On Smarandache's Podaire Theorem

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Let A', B', C' be the feet of the altitudes of an acute-angled triangle ABC ($A' \in BC$, $B' \in AC$, $C' \in AB$). Let a', b', b' denote the sides of the podaire triangle $A'B'C'$. Smarandache's Podaire theorem [2] (see [1]) states that

$$\sum a'b' \leq \frac{1}{4} \sum a^2 \tag{1}$$

where a, b, c are the sides of the triangle ABC. Our aim is to improve (1) in the following form:

$$\sum a'b' \leq \frac{1}{3} \left(\sum a'\right)^2 \leq \frac{1}{12} \left(\sum a\right)^2 \leq \frac{1}{4} \sum a^2. \tag{2}$$

First we need the following auxiliary proposition.

Lemma. *Let p and p' denote the semi-perimeters of triangles ABC and $A'B'C'$, respectively. Then*

$$p' \leq \frac{p}{2}. \tag{3}$$

Proof. Since $AC' = b \cos A$, $AB' = c \cos A$, we get

$$C'B' = AB'^2 + AC'^2 - 2AB' \cdot AC' \cdot \cos A = a^2 \cos^2 A,$$

so $C'B' = a \cos A$. Similarly one obtains

$$A'C' = b \cos B, \quad A'B' = c \cos C.$$

Therefore

$$p' = \frac{1}{2} \sum A'B' = \frac{1}{2} \sum a \cos A = \frac{R}{2} \sum \sin 2A = 2R \sin A \sin B \sin C$$

(where R is the radius of the circumcircle). By $a = 2R\sin A$, etc. one has

$$p' = 2R\prod \frac{a}{2R} = \frac{S}{R},$$

where $S = area(ABC)$. By $p = \frac{S}{r}$ (r = radius of the incircle) we obtain

$$p' = \frac{r}{R}p. \tag{4}$$

Now, Euler's inequality $2r \leq R$ gives relation (3).

For the proof of (2) we shall apply the standard algebraic inequalities

$$3(xy + xz + yz) \leq (x + y + z)^2 \leq 3(x^2 + y^2 + z^2).$$

Now, the proof of (2) runs as follows:

$$\sum a'b' \leq \frac{1}{3}\left(\sum a'\right)^2 = \frac{1}{3}(2p')^2 \leq \frac{1}{3}p^2 = \frac{1}{3}\frac{\left(\sum a\right)^2}{4} \leq \frac{1}{4}\sum a^2.$$

Remark. Other properties of the podaire triangle are included in a recent paper of the author ([4]), as well as in his monograph [3].

References

[1] F. Smarandache, *Problèmes avec et sans problemes*, Ed. Sompress, Fes, Marocco, 1983.

[2] www.gallup.unm.edu/~smarandache

[3] J. Sándor, *Geometric inequalities* (Hungarian), Ed. Dacia, Cluj, 1988.

[4] J. Sándor, *Relations between the elements of a triangle and its podaire triangle*, Mat. Lapok 9/2000, pp.321-323.