

# A problem related to the Smarandache $n$ -ary power sieve

Juanli Su<sup>†‡</sup>

<sup>†</sup> Department of Mathematics, Northwest University, Xi'an, Shaanxi, P.R.China

<sup>‡</sup> Yangling Vocational And Technical College, Yangling, Shaanxi, P.R.China

**Abstract** For any fixed positive integer  $k \geq 2$ , the power  $k$  sieve is defined as following: Starting to count on the natural numbers set at any step from 1: - delete every  $2^k$ -th numbers; - delete, from the remaining ones, every  $3^k$ -th numbers  $\dots$ , and so on: delete, from the remaining ones, every  $n^k$ -th numbers,  $n = 2, 3, 4, \dots$ . In this paper, we study the following two problems:

(A) Are there an infinity of primes that belong to this sequence?

(B) Are there an infinity of numbers of this sequence which are not prime?

Then we using the elementary methods to study these problems, and prove that the problem (B) is true.

**Keywords** The power  $k$  sieve, asymptotic formula, elementary method.

## §1. Introduction and result

For any fixed positive integer  $n \geq 2$ , the famous F.Smarandache  $n$ -ary power sieve is defined as following:

Starting to count on the natural numbers set at any step from 1.

-delete every  $n$ -th numbers,

-delete, from the remaining ones, every  $n^2$ -th numbers,  $\dots$ , and so on: delete, from the remaining ones, every  $n^k$ -th numbers,  $k = 1, 2, 3, \dots$ . For example, if  $n = 2$ , then we call this sieve as Binary Sieve:

1, 3, 5, 9, 11, 13, 17, 21, 25, 27, 29, 33, 35, 37, 43, 49, 51, 53, 57, 59, 65, 67, 69, 73, 75, 77, 81, 85, 89,  $\dots$ .

Simultaneously, if  $n = 3$ , then call the sieve as Trinary Sieve:

1, 2, 4, 5, 7, 8, 10, 11, 14, 16, 17, 19, 20, 22, 23, 25, 28, 29, 31, 32, 34, 35, 37, 38, 41, 43, 46, 47, 49, 50,  $\dots$ .

In reference [1] and [2], Professor F.Smarandache asked us to study the properties of the  $n$ -ary power sieve sequence. At the same time, he also proposed the following two conjectures:

(a) There are an infinity of primes that belong to this sequence.

(b) There are an infinity of numbers of this sequence which are not prime.

About these two conjectures, Yi Yuan [3] had studied them, and proved that the conjecture (b) is correct.

In this paper, we define another sequence related the Smarandache  $n$ -ary power sieve (we called it as the power  $k$  sieve) as follows:

Starting to count on the natural numbers set at any step from 1: - delete every  $2^k$ -th numbers; - delete, from the remaining ones, every  $3^k$ -th numbers  $\dots$ , and so on: delete, from the remaining ones, every  $n^k$ -th numbers,  $n = 2, 3, 4, \dots$ . Then, two similar problems here can be proposed naturally when we studying the properties of this sequence:

(A) Are there an infinity of primes that belong to the power  $k$  sieve sequence?

(B) Are there an infinity of numbers of the power  $k$  sieve sequence which are not prime?

In this paper, we use the elementary method to study these two problems, and obtain an interesting asymptotic formula. As a corollary of our result, we solved the problem (B). That is, we shall prove the following:

**Theorem.** Let  $k \geq 2$  be a fixed positive integer,  $A$  denotes the set of all power  $k$  sieve sequence. Then for any real  $x > 1$ , we have the asymptotic formula

$$\sum_{\substack{n \leq x \\ n \in A}} 1 = c(k) \cdot x + O\left(x^{\frac{1}{k}}\right),$$

where  $c(k) = \prod_{n=2}^{\infty} \left(1 - \frac{1}{n^k}\right)$  is a positive constant, and  $c(2) = \frac{1}{2}$ .

For any real number  $x > 1$ , from the Prime Number Theorem (see reference [6]) we know that there are at most  $O\left(\frac{x}{\ln x}\right)$  primes in the interval  $[1, x]$ , so from our Theorem we know that there are an infinity of numbers of the power  $k$  sieve sequence which are not prime. Therefore, the problem (B) is true.

## §2. Proof of the theorem

In this section, we shall complete the proof of our Theorem directly. Let  $k \geq 2$  be a fixed positive integer. For any positive integer  $x > 1$ , let  $U_h(x)$  denotes the number of all remaining ones when delete  $i^k$ -th numbers ( $i = 2, 3, \dots, h$ ) in the interval  $[1, x]$ . Then we have

$$x - \frac{x}{2^k} \leq U_1(x) = x - \left[\frac{x}{2^k}\right] \leq x - \frac{x}{2^k} + \frac{2^k - 1}{2^k}$$

or

$$U_1(x) = \left(1 - \frac{1}{2^k}\right) \cdot x + R_1(x) \quad \text{with } |R_1(x)| \leq 1.$$

$$U_1(x) - \frac{U_1(x)}{3^k} \leq U_2(x) = U_1(x) - \left[\frac{U_1(x)}{3^k}\right] \leq U_1(x) - \frac{U_1(x)}{3^k} + \frac{3^k - 1}{3^k}$$

or

$$U_2(x) = \left(1 - \frac{1}{2^k}\right) \left(1 - \frac{1}{3^k}\right) \cdot x + R_2(x) \quad \text{with } |R_2(x)| \leq |R_1(x)| + 1 \leq 2.$$

Generally, for any positive integer  $h \geq 2$ , we have

$$U_h(x) = \left(1 - \frac{1}{2^k}\right) \left(1 - \frac{1}{3^k}\right) \cdots \left(1 - \frac{1}{h^k}\right) \cdot x + R_h(x) \quad \text{with } |R_h(x)| \leq h.$$

Taking  $m = \left\lceil x^{\frac{1}{k}} \right\rceil$ , if  $h > m$ , then  $h^k > x$ . So we have

$$\sum_{\substack{n \leq x \\ n \in A}} 1 = U_m(x) = x \cdot \prod_{h=1}^m \left(1 - \frac{1}{h^k}\right) + R_m(x). \quad (1)$$

Note that  $|R_m(x)| \leq m \leq x^{\frac{1}{k}}$  and

$$\prod_{h=1}^m \left(1 - \frac{1}{h^k}\right) = \prod_{h=1}^{\infty} \left(1 - \frac{1}{h^k}\right) + O\left(m^{-(k-1)}\right) = \prod_{h=2}^{\infty} \left(1 - \frac{1}{h^k}\right) + O\left(x^{-\frac{k-1}{k}}\right).$$

From (1) we may immediately get the asymptotic formula

$$\sum_{\substack{n \leq x \\ n \in A}} 1 = c \cdot x + O\left(x^{\frac{1}{k}}\right),$$

where  $c = c(k) = \prod_{n=2}^{\infty} \left(1 - \frac{1}{n^k}\right)$  is a positive constant, and

$$c(2) = \prod_{n=2}^{\infty} \left(1 - \frac{1}{n^2}\right) = \frac{2^2 - 1}{2^2} \cdot \frac{3^2 - 1}{3^2} \cdot \frac{4^2 - 1}{4^2} \cdots \frac{n^2 - 1}{n^2} \cdots = \frac{1}{2}.$$

This completes the proof of Theorem.

## References

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