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Smarandache's Conjecture on Consecutive Primes

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Abstract: Let p and q two consecutive prime numbers, where $q > p$. Smarandache's conjecture states that the nonlinear equation $q^x - p^x = 1$ has solutions > 0.5 for any p and q consecutive prime numbers. This article describes the conditions that must be fulfilled for Smarandache's conjecture to be true.

Key Words: Smarandache conjecture, Smarandache constant, prime, gap of consecutive prime.

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§1. Introduction

We note $\mathbb{P}_{\geqslant k} = \{p \mid p \text{ prime number}, \ p \geqslant k\}$ and two consecutive prime numbers $p_n, p_{n+1} \in \mathbb{P}_{\geqslant 2}$.

Smarandache Conjecture *The equation*

$$p_{n+1}^x - p_n^x = 1 \ , \tag{1.1}$$

has solutions > 0.5, *for any* $n \in \mathbb{N}^*$ ([18], [25]).

Smarandache's constant([18], [29]) is $c_S \approx 0.567148130202539\cdots$, the solution for the equation

$$127^x - 113^x = 1 \ .$$

Smarandache Constant Conjecture *The constant c_S is the smallest solution of equation (1.1) for any $n \in \mathbb{N}^*$.*

The function that counts the the prime numbers p, $p \leqslant x$, was denoted by Edmund Landau in 1909, by π ([10], [27]). The notation was adopted in this article.

We present some conjectures and theorems regarding the distribution of prime numbers.

Legendre Conjecture([8], [20]) *For any $n \in \mathbb{N}^*$ there is a prime number p such that*

$$n^2 < p < (n+1)^2 \ .$$

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The smallest primes between n^2 and $(n+1)^2$ for $n = 1, 2, \cdots$, are 2, 5, 11, 17, 29, 37, 53, 67, 83, $\cdots$, [24, A007491].

The largest primes between n^2 and $(n+1)^2$ for $n = 1, 2, \cdots$, are 3, 7, 13, 23, 31, 47, 61, 79, 97, $\cdots$, [24, A053001].

The numbers of primes between n^2 and $(n+1)^2$ for $n = 1, 2, \cdots$ are given by 2, 2, 2, 3, 2, 4, 3, 4, $\cdots$, [24, A014085].

Bertrand Theorem *For any integer n, $n > 3$, there is a prime p such that $n < p < 2(n-1)$.*

Bertrand formulated this theorem in 1845. This assumption was proven for the first time by Chebyshev in 1850. Ramanujan in 1919 ([19]), and Erdös in 1932 ([5]), published two simple proofs for this theorem.

Bertrand's theorem stated that: *for any $n \in \mathbb{N}^*$ there is a prime p, such that $n < p < 2n$.* In 1930, Hoheisel, proved that there is $\theta \in (0, 1)$ ([9]), such that

$$\pi(x + x^\theta) - \pi(x) \approx \frac{x^\theta}{\ln(x)} \ . \tag{1.2}$$

Finding the smallest interval that contains at least one prime number p, was a very hot topic. Among the most recent results belong to Andy Loo whom in 2011 ([11]) proved any for $n \in \mathbb{N}^*$ there is a prime p such that $3n < p < 4n$. Even ore so, we can state that, if Riemann's hypothesis

$$\pi(x) = \int_2^x \frac{du}{\ln(u)} + O(\sqrt{x}\ln(x)) \ , \tag{1.3}$$

stands, then in (1.2) we can consider $\theta = 0.5 + \varepsilon$, according to Maier ([12]).

Brocard Conjecture([17,26]) *For any $n \in \mathbb{N}^*$ the inequality*

$$\pi(p_{n+1}^2) - \pi(p_n^2) \geqslant 4$$

holds.

Legendre's conjecture stated that between p_n^2 and a^2, where $a \in (p_n, p_{n+1})$, there are at least two primes and that between a^2 and p_{n+1}^2 there are also at least two prime numbers. Namely, is Legendre's conjecture stands, then there are at least four prime numbers between p_n^2 and p_{n+1}^2.

Concluding, if Legendre's conjecture stands then Brocard's conjecture is also true.

Andrica Conjecture([1],[13],[17]) *For any $n \in \mathbb{N}^*$ the inequality*

$$\sqrt{p_{n+1}} - \sqrt{p_n} < 1 \ , \tag{1.4}$$

stands.

The relation (1.4) is equivalent to the inequality

$$\sqrt{p_n + g_n} < \sqrt{p_n} + 1 \ , \tag{1.5}$$

where we denote by $g_n = p_{n+1} - p_n$ the gap between p_{n+1} and p_n. Squaring (1.5) we obtain the equivalent relation

$$g_n < 2\sqrt{p_n} + 1 \ . \tag{1.6}$$

Therefore Andrica's conjecture equivalent form is: *for any* $n \in \mathbb{N}^*$ *the inequality* (1.6) *is true.*

In 2014 Paz ([17]) proved that if Legendre's conjecture stands then Andirca's conjecture is also fulfilled. Smarandache's conjecture is a generalization of Andrica's conjecture ([25]).

Cramér Conjecture([4, 7, 21, 23]) *For any* $n \in \mathbb{N}^*$

$$g_n = O(\ln(p_n)^2) \ , \tag{1.7}$$

where $g_n = p_{n+1} - p_n$, namely

$$\limsup_{n\to\infty} \frac{g_n}{\ln(p_n)^2} = 1 \ .$$

Cramér proved that

$$g_n = O\big(\sqrt{p_n}\ln(p_n)\big) \ ,$$

a much weaker relation (1.7), by assuming Riemann hypothesis (1.3) to be true.

Westzynthius proved in 1931 that the gaps g_n grow faster then the prime numbers logarithmic curve ([30]), namely

$$\limsup_{n\to\infty} \frac{g_n}{\ln(p_n)} = \infty \ .$$

Cramér-Granville Conjecture *For any* $n \in \mathbb{N}^*$

$$g_n < R \cdot \ln(p_n)^2 \ , \tag{1.8}$$

stands for $R > 1$, where $g_n = p_{n+1} - p_n$.

Using Maier's theorem, Granville proved that Cramér's inequality (1.8) does not accurately describe the prime numbers distribution. Granville proposed that $R = 2e^{-\gamma} \approx 1.123\cdots$ considering the small prime numbers ([6, 13]) (a prime number is considered small if $p < 10^6$, [3]).

Nicely studied the validity of Cramér-Grandville's conjecture, by computing the ratio

$$R = \frac{\ln(p_n)}{\sqrt{g_n}} \ ,$$

using large gaps. He noted that for this kind of gaps $R \approx 1.13\cdots$. Since $1/R^2 < 1$, using the ratio R we can not produce a proof for Cramér-Granville's conjecture ([14]).

Oppermann Conjecture([16],[17]) *For any* $n \in \mathbb{N}^*$, *the intervals*

$$[n^2 - n + 1, n^2 - 1] \ \text{ and } \ [n^2 + 1, n^2 + n]$$

contain at least one prime number p.

Firoozbakht Conjecture *For any $n \in \mathbb{N}^*$ we have the inequality*

$$\sqrt[n+1]{p_{n+1}} < \sqrt[n]{p_n} \tag{1.9}$$

or its equivalent form

$$p_{n+1} < p_n^{1+\frac{1}{n}} \ .$$

If Firoozbakht's conjecture stands, then for any $n > 4$ we the inequality

$$g_n < \ln(p_n)^2 - \ln(p_n) \ , \tag{1.10}$$

is true, where $g_n = p_{n+1} - p_n$. In 1982 Firoozbakht verified the inequality (1.10) using maximal gaps up to 4.444×10^{12} ([22]), namely close to the 48th position in Table 1.

Currently the table was completed up to the position 75 ([15, 24]).

Paz Conjecture([17]) *If Legendre's conjecture stands then:*

(1) *The interval $[n, n + 2\lfloor\sqrt{n}\rfloor + 1]$ contains at least one prime number p for any $n \in \mathbb{N}^*$;*

(2) *The interval $[n - \lfloor\sqrt{n}\rfloor + 1, n]$ or $[n, n + \lfloor\sqrt{n}\rfloor - 1]$ contains at leas one prime number p, for any $n \in \mathbb{N}^*$, $n > 1$.*

Remark 1.1 According to Case (1) and (2), if Legendre's conjecture holds, then Andrica's conjecture is also true ([17]).

Conjecture Wolf *Furthermore, the bounds presented below suggest yet another growth rate, namely, that of the square of the so-called* Lambert W function*. These growth rates differ by very slowly growing factors (like $ln(ln(p_n))$). Much more data is needed to verify empirically which one is closer to the true growth rate.*

Let $P(g)$ be the least prime such that $P(g) + g$ is the smallest prime larger than $P(g)$. The values of $P(g)$ are bounded, for our empirical data, by the functions

$$P_{min}(g) = 0.12 \cdot \sqrt{g} \cdot e^{\sqrt{g}} \ ,$$

$$P_{max}(g) = 30.83 \cdot \sqrt{g} \cdot e^{\sqrt{g}} \ .$$

For large g, there bounds are in accord with a conjecture of Marek Wolf ([15, 31, 32]).

§2. Proof of Smarandache Conjecture

In this article we intend to prove that there are no equations of type (1.1), in respect to x with solutions $\leqslant 0.5$ for any $n \in \mathbb{N}^*$.

Let $f : [0, 1] \to \mathbb{R}$,

$$f(x) = (p + g)^x - p^x - 1 \ , \tag{2.1}$$

where $p \in \mathbb{P}_{\geqslant 3}$, $g \in \mathbb{N}^*$ and g the gap between p and the consecutive prime number $p + g$. Thus the equation

$$(p+g)^x - p^x = 1 \ . \tag{2.2}$$

is equivalent to equation (1.1).

Since for any $p \in \mathbb{P}_{\geqslant 3}$ we have $g \geqslant 2$ (if *Goldbach's conjecture* is true, then $g = 2 \cdot \mathbb{N}^{*}$[1]).

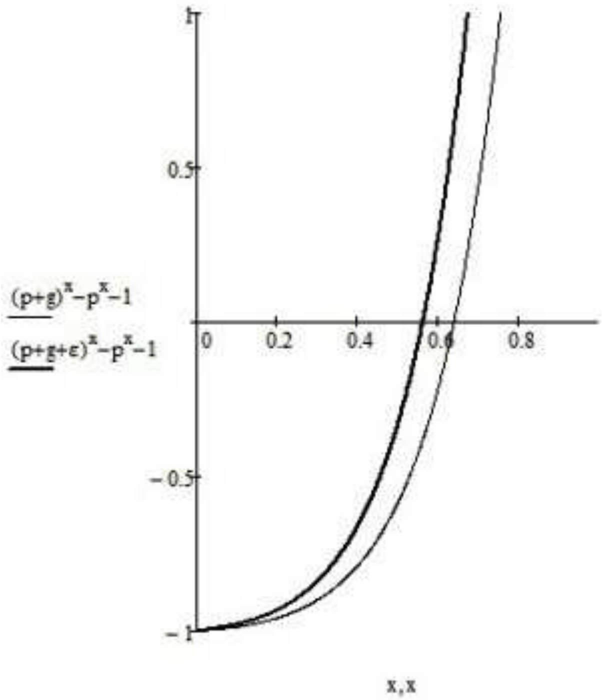

Figure 1 The functions (2.1) and $(p+g+\varepsilon)^x - p^x - 1$ for $p = 89$, $g = 8$ and $\varepsilon = 5$

Theorem 2.1 *The function f given by (2.1) is strictly increasing and convex over its domain.*

Proof If we compute the first and second derivative of function f, namely

$$f'(x) = \ln(p+g)(p+g)^x - \ln(p)p^x$$

and

$$f''(x) = \ln(p+g)^2(p+g)^x - \ln(p)^2 p^x \ .$$

it follows that $f'(x) > 0$ and $f''(x) > 0$ over $[0, 1]$, thus function f is strictly increasing and convex over its domain. □

Corollary 2.2 *Since $f(0) = -1 < 0$ and $f(1) = g - 1 > 0$ because $g \geqslant 2$ if $p \in \mathbb{P}_{\geqslant 3}$ and, also since function f is strictly monotonically increasing function it follows that equation (2.2) has a unique solution over the interval* $(0, 1)$.

[1] $2 \cdot \mathbb{N}^{*}$ is the set of all even natural numbers

Theorem 2.3 *For any g that verifies the condition $2 \leqslant g < 2\sqrt{p}+1$, function $f(0.5) < 0$.*

Proof The inequality $\sqrt{p+g} - \sqrt{p} - 1 < 0$ in respect to g had the solution $-p \leqslant g < 2\sqrt{p}+1$. Considering the give condition it follows that for a given g that fulfills $2 \leqslant g < 2\sqrt{p}+1$ we have $f(0.5) < 0$ for any $p \in \mathbb{P}_{\geqslant 3}$. □

Remark 2.4 The condition $g < 2\sqrt{p}+1$ represent Andrica's conjecture (1.6).

Theorem 2.5 *Let $p \in \mathbb{P}_{\geqslant 3}$ and $g \in \mathbb{N}^*$, then the equation (2.2) has a greater solution s then s_ε, the solution for the equation $(p+g+\varepsilon)^x - p^x - 1 = 0$, for any $\varepsilon > 0$.*

Proof Let $\varepsilon > 0$, then $p+g+\varepsilon > p+g$. It follows that $(p+g+\varepsilon)^x - p^x - 1 > (p+g)^x - p^x - 1$, for any $x \in [0,1]$. Let s be the solution to equation (2.2), then there is $\delta > 0$, that depends on ε, such that $(p+g+\varepsilon)^{s-\delta} - p^{s-\delta} - 1 = 0$. Therefore s, the solution for equation (2.2), is greater that the solution $s_\varepsilon = s - \delta$ for the equation $(p+g+\varepsilon)^x - p^x - 1 = 0$, see Figure 1. □

Theorem 2.6 *Let $p \in \mathbb{P}_{\geqslant 3}$ and $g \in \mathbb{N}^*$, then $s < s_\varepsilon$, where s is the equation solution (2.2) and s_ε is the equation solution $(p+\varepsilon+g)^x - (p+\varepsilon)^x - 1 = 0$, for any $\varepsilon > 0$.*

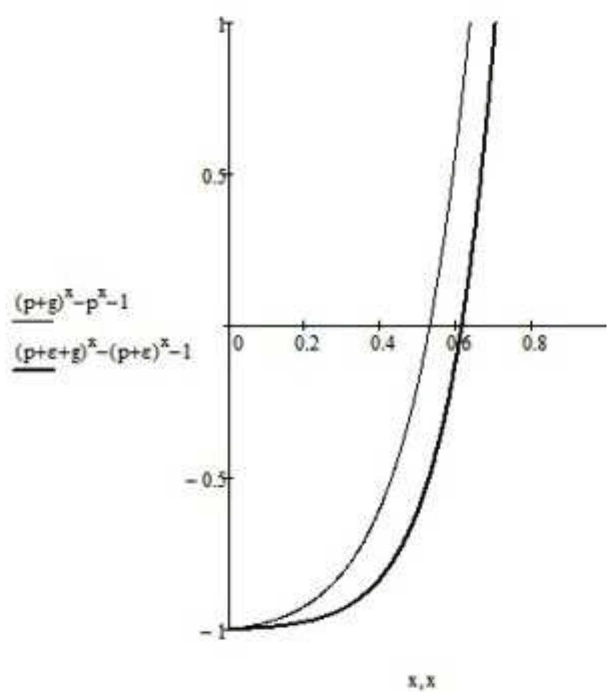

Figure 2 The functions (2.1) and $(p+\varepsilon+g)^x - (p+\varepsilon)^x - 1$ for $p = 113, \varepsilon = 408, g = 14$

Proof Let $\varepsilon > 0$, Then $p+\varepsilon+g > p+g$, from which it follows that $(p+\varepsilon+g)^x - (p+\varepsilon)^x - 1 < (p+g)^x - p^x - 1$, for any $x \in [0,1]$ (see Figure 2). Let s the equation solution (2.2), then there $\delta > 0$, which depends on ε, so $(p+\varepsilon+g)^{s+\delta} - (p+\varepsilon)^{s+\delta} - 1 = 0$. Therefore the solution s, of the equation (2.2), is lower than the solution $s_\varepsilon = s+\delta$ of the equation $(p+\varepsilon+g)^x - (p+\varepsilon)^x - 1 = 0$, see Figure 2. □

Remark 2.7 Let p_n and p_{n+1} two prime numbers in Table maximal gaps corresponding the maximum gap g_n. The Theorem 2.6 allows us to say that all solutions of the equation $(q+\gamma)^x - q^x = 1$, where $q \in \{p_n, \cdots, p_{n+1}-2\}$ and $\gamma < g_n$ solutions are smaller that the solution of the equation $p_{n+1}^x - p_n^x = 1$, see Figure 2.

Let:

(1) $g_A(p) = 2\sqrt{p}+1$, Andrica's gap function ;

(2) $g_{CG}(p) = 2 \cdot e^{-\gamma} \cdot \ln(p)^2$, Cramér-Grandville's gap function ;

(3) $g_F(p) = g_1(p) = \ln(p)^2 - \ln(p)$, Firoozbakht's gap function;

(4) $g_c(p) = \ln(p)^2 - c \cdot \ln(p)$, where $c = 4(2\ln(2)-1) \approx 1.545\cdots$,

(5) $g_b(p) = \ln(p)^2 - b \cdot \ln(p)$, where $b = 6(2\ln(2)-1) \approx 2.318\cdots$.

Theorem 2.8 *The inequality $g_A(p) > g_\alpha(p)$ is true for:*

(1) $\alpha = 1$ *and* $p \in \mathbb{P}_{\geqslant 3} \setminus \{7, 11, \cdots, 41\}$*;*

(2) $\alpha = c = 4(2\ln(2)-1)$ *and* $p \in \mathbb{P}_{\geqslant 3}$*;*

(3) $\alpha = b = 6(2\ln(2)-1)$ *and* $p \in \mathbb{P}_{\geqslant 3}$ *and the function* g_A *increases at at a higher rate then function* g_b*.*

Proof Let the function

$$d_\alpha(p) = g_A(p) - g_\alpha(p) = 1 + 2\sqrt{p} + \alpha \cdot \ln(p) - \ln(p)^2$$

The derivative of function d_α is

$$d'_\alpha(p) = \frac{\alpha - 2\ln(p) + \sqrt{p}}{p} .$$

The analytical solutions for function d'_1 are $5.099\cdots$ and $41.816\cdots$. At the same time, $d'_1(p) < 0$ for $\{7, 11, \cdots, 41\}$ and $d'_1(p) > 0$ for $p \in \mathbb{P}_{\geqslant 3} \setminus \{7, 11, \cdots, 41\}$, meaning that the function d_1 is strictly increasing only over $p \in \mathbb{P}_{\geqslant 3} \setminus \{7, 11, \cdots, 41\}$ (see Figure 3).

For $\alpha = c = 4(2\ln(2)-1) \approx 1.5451774444795623\cdots$, $d'_c(p) > 0$ for any $p \in \mathbb{P}_{\geqslant 3}$, ($d'_c$ is nulled for $p = 16$, but $16 \notin \mathbb{P}_{\geqslant 3}$), then function d_c is strictly increasing for $p \in \mathbb{P}_{\geqslant 3}$ (see Figure c). Because function d_c is strictly increasing and $d_c(3) = \ln(3)\big(8\ln(2) - 4 - \ln(3)\big) + 2\sqrt{3} + 1 \approx 4.954\cdots$, it follows that $d_c(p) > 0$ for any $p \in \mathbb{P}_{\geqslant 3}$.

In $\alpha = b = 6(2\ln(2)-1) \approx 2.3177661667193434\cdots$, function d_b is increasing fastest for any $p \in \mathbb{P}_{\geqslant 3}$ (because $d'_b(p) > d'_\alpha(p)$ for any $p \in \mathbb{P}_{\geqslant 3}$ and $\alpha \geqslant 0$, $\alpha \neq b$). Since $d'_b(p) > 0$ for any $p \in \mathbb{P}_{\geqslant 3}$ and because

$$d_b(3) = \ln(3)\big(12\ln(2) - 6 - \ln(3)\big) + 2\sqrt{3} + 1 \approx 5.803479047342222\cdots .$$

It follows that $d_b(p) > 0$ for any $p \in \mathbb{P}_{\geqslant 3}$ (see Figure 3). □

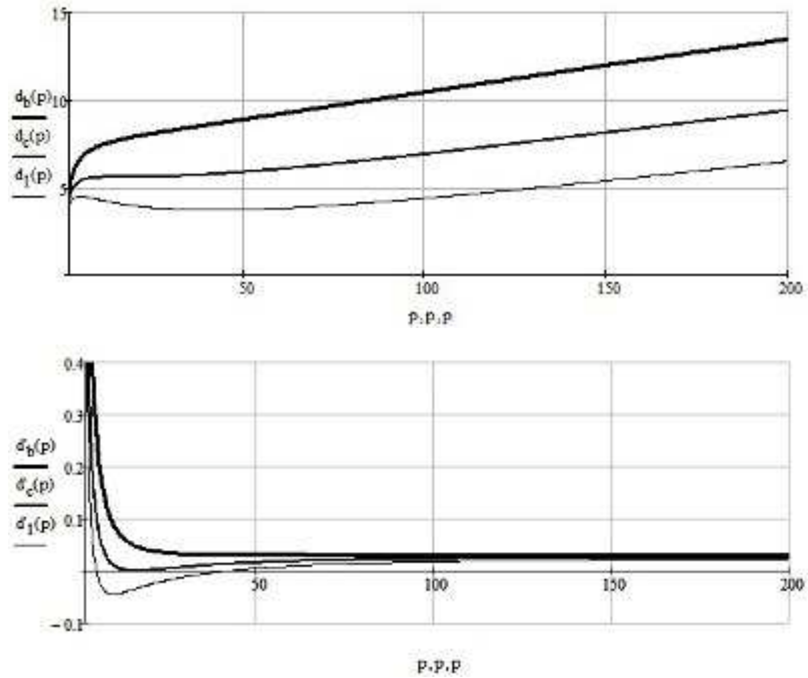

Figure 3 d_α and d'_α functions

Remark 2.9 In order to determine the value of c, we solve the equation $d'_\alpha(p) = 0$ in respect to α. The solution α in respect to p is $\alpha(p) = 2\ln(p) - \sqrt{p}$. We determine p, the solution of $\alpha'(p) = \frac{4-\sqrt{p}}{2p}$. Then it follows that $c = \alpha(16) = 4(2\ln(2) - 1)$.

Remark 2.10 In order to find the value for b, we solve the equation $d''_\alpha(p) = 0$ in respect to α. The solution α in respect to p is $\alpha(p) = 2\ln(p) - \dfrac{\sqrt{p}}{2} - 2$. We determine p, the solution of $\alpha'(p) = \dfrac{8-\sqrt{p}}{4p}$. It follows that $b = \alpha(8) = 6(2\ln(2) - 1)$.

Since function d_b manifests the fastest growth rate we can state that the function g_A increases more rapidly then function g_b.

Let $h(p, g) = f\,(0.5) = \sqrt{p+g} - \sqrt{p} - 1$.

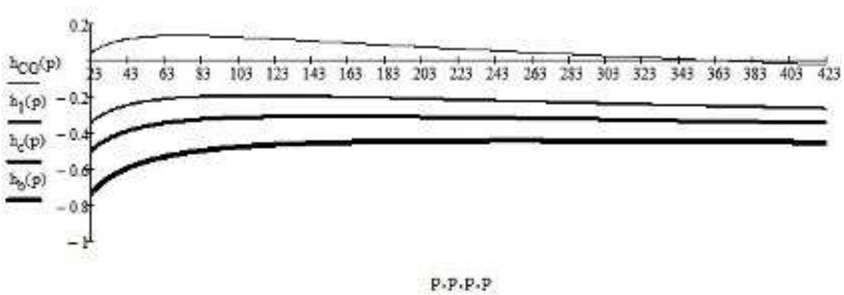

Figure 4 Functions h_b, h_c, h_F and h_{CG}

Theorem 2.11 *For*

$$h_{CG}(p) = h(p, g_{CG}(p)) = \sqrt{p + 2e^{-\gamma}\ln(p)^2} - \sqrt{p} - 1$$

$h_{CG}(p) < 0$ *for* $p \in \{3, 5, 7, 11, 13, 17\} \cup \{359, 367, \cdots\}$ *and*

$$\lim_{p\to\infty} h_{CG}(p) = -1\ .$$

Proof The theorem can be proven by direct computation, as observed in the graph from Figure 4. □

Theorem 2.12 *The function*

$$h_F(p) = h_1(p) = h(p, g_F(p)) = \sqrt{p + \ln(p)^2 - \ln(p)} - \sqrt{p} - 1$$

reaches its maximal value for $p = 111.152\cdots$ *and* $h_F(109) = -0.201205\cdots$ *while* $h_F(113) = -0.201199\cdots$ *and*

$$\lim_{p\to\infty} h_F(p) = -1\ .$$

Proof Again, the theorem can be proven by direct calculation as one can observe from the graph in Figure 4. □

Theorem 2.13 *The function*

$$h_c(p) = h(p, g_c(p)) = \sqrt{p + \ln(p)^2 - c\ln(p)} - \sqrt{p} - 1$$

reaches its maximal value for $p = 152.134\cdots$ *and* $h_c(151) = -0.3105\cdots$ *while* $h_c(157) = -0.3105\cdots$ *and*

$$\lim_{p\to\infty} h_c(p) = -1\ .$$

Proof Again, the theorem can be proven by direct calculation as one can observe from the graph in Figure 4. □

Theorem 2.14 *The function*

$$h_B(p) = h(p, g_B(p)) = \sqrt{\ln(p)^2 - b\ln(p) + p} - \sqrt{p} - 1$$

reaches its maximal value for $p = 253.375\cdots$ *and* $h_B(251) = -0.45017\cdots$ *while* $h_B(257) = -0.45018\cdots$ *and*

$$\lim_{p\to\infty} h_B(p) = -1\ .$$

Proof Again, the theorem can be proven by direct calculation as one can observe from the graph in Figure 4. □

Table 1: Maximal gaps [24, 14, 15]

#	n	p_n	g_n
1	1	2	1
2	2	3	2
3	4	7	4
4	9	23	6
5	24	89	8
6	30	113	14
7	99	523	18
8	154	887	20
9	189	1129	22
10	217	1327	34
11	1183	9551	36
12	1831	15683	44
13	2225	19609	52
14	3385	31397	72
15	14357	155921	86
16	30802	360653	96
17	31545	370261	112
18	40933	492113	114
19	103520	1349533	118
20	104071	1357201	132
21	149689	2010733	148
22	325852	4652353	154
23	1094421	17051707	180
24	1319945	20831323	210
25	2850174	47326693	220
26	6957876	122164747	222
27	10539432	189695659	234
28	10655462	191912783	248
29	20684332	387096133	250
30	23163298	436273009	282
31	64955634	1294268491	288

#	n	p_n	g_n
32	72507380	1453168141	292
33	112228683	2300942549	320
34	182837804	3842610773	336
35	203615628	4302407359	354
36	486570087	10726904659	382
37	910774004	20678048297	384
38	981765347	22367084959	394
39	1094330259	25056082087	456
40	1820471368	42652618343	464
41	5217031687	127976334671	468
42	7322882472	182226896239	474
43	9583057667	241160624143	486
44	11723859927	297501075799	490
45	11945986786	303371455241	500
46	11992433550	304599508537	514
47	16202238656	416608695821	516
48	17883926781	461690510011	532
49	23541455083	614487453523	534
50	28106444830	738832927927	540
51	50070452577	1346294310749	582
52	52302956123	1408695493609	588
53	72178455400	1968188556461	602
54	94906079600	2614941710599	652
55	251265078335	7177162611713	674
56	473258870471	13829048559701	716
57	662221289043	19581334192423	766
58	1411461642343	42842283925351	778
59	2921439731020	90874329411493	804
60	5394763455325	171231342420521	806
61	6822667965940	218209405436543	906
62	35315870460455	1189459969825483	916
63	49573167413483	1686994940955803	924
64	49749629143526	1693182318746371	1132

#	n	p_n	g_n
65	1175661926421598	43841547845541059	1184
66	1475067052906945	55350776431903243	1198
67	2133658100875638	80873624627234849	1220
68	5253374014230870	203986478517455989	1224
69	5605544222945291	218034721194214273	1248
70	7784313111002702	305405826521087869	1272
71	8952449214971382	352521223451364323	1328
72	10160960128667332	401429925999153707	1356
73	10570355884548334	418032645936712127	1370
74	20004097201301079	804212830686677669	1442
75	34952141021660495	1425172824437699411	1476

We denote by $a_n = \lfloor g_A(p_n) \rfloor$ (Andrica's conjecture), by $cg_n = \lfloor g_{CG}(p_n) \rfloor$ (Cramér-Grandville's conjecture) by $f_n = \lfloor g_F(p_n) \rfloor$ (Firoozbakht's conjecture), by $c_n = \lfloor g_c(p_n) \rfloor$ and $b_n = \lfloor g_b(p_n) \rfloor$.

The columns of Table 2 represent the values of the maximal gaps a_n, cg_n, f_n, c_n, b_n and g_n, [14, 2, 28, 15]. Note the Cramér-Grandville's conjecture as well as Firoozbakht's conjecture are confirmed when $n \geqslant 9$ (for $p_9 = 23$, the forth row in the table of *maximal gaps*).

Table 2: Approximative values of maximal gaps

#	a_n	cg_n	f_n	c_n	b_n	g_n
1	3	0	-1	-1	-2	1
2	4	1	0	-1	-2	2
3	6	4	1	0	-1	4
4	10	11	6	4	2	6
5	19	22	15	13	9	8
6	22	25	17	15	11	14
7	46	43	32	29	24	18
8	60	51	39	35	30	20
9	68	55	42	38	33	22
10	73	58	44	40	35	34
11	196	94	74	69	62	36

#	a_n	cg_n	f_n	c_n	b_n	g_n
12	251	104	83	78	70	44
13	281	109	87	82	74	52
14	355	120	96	91	83	72
15	790	160	131	123	115	86
16	1202	183	150	143	134	96
17	1217	184	151	144	134	112
18	1404	192	158	151	141	114
19	2324	223	185	177	166	118
20	2330	223	185	177	166	132
21	2837	236	196	188	177	148
22	4314	264	220	211	200	154
23	8259	311	260	251	238	180
24	9129	318	267	257	244	210
25	13759	350	294	285	271	220
26	22106	389	328	317	303	222
27	27547	407	344	333	319	234
28	27707	408	344	334	319	248
29	39350	439	371	360	345	250
30	41775	444	375	365	349	282
31	71952	494	419	407	391	288
32	76241	499	423	412	396	292
33	95937	521	443	431	414	320
34	123978	546	464	452	435	336
35	131186	552	469	457	440	354
36	207142	598	510	497	479	382
37	287598	633	540	527	509	384
38	299113	637	544	531	512	394
39	316583	643	549	536	517	456
40	413051	672	574	561	542	464
41	715476	734	628	614	594	468
42	853761	754	646	632	612	474
43	982163	771	660	646	626	486
44	1090874	783	671	657	636	490

#	a_n	cg_n	f_n	c_n	b_n	g_n
45	1101584	784	672	658	637	500
46	1103811	785	672	658	637	514
47	1290905	803	689	674	653	516
48	1358957	810	694	679	659	532
49	1567786	827	709	694	673	534
50	1719108	838	719	704	683	540
51	2320599	875	752	736	715	582
52	2373770	878	754	739	717	588
53	2805843	899	773	757	735	602
54	3234157	918	788	773	751	652
55	5358046	983	846	830	807	674
56	7437486	1028	885	868	845	716
57	8850161	1051	906	889	865	766
58	13090804	1106	953	936	912	778
59	19065606	1159	1000	983	958	804
60	26171079	1206	1041	1023	998	806
61	29543826	1224	1057	1039	1013	906
62	68977097	1353	1170	1151	1124	916
63	82146088	1380	1194	1175	1148	924
64	82296594	1380	1194	1175	1148	1132
65	418767467	1648	1430	1409	1379	1184
66	470534915	1668	1447	1426	1396	1198
67	568765768	1701	1476	1455	1425	1220
68	903297246	1783	1548	1526	1496	1224
69	933883765	1789	1553	1532	1501	1248
70	1105270694	1820	1580	1558	1527	1272
71	1187469955	1833	1592	1570	1538	1328
72	1267169959	1844	1602	1580	1549	1356
73	1293108884	1848	1605	1583	1552	1370
74	1793558286	1908	1658	1636	1604	1442
75	2387612050	1962	1705	1682	1650	1476

Table 2, the graphs in 5 and 6 stand proof that

$$g_n = p_{n+1} - p_n < \ln(p_n)^2 - c \cdot \ln(p_n) \ , \tag{2.3}$$

for $p \in \{89, 113, \cdots, 1425172824437699411\}$. By Theorem 2.6 we can say that inequality (2.3) is true for any $p \in \mathbb{P}_{\geqslant 89} \setminus \mathbb{P}_{\geqslant 1425172824437699413}$.

This valid statements in respect to the inequality (2.3) allows us to consider the following hypothesis.

Conjecture 2.1 *The relation (2.3) is true for any $p \in \mathbb{P}_{\geqslant 29}$.*

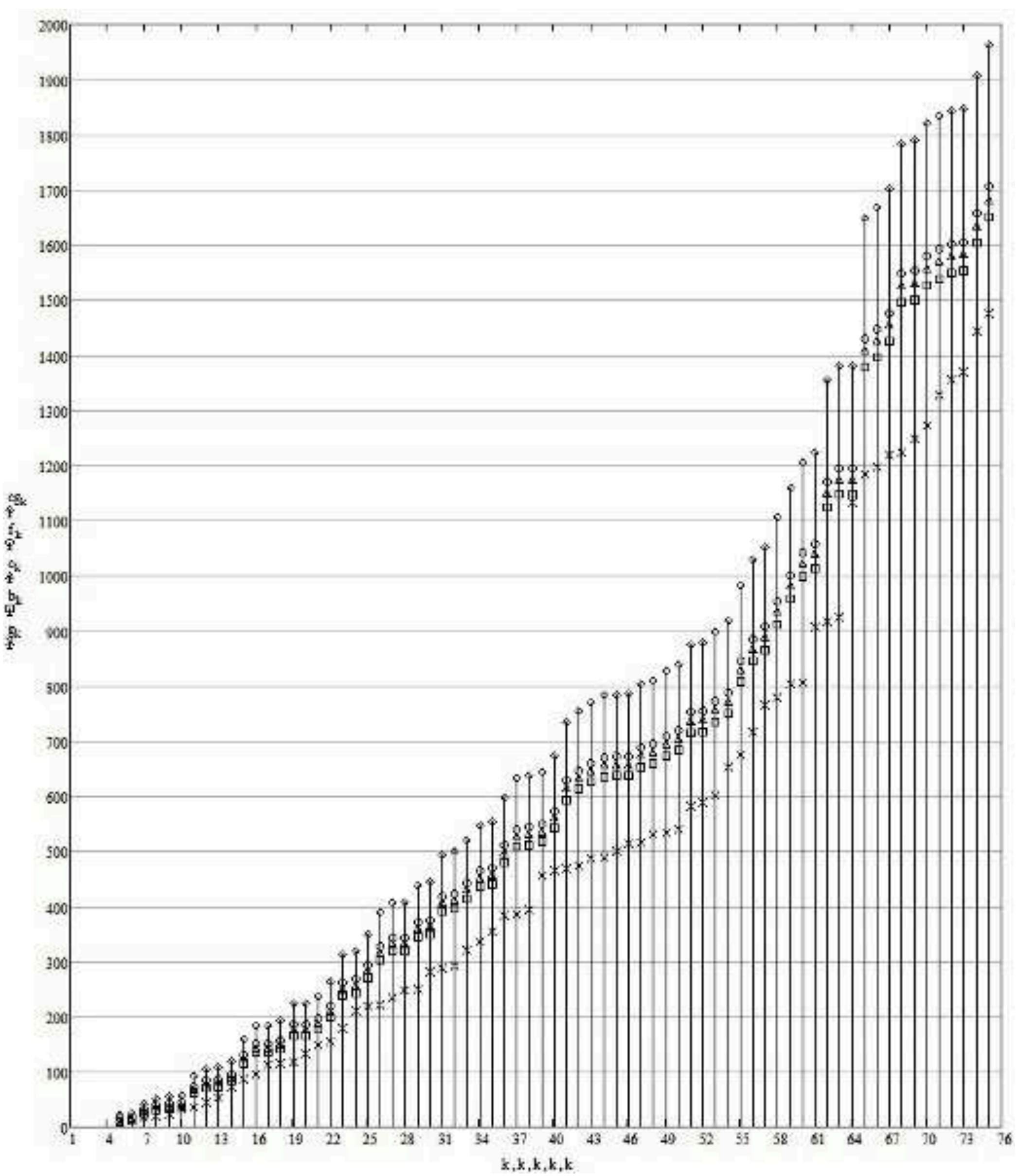

Figure 5 Maximal gaps graph

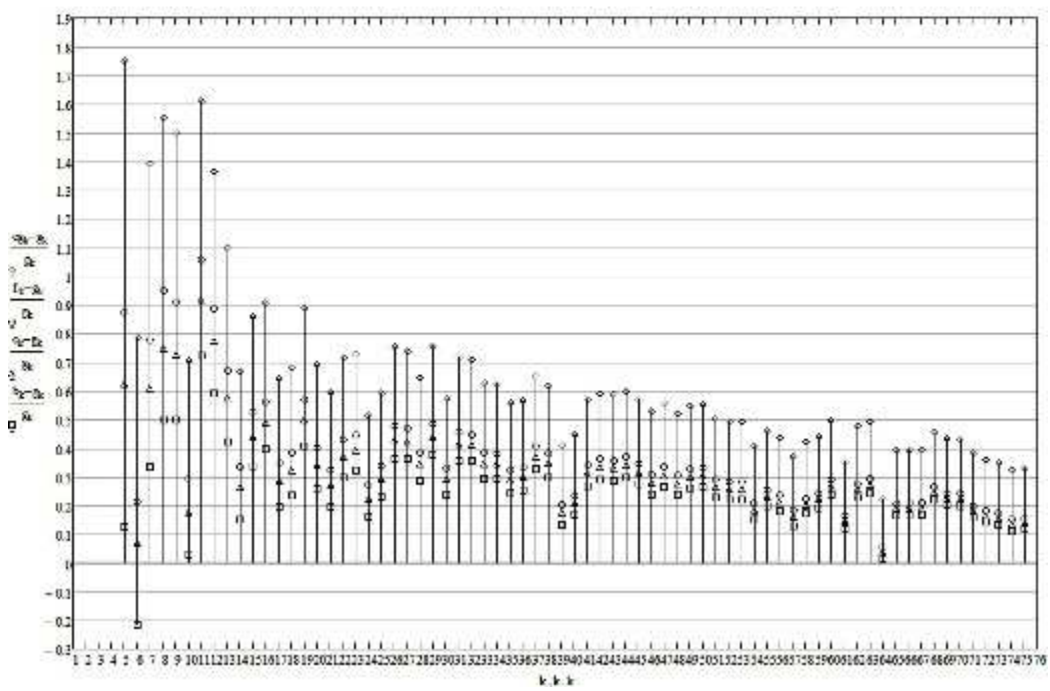

Figure 6 Relative errors of cg, f, c and b in respect to g

Let $g_\alpha : \mathbb{P}_{\geqslant 3} \to \mathbb{R}_+$,

$$g_\alpha(p) = \ln(p)^2 - \alpha \cdot \ln(p)$$

and $h_\alpha : \mathbb{P}_{\geqslant 3} \times [0,1] \to \mathbb{R}$, with p fixed,

$$h_\alpha(p,x) = (p + g_\alpha(p))^x - p^x - 1$$

that, according to Theorem 2.1, is strictly increasing and convex over its domain, and according to the Corollary 2.2 has a unique solution over the interval $[0,1]$.

We solve the following equation, equivalent to (2.2)

$$h_c(p,x) = \left(p + \ln(p)^2 - c\ln(p)\right)^x - p^x - 1 = 0 \ , \tag{2.4}$$

in respect to x, for any $p \in \mathbb{P}_{\geqslant 29}$. In accordance to Theorem 2.5 the solution for equation (2.2) is greater then the solution to equation (2.4). Therefore if we prove that the solutions to equation (2.4) are greater then 0.5 then, even more so, the solutions to (2.2) are greater then 0.5 .

For equation $h_\alpha(p,x) = 0$ we consider the secant method, with the initial iterations x_0 and x_1 (see Figure 7). The iteration x_2 is given by

$$x_2 = \frac{x_1 \cdot h_\alpha(p,x_0) - x_0 \cdot h_\alpha(p,x_1)}{h_\alpha(p,x_1) - h_\alpha(p,x_0)} \ . \tag{2.5}$$

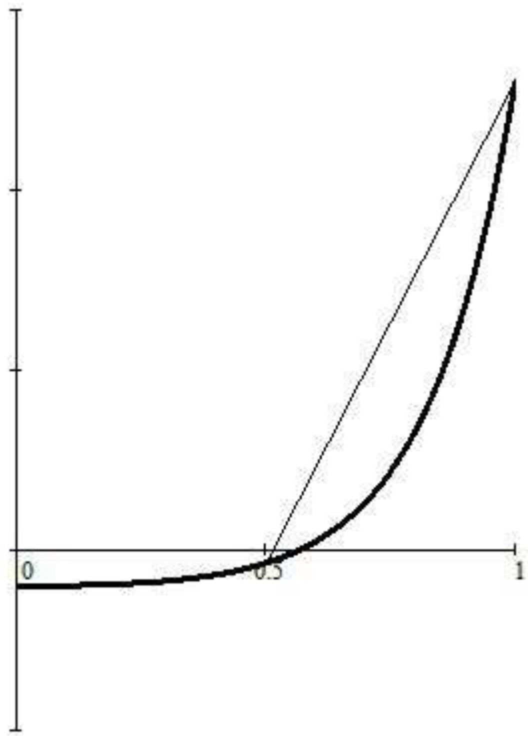

Figure 7 Function f and the secant method

If Andrica's conjecture, $\sqrt{p+g}-\sqrt{p}-1<0$ for any $p\in\mathbb{P}_{\geqslant 3}$, $g\in\mathbb{N}^*$ and $p>g\geqslant 2$, is true, then $h_\alpha(p,0.5)<0$ (according to Remark 1.1 if Legendre's conjecture is true then Andrica's conjecture is also true), and $h_\alpha(p,1)>0$. Since function $h_\alpha(p,\cdot)$ is strictly increasing and convex, iteration x_2 **approximates the solution to the equation** $h_\alpha(p,x)=0$, (in respect to x). Some simple calculation show that a the solution x_2 in respect to h_α, p, x_0 and x_1 is:

$$a(p,h_\alpha,x_0,x_1)=\frac{x_1\cdot h_\alpha(p,x_0)-x_0h-\alpha(p,x_1)}{h_\alpha(p,x_1)-h_\alpha(p,x_0)}\ . \tag{2.6}$$

Let $a_\alpha(p)=a(p,h_\alpha,0.5,1)$, then

$$a_\alpha(p)=\frac{1}{2}+\frac{1+\sqrt{p}-\sqrt{\ln(p)^2-\alpha\ln(p)+p}}{2\big(\ln(p)^2-\alpha\ln(p)+\sqrt{p}-\sqrt{\ln(p)^2-\alpha\ln(p)+p}\big)}\ . \tag{2.7}$$

Theorem 2.15 *The function $a_c(p)$, that approximates the solution to equation (2.4) has values in the open interval $(0.5,1)$ for any $p\in\mathbb{P}_{\geqslant 29}$.*

Proof According to Theorem 2.8 for $\alpha=c=4(2\ln(2)-1)$ we have $\ln(p)^2-c\cdot\ln(p)<2\sqrt{p}+1$ for any $p\in\mathbb{P}_{\geqslant 29}$.

We can rewrite function a_c as

$$a_c(p)=\frac{1}{2}+\frac{1+\sqrt{p}-\sqrt{p+c}}{2\big(c+\sqrt{p}-\sqrt{p+c}\big)}\ .$$

which leads to

$$\frac{1+\sqrt{p}-\sqrt{p+c}}{2\big(c+\sqrt{p}-\sqrt{p+c}\big)} > 0 \ ,$$

it follows that $a_c(p) > \frac{1}{2}$ for $p \in \mathbb{P}_{\geqslant 3}$ (see Figure 8) and we have

$$\lim_{p\to\infty} a_c(p) = \frac{1}{2} \ . \qquad \square$$

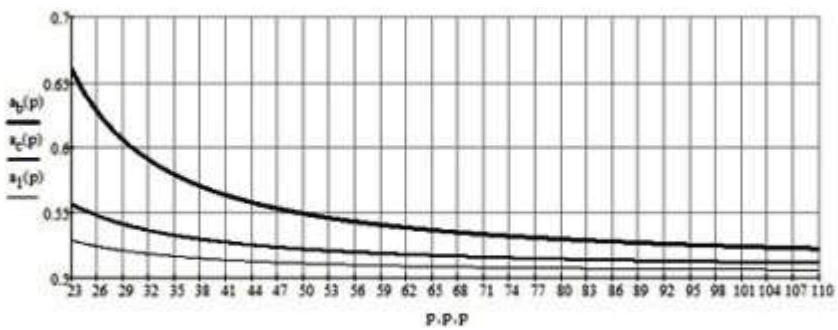

Figure 8 The graphs for functions a_b, a_c and a_1

For $p \in \{2, 3, 5, 7, 11, 13, 17, 19, 23\}$ and the respective gaps we solve the following equations (2.2).

$$\begin{cases} (2+1)^x - 2^x = 1 \ , & s = 1 \\ (3+2)^x - 3^x = 1 \ , & s = 0.7271597432435757\cdots \\ (5+2)^x - 5^x = 1 \ , & s = 0.7632032096058607\cdots \\ (7+4)^x - 7^x = 1 \ , & s = 0.5996694211239202\cdots \\ (11+2)^x - 11^x = 1 \ , & s = 0.8071623463868518\cdots \\ (13+4)^x - 13^x = 1 \ , & s = 0.6478551304201904\cdots \\ (17+2)^x - 17^x = 1 \ , & s = 0.8262031187421179\cdots \\ (19+4)^x - 19^x = 1 \ , & s = 0.6740197879899883\cdots \\ (23+6)^x - 23^x = 1 \ , & s = 0.6042842019286720\cdots \ . \end{cases} \tag{2.8}$$

Corollary 2.9 *We proved that the approximative solutions of equation (2.4) are* > 0.5 *for any* $n \geqslant 10$*, then the solutions of equation (2.2) are* > 0.5 *for any* $n \geqslant 10$ *. If we consider the exceptional cases (2.8) we can state that the equation (1.1) has solutions in* $s \in (0.5, 1]$ *for any* $n \in \mathbb{N}^*$.

§3. Smarandache Constant

We order the solutions to equation (2.2) in Table 1 using the maximal gaps.

Table 3: Equation (2.2) solutions

p	g	solution for (2.2)
113	14	0.5671481305206224...
1327	34	0.5849080865740931...
7	4	0.5996694211239202...
23	6	0.6042842019286720...
523	18	0.6165497314215637...
1129	22	0.6271418980644412...
887	20	0.6278476315319166...
31397	72	0.6314206007048127...
89	8	0.6397424613256825...
19609	52	0.6446915279533268...
15683	44	0.6525193297681189...
9551	36	0.6551846556887808...
155921	86	0.6619804741301879...
370261	112	0.6639444999972240...
492113	114	0.6692774164975257...
360653	96	0.6741127001176469...
1357201	132	0.6813839139412406...
2010733	148	0.6820613370357171...
1349533	118	0.6884662952427394...
4652353	154	0.6955672852207547...
20831323	210	0.7035651178160084...
17051707	180	0.7088121412466053...
47326693	220	0.7138744163020114...
122164747	222	0.7269826061830018...
3	2	0.7271597432435757...
191912783	248	0.7275969819805509...
189695659	234	0.7302859105830866...
436273009	282	0.7320752818323865...
387096133	250	0.7362578381533295...
1294268491	288	0.7441766589716590...
1453168141	292	0.7448821415605216...

p	g	solution for (2.2)
2300942549	320	0.7460035467176455...
4302407359	354	0.7484690049408947...
3842610773	336	0.7494840618593505...
10726904659	382	0.7547601234459729...
25056082087	456	0.7559861641728429...
42652618343	464	0.7603441937898209...
22367084959	394	0.7606955951728551...
20678048297	384	0.7609716068556747...
127976334671	468	0.7698203623795380...
182226896239	474	0.7723403816143177...
304599508537	514	0.7736363009251175...
241160624143	486	0.7737508697071668...
303371455241	500	0.7745991865337681...
297501075799	490	0.7751693424982924...
461690510011	532	0.7757580339651479...
416608695821	516	0.7760253389165942...
614487453523	534	0.7778809828805762...
1408695493609	588	0.7808871027951452...
1346294310749	582	0.7808983645683428...
2614941710599	652	0.7819658004744228...
1968188556461	602	0.7825687226257725...
7177162611713	674	0.7880214782837229...
13829048559701	716	0.7905146362137986...
19581334192423	766	0.7906829063252424...
42842283925351	778	0.7952277512573828...
90874329411493	804	0.7988558653770882...
218209405436543	906	0.8005126614171458...
171231342420521	806	0.8025304565279002...
1693182318746371	1132	0.8056470803187964...
1189459969825483	916	0.8096231085041140...
1686994940955803	924	0.8112057874892308...
43841547845541060	1184	0.8205327998695296...
55350776431903240	1198	0.8212591131062218...

p	g	solution for (2.2)
80873624627234850	1220	0.8224041089823987...
218034721194214270	1248	0.8258811322716928...
352521223451364350	1328	0.8264955008480679...
1425172824437699300	1476	0.8267652954810718...
305405826521087900	1272	0.8270541728027422...
203986478517456000	1224	0.8271121951019150...
418032645936712100	1370	0.8272229385637846...
401429925999153700	1356	0.8272389079572986...
804212830686677600	1442	0.8288714147741382...
2	1	1

§4 Conclusions

Therefore, if Legendre's conjecture is true then Andrica's conjecture is also true according to Paz [17]. Andrica's conjecture validated the following sequence of inequalities $a_n > cg_n > f_n > c_n > b_n > g_n$ for any n natural number, $5 \leqslant n \leqslant 75$, in Tables 2. The inequalities $c_n < g_n$ for any natural n, $5 \leqslant n \leqslant 75$, from Table 2 allows us to state Conjecture 2.1.

If Legendre's conjecture and Conjecture 2.1 are true, then Smarandache's conjecture is true.

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