1 = C_2$. Consequently $\lim_{n\to\infty} C_n = C$ exists.

□

Calculations

A *UBASIC* program has been developed to implement the theory of Smarandache general continued fractions. The same program can be used for classical continued fractions since these correspond to the special case of a general continued fraction where $r_1 = r_2 = \ldots = r_n = 1$.

The complete program used in the calculations is given below. The program applies equally well to simple continued fractions by setting all element of the array R equals to 1.

```
10 point 10
20 dim Q(25),R(25),A25),B25)
30 input "Number of decimal places of accuracy: ";D
40 input "Number of input terms for R (one more for Q) ";N%
50 cls
60 for I%=0 to N%:read Q(I%):next
70 data                                          'The relevant data q0, q1, ...
80 for I%=1 to N%:read R(I%):next
90 data                                          'The relevant data for r1,r2, ...
100 print tab(10);"Smarandache Generalized Continued Fraction"
110 print tab(10);"Sequence Q:";
120 for I%=0 to 6:print Q(I%);:next:print "  ETC"
130 print tab(10);"Sequence R:";
140 for I%=1 to 6:print R(I%);:next:print "  ETC"
```

```
150  print tab(10);"Number of decimal places of accuracy: ";D
160  A(0)=Q(0):B(0)=1                                          'Initiating recurrence algorithm
170  A(1)=Q(0)*Q(1)+R(1):B(1)=Q(1)
180  Delta=1:M=1                                               'M=loop counter
190  while abs(Delta)>10^(-D)                                  'Convergens check
200  inc M
210  A(M)=Q(M)*A(M-1)+R(M)*A(M-2)                              'Recurrence
220  B(M)=Q(M)*B(M-1)+R(M)*B(M-2)
230  Delta=A(M)/B(M)-A(M-1)/B(M-1)                             'Cm-Cm-1
240  wend
250  print tab(10);"An/Bn=";:print using(2,20),A(M)/B(M)       'Cn in decimalform
260  print tab(10);"An/Bn=";:print A(M);"/";B(M)               'Cn in fractional form
270  print tab(10);"Delta=";:print using(2,20),Delta;          'Delta=Last difference
280  print " for n=";M                                         'n=number of iterations
290  print
300  end
```

To illustrate the behaviour of the convergents C_n have been calculated for $q_1=q_2= \ldots = q_n=1$ and $r_1=r_2= \ldots = r_n=10$. The iteration of C_n is stopped when $\Delta_n= |C_n-C_{n-1}|<0.01$. Table 1 shows the result which is illustrated in figure 2. The factor $(-1)^{n-1}$ in (13) produces an oscillating behaviour with diminishing amplitude approaching $\lim_{n\to\infty} C_n = C$

Table 1. Value of convergents C_n for $q\varepsilon\{1,1,\ldots\}$ and $r\varepsilon\{10,10,\ldots\}$

n	1	2	3	4	5	6	7	8	9	10	11
C_n	11	1.91	6.24	2.6	4.84	3.07	4.26	3.35	3.99	3.51	3.85
n	12	13	14	15	16	17	18	19	20	21	22
C_n	3.6	3.78	3.65	3.74	3.67	3.72	3.69	3.71	3.69	3.71	3.7

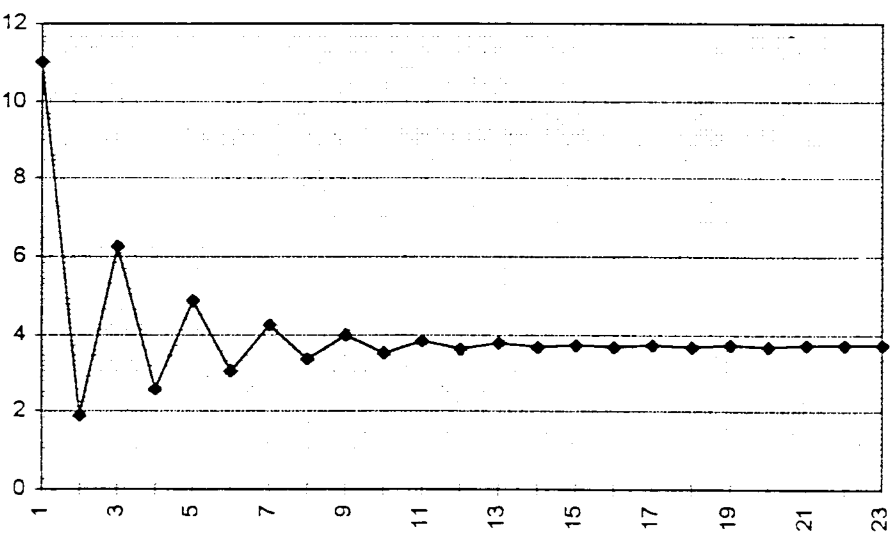

Figure 2. C_n as a function of n

A number of sequences, given below, will be substituted into the recurrence relations (10) and the convergence estimate (13).

$S_1=\{1, 1, 1, \ldots\ldots\}$
$S_2=\{1, 2, 1, 2, 1, 2, \ldots\ldots\}$
$S_3=\{3, 3, 3, 3, 3, 3, \ldots\ldots\}$
$S_4=\{1, 12, 123, 1234, 12345, 123456, \ldots\ldots\}$ Smarandache Consecutive Sequence.
$S_5=\{1, 21, 321, 4321, 54321, 654321, \ldots\ldots\}$ Smarandache Reverse Sequence.
CS1={1,1,2,8,9,10,512,513,514,520,521,522,729,730,731,737,738, ...
NCS1={1,2,3,4,5,6,7,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,29,30,...

The Smarandache CS1 sequence definition: CS1(n) is the smallest number, strictly greater than the previous one (for n≥3), which is the cubes sum of one or more previous distinct terms of the sequence.
The Smarandache NCS1 sequence definition: NCS1(n) is the smallest number, strictly greater than the previous one, which is NOT the cubes sum of one or more previous distinct terms of the sequence.
These sequences have been randomly chosen form a large number of Smarandache sequences [5].

As expected the last fraction in table 2 converges much slower than the previous one. These general continued fractions are, of course, very artificial as are the sequences on which they are based. As is often the case in empirical number theory it is not the individual figures or numbers which are of interest but the general behaviour of numbers and sequences under certain operations. In the next section we will carry out some experiments with simple continued fractions.

Experiments with Simple Continued Fractions

The theory of simple continued fractions is covered in standard textbooks. Without proof we will therefore make use of some of this theory to make some more calculations. We will first make use of the fact that

> *There is a one to one correspondence between irrational numbers and infinite simple continued fractions.*

The approximations given in table 2 expressed as simple continued fractions would therefore show how these are related to the corresponding general continued fractions.

Table 2. Calculation of general continued fractions

Q	R	n	Δ_n	C_n (dec.form)	C_n (fraction)
S_1	S_1	18	$-9\cdot10^{-8}$	1.6180339	$\frac{6765}{4181}$
S_2	S_1	13	$8\cdot10^{-8}$	1.3660254	$\frac{7953}{5822}$
S_2	S_3	22	$-9\cdot10^{-8}$	1.8228756	$\frac{1402652240}{769472267}$
S_4	S_1	2	$-7\cdot10^{-6}$	1.04761	$\frac{7063}{6742}$
		3	$5\cdot10^{-12}$	1.04761198457	$\frac{30519245}{29132203}$
		4	$-2\cdot10^{-20}$	1.0476119845794017019	$\frac{1657835914708}{1582490405905}$
S_4	S_5	2	$-1\cdot10^{-3}$	1.082	$\frac{540}{499}$
		4	$-7\cdot10^{-10}$	1.082166760	$\frac{8245719435}{7619638429}$
		6	$-1\cdot10^{-19}$	1.08216676051416702768	$\frac{418939686644589150004}{387130433063328840289}$
S_5	S_1	2	$-7\cdot10^{-6}$	1.04761	$\frac{7063}{6742}$
		3	$5\cdot10^{-12}$	1.04761198457	$\frac{30519245}{29132203}$
		4	$-2\cdot10^{-20}$	1.04761198457940170194	$\frac{1657835914708}{1582490405905}$
S_5	S_4	2	$-8\cdot10^{-5}$	1.0475	$\frac{2358}{2251}$
		3	$7\cdot10^{-9}$	1.04753443	$\frac{2547455}{2431858}$
		5	$1\cdot10^{-20}$	1.04753443663236268392	$\frac{60363763803209222}{57624610411155561}$
CS1	NCS1	6	$-1\cdot10^{-7}$	1.540889	$\frac{1376250}{893153}$
		7	$3\cdot10^{-12}$	1.54088941088	$\frac{1412070090}{916399373}$
		9	$-1\cdot10^{-20}$	1.54088941088788795255	$\frac{377447939426190}{244954593599743}$
NCS1	CS1	16	$-5\cdot10^{-5}$	0.6419	$\frac{562791312666017539}{876693583206100846}$

Table 3. Some general continued fractions converted to simple continued fractions

Q	R	C_n (dec.form)	C_n (Simple continued fraction sequence)
S_4	S_5	1.08216676051416702768 (corresponding to 6 terms)	1,12,5,1,6,1,1,1,48,7,2,1,20,2,1,5,1,2,1,1,9,1, 1,10,1,1,7,1,3,1,7,2,1,3,31,1,2,6,38,2 (39 terms)
S_5	S_4	1.04753443663236268392 (corresponding to 5 terms)	1,21,26,1,3,26,10,4,4,19,1,2,2,1,8,8,1,2,3,1, 10,1,2,1,2,3,1,4,1,8 (29 terms)
CS1	NCS1	1.54088941088788795255 (corresponding to 9 terms)	1,1,15,1,1,1,1,2,4,17,1,1,3,13,4,2,2,2,5,1,6,2, 2,9,2,15,1.51 (28 terms)

These sequences show no special regularities. As can be seen from table 3 the number of terms required to reach 20 decimals is much larger than for the corresponding general continued fractions.

A number of Smarandache periodic sequences were explored in the author's book *Computer Analysis of Number Sequences* [6]. An interesting property of simple continued fractions is that

> *A periodic continued fraction is a quadratic surd, i.e. an irrational root of a quadratic equation with integral coefficients.*

In terms of A_n and B_n, which for simple continued fractions are defined through

$$A_0=q_0,\quad A_1=q_0q_1+1,\qquad A_n=q_nA_{n-1}+A_{n-2}$$

$$B_0=1,\quad B_1=q_1,\qquad B_n=q_nB_{n-1}+B_{n-2} \tag{15}$$

the quadratic surd is found from the quadratic equation

$$B_nx^2+(B_{n-1}-A_n)x-A_{n-1}=0 \tag{16}$$

where n is the index of the last term in the periodic sequence. The relevant quadratic surd is

$$x=\frac{A_n-B_{n-1}+\sqrt{A_n^2+B_{n-1}^2-2A_nB_{n-1}-4A_{n-1}B_n}}{2B_n} \tag{17}$$

An example has been chosen from each of the following types of Smarandache periodic sequences:

1. The Smarandache two-digit periodic sequence:

Definition: Let N_k be an integer of at most two digits. N_k' is defined through

$$N_k' = \begin{cases} \text{the reverse of } N_k \text{ if } N_k \text{ is a two digit integer} \\ N_k\cdot 10 \text{ if } N_k \text{ is a one digit integer} \end{cases}$$

N_{k+1} is then determined by

$$N_{k+1}=|N_k-N_k'|$$

The sequence is initiated by an arbitrary two digit integer N_1 with unequal digits.

One such sequence is Q={9, 81, 63, 27, 45}. The corresponding quadratic equation is

$$6210109x^2-55829745x-1242703=0$$

2. The Smarandache Multiplication Periodic Sequence:

Definition: Let c>1 be a fixed integer and N_0 and arbitrary positive integer. N_{k+1} is derived from N_k by multiplying each digit x of N_k by c retaining only the last digit of the product cx to become the corresponding digit of N_{k+1}.

For c=3 we have the sequence Q={1, 3, 9, 7} with the corresponding quadratic equation

$$199x^2-235x-37=0$$

3. The Smarandache Mixed Composition Periodic Sequence:

Definition. Let N_0 be a two-digit integer $a_1\cdot10+a_0$. If $a_1+a_0<10$ then $b_1= a_1+a_0$ otherwise $b_1= a_1+a_0+1$. $b_0=|a_1-a_0|$. We define $N_1=b_1\cdot10+b_0$. N_{k+1} is derived from N_k in the same way.

One of these sequences is Q={18, 97, 72, 95, 54, 91} with the quadratic equation

$$3262583515x^2-58724288064x-645584400=0$$

and the relevant quadratic surd

$$x=\frac{58724288064+\sqrt{3456967100707577532096}}{6525167030}$$

The above experiments were carried out with *UBASIC* programs. An interesting aspect of this was to check the correctness by converting the quadratic surd back to the periodic sequence.

There are many interesting calculations to carry out in this area. However, this study will finish by this equality between a general continued fraction convergent and a simple continued fraction convergent.

[1,12,123,1234,12345,123456,1234567,*1,21,321,4321,54321,654321*]=
[1,12,5,1,6,1,1,1,48,7,2,1,20,2,1,5,1,2,1,1,9,1, 1,10,1,1,7,1,3,1,7,2,1,3,31,1,2,6,38,2]

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