Scientia Magna
Vol. 4 (2008), No. 4, 36-39

On the Smarandache sequences

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Abstract In this paper, we use the elementary method to study the convergence of the Smarandache alternate consecutive, reverse Fibonacci sequence and Smarandache multiple sequence.

Keywords Convergence, Smarandache sequences, elementary method.

§1. Introduction and results

For any positive integer n, the Smarandache alternate consecutive and reverse Fibonacci sequence $a(n)$ is defined as follows: $a(1) = 1,\ a(2) = 11,\ a(3) = 112,\ a(4) = 3211,\ a(5) = 11235,\ a(6) = 853211,\ a(7) = 11235813,\ a(8) = 2113853211,\ a(9) = 112358132134, \cdots$. The Smarandache multiple sequence $b(n)$ is defined as: $b(1) = 1,\ b(2) = 24,\ b(3) = 369,\ b(4) = 481216,\ b(5) = 510152025,\ b(6) = 61218243036,\ b(7) = 7142128354249, b(8) = 816243240485664, b(9) = 91827364554637281, \cdots$.

These two sequences were both proposed by professor F.Smarandache in reference [1], where he asked us to study the properties of these two sequences.

About these problems, it seems that none had studied it, at least we have not seen any related papers before. However, in reference [1] (See chapter III, problem 6 and problem 21), professor Felice Russo asked us to study the convergence of

$$\frac{a(n)}{a(n+1)}, \quad \lim_{k\to\infty}\sum_{n=1}^{k}\frac{b(n)}{b(n+1)}$$

and other properties.

The main purpose of this paper is using the elementary method to study these problems, and give some interesting conclusions. That is, we shall prove the following:

Theorem 1. For Smarandache alternate consecutive and reverse Fibonacci sequence $a(n)$, we have $\lim_{n\to\infty}\frac{a(n)}{a(n+1)} = 0$.

Theorem 2. For Smarandache multiple sequence $b(n)$, the series $\sum_{n=1}^{\infty}\frac{b(n)}{b(n+1)}$ is convergent.

§2. Proof of the theorems

In this section, we shall using elementary method to prove our Theorems. First we prove Theorem 1. If n is an odd number, then from the definition of $a(n)$ we know that $a(n)$ can be written in the form:

$$a(n) = F(1)F(2)\cdots F(n) \quad \text{and} \quad a(n+1) = F(n+1)F(n)\cdots F(1),$$

where $F(n)$ be the Fibonacci sequence.

Let α_n denote the number of the digits of $F(n)$ in base 10, then

$$a(n) = F(n) + F(n-1)\cdot 10^{\alpha_n} + F(n-2)\cdot 10^{\alpha_n+\alpha_{n-1}} + \cdots + F(1)\cdot 10^{\alpha_n+\alpha_{n-1}+\cdots+\alpha_2}$$

and

$$a(n+1) = F(1) + F(2)\cdot 10^{\alpha_1} + F(3)\cdot 10^{\alpha_1+\alpha_2} + \cdots + F(n+1)\cdot 10^{\alpha_1+\alpha_2+\cdots+\alpha_n}.$$

So

$$\begin{aligned}\frac{a(n)}{a(n+1)} &\leq \frac{F(n) + F(n-1)\cdot 10^{\alpha_n} + \cdots + F(1)\cdot 10^{\alpha_n+\alpha_{n-1}+\cdots+\alpha_2}}{F(n+1)\cdot 10^{\alpha_1+\alpha_2+\cdots\alpha_n}} \\ &= \frac{\dfrac{F(n)}{10^{\alpha_1+\cdots+\alpha_n}} + \dfrac{F(n-1)}{10^{\alpha_1+\cdots+\alpha_{n-1}}} + \cdots + \dfrac{F(1)}{10^{\alpha_1}}}{F(n+1)}.\end{aligned}$$

For $1 \leq k \leq n$, since the number of the digits of $F(k)$ in base 10 is α_k, we can suppose $F(k) = a_1 \cdot 10^{\alpha_k - 1} + a_2 \cdot 10^{\alpha_k-2} + \cdots + a_{\alpha_k}$, $0 \leq a_i \leq 9$ and $1 \leq i \leq \alpha_k$. Therefore, we have

$$\begin{aligned}\frac{F(k)}{10^{\alpha_1+\alpha_2+\cdots+\alpha_k}} &= \frac{a_1\cdot 10^{\alpha_k-1} + a_2\cdot 10^{\alpha_k-2} + \cdots + a_{\alpha_k}}{10^{\alpha_1+\alpha_2+\cdots+\alpha_k}} \\ &\leq \frac{9\cdot(1+10^{-1}+10^{-2}+\cdots+10^{1-\alpha_k})}{10^{\alpha_1+\alpha_2+\cdots+\alpha_{k-1}+1}} \\ &= \frac{10\cdot(1-10^{-\alpha_k})}{10^{\alpha_1+\alpha_2+\cdots+\alpha_{k-1}+1}} \\ &\leq \frac{1}{10^{\alpha_1+\alpha_2+\cdots+\alpha_{k-1}}} \leq \frac{1}{10^{k-1}}.\end{aligned}$$

Thus,

$$0 \leq \frac{a(n)}{a(n+1)} \leq \frac{1+10^{-1}+10^{-2}+\cdots+10^{1-n}}{F(n+1)} \leq \frac{10}{9F(n+1)} \to 0, \text{ as } n \to \infty.$$

That is to say,

$$\lim_{n\to\infty} \frac{a(n)}{a(n+1)} = 0.$$

If n be an even number, then we also have

$$a(n) = F(1) + F(2)\cdot 10^{\alpha_1} + F(3)\cdot 10^{\alpha_1+\alpha_2} + \cdots + F(n)\cdot 10^{\alpha_1+\alpha_2+\cdots+\alpha_{n-1}}$$

and

$$a(n+1) = F(n+1) + F(n)\cdot 10^{\alpha_{n+1}} + F(n-1)\cdot 10^{\alpha_{n+1}+\alpha_n} + \cdots + F(1)\cdot 10^{\alpha_{n+1}+\alpha_n+\cdots+\alpha_2}.$$

We can use the similar methods to evaluate the value of $\frac{a(n)}{a(n+1)}$. And

$$\frac{a(n)}{a(n+1)} \leq \frac{F(1)+F(2)\cdot 10^{\alpha_1}+F(3)\cdot 10^{\alpha_1+\alpha_2}+\cdots+F(n)\cdot 10^{\alpha_1+\alpha_2+\cdots+\alpha_{n-1}}}{10^{\alpha_2+\alpha_3\cdots+\alpha_{n+1}}}.$$

For every $1 \leq k \leq n$, similarly, let $F(k) = a_1 \cdot 10^{\alpha_k-1} + a_2 \cdot 10^{\alpha_k-2} + \cdots + a_{\alpha_k}$, we have

$$\begin{aligned}
\frac{F(k)\cdot 10^{\alpha_1+\alpha_2+\cdots+\alpha_{k-1}}}{10^{\alpha_2+\alpha_3+\cdots+\alpha_{n+1}}} &= \frac{(a_1\cdot 10^{\alpha_k-1}+a_2\cdot 10^{\alpha_k-2}+\cdots+a_{\alpha_k})\cdot 10^{\alpha_1+\alpha_2+\cdots+\alpha_{k-1}}}{10^{\alpha_2+\alpha_3+\cdots+\alpha_{n+1}}} \\
&\leq \frac{9(1+10^{-1}+10^{-2}+\cdots+10^{1-\alpha_k})}{10^{\alpha_{k+1}+\alpha_{k+2}+\cdots+\alpha_{n+1}}} \\
&\leq \frac{1}{10^{\alpha_{k+1}+\alpha_{k+2}+\cdots+\alpha_{n+1}-1}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
0 \leq \frac{a(n)}{a(n+1)} &\leq \frac{1}{10^{\alpha_2+\alpha_3+\cdots+\alpha_{n+1}-1}}+\frac{1}{10^{\alpha_3+\alpha_4+\cdots+\alpha_{n+1}-1}}+\cdots+\frac{1}{10^{\alpha_{n+1}-1}} \\
&\leq \frac{1}{10^{\alpha_{n+1}-1}}(1+10^{-1}+10^{-2}+\cdots+10^{1-n)}) \\
&\leq \frac{100}{9\cdot 10^{\alpha_{n+1}}} \to 0, \quad \text{as } n\to\infty.
\end{aligned}$$

So $\lim_{n\to\infty}\frac{a(n)}{a(n+1)} = 0$. This proves Theorem 1.

Now we prove Theorem 2. For the sequence $b(n) = n(2n)(3n)\cdots(n\cdot n)$, let $\gamma(n)$ denote the number of the digits of n^2 in base 10, then by observation, we can obtain $\gamma(2) = \gamma(3) = 1$, $\gamma(4) = 2, \gamma(10) = 3, \gamma(40) = 4, \gamma(100) = 5, \gamma(400) = 6, \gamma(1000) = 7, \cdots$. When n ranges from $4\cdot 10^{\alpha}$ to $10^{\alpha+1}$, $\gamma(n)$ increases. That is to say, $\gamma(4\cdot 10^{\alpha}) = 2\alpha+2$, and $\gamma(10^{\alpha+1}) = 2\alpha+3$, where $\alpha = 0,1,2,\cdots$.

For every positive integer n, it is obvious that $k\cdot n \leq k\cdot(n+1)$, where $1\leq k\leq n$. So we can evaluate $\frac{b(n)}{b(n+1)}$ as

$$\begin{aligned}
\frac{b(n)}{b(n+1)} &= \frac{n(2n)(3n)\cdots(n\cdot n)}{(n+1)(2(n+1))\cdots(n\cdot(n+1))((n+1)\cdot(n+1))} \\
&\leq \frac{b(n)}{b(n)\cdot 10^{\gamma(n+1)}} = \frac{1}{10^{\gamma(n+1)}},
\end{aligned}$$

thus,

$$\sum_{n=1}^{\infty}\frac{b(n)}{b(n+1)} \leq \sum_{n=1}^{\infty}\frac{1}{10^{\gamma(n+1)}}.$$

If $4\cdot 10^{\alpha} \leq n \leq 10^{\alpha+1}-1$, then $2\alpha+2 \leq \gamma(n) \leq 2\alpha+3$, where $\alpha = 0,1,2,3,\cdots$. In addition, if $10^{\alpha+1} \leq n \leq 4\cdot 10^{\alpha+1}-1$, then $2\alpha+3 \leq \gamma(n) \leq 2\alpha+4$, where $\alpha = 0,1,2,3,\cdots$.

Therefore, we can get

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{10^{\gamma(n+1)}} &= \frac{2}{10}+\frac{1}{10^{\gamma(4)}}+\frac{1}{10^{\gamma(5)}}+\cdots+\frac{1}{10^{\gamma(9)}}+\frac{1}{10^{\gamma(10)}}+\cdots+\frac{1}{10^{\gamma(39)}}\\
&\quad +\cdots+\frac{1}{10^{\gamma(4\cdot 10^{\alpha})}}+\cdots+\frac{1}{10^{\gamma(10^{\alpha+1}-1)}}\\
&\quad +\frac{1}{10^{\gamma(10^{\alpha+1})}}+\cdots+\frac{1}{10^{\gamma(4\cdot 10^{\alpha+1}-1)}}+\cdots\\
&= \frac{2}{10}+\sum_{\alpha=0}^{\infty}\left(\frac{1}{10^{\gamma(4\cdot 10^{\alpha})}}+\cdots+\frac{1}{10^{\gamma(10^{\alpha+1}-1)}}\right)\\
&\quad +\sum_{\alpha=0}^{\infty}\left(\frac{1}{10^{\gamma(10^{\alpha+1})}}+\cdots+\frac{1}{10^{\gamma(4\cdot 10^{\alpha+1}-1)}}\right)\\
&\le \frac{1}{5}+\sum_{\alpha=0}^{\infty}\frac{6\cdot 10^{\alpha}}{10^{2\alpha+2}}+\sum_{\alpha=0}^{\infty}\frac{3\cdot 10^{\alpha+1}}{10^{2\alpha+3}}\\
&= \frac{1}{5}+\sum_{\alpha=0}^{\infty}\frac{6}{10^{\alpha+2}}+\sum_{\alpha=0}^{\infty}\frac{3}{10^{\alpha+2}}=\frac{3}{10}.
\end{aligned}$$

So the series $\sum_{n=1}^{\infty}\frac{b(n)}{b(n+1)}$ is convergent. This completes the proof of Theorem 2.

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