

## $N^*C^*$ – Smarandache Curves of Mannheim Curve Couple According to Frenet Frame

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**Abstract:** In this paper, when the unit Darboux vector of the partner curve of Mannheim curve are taken as the position vectors, the curvature and the torsion of Smarandache curve are calculated. These values are expressed depending upon the Mannheim curve. Besides, we illustrate example of our main results.

**Key Words:** Mannheim curve, Mannheim partner curve, Smarandache Curves, Frenet invariants.

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### §1. Introduction

A regular curve in Minkowski space-time, whose position vector is composed by Frenet frame vectors on another regular curve, is called a Smarandache curve ([12]). Special Smarandache curves have been studied by some authors .

Melih Turgut and Süha Yılmaz studied a special case of such curves and called it Smarandache  $TB_2$  curves in the space  $E_1^4$  ([12]). Ahmad T.Ali studied some special Smarandache curves in the Euclidean space. He studied Frenet-Serret invariants of a special case ([1]). Muhammed Çetin , Yılmaz Tunçer and Kemal Karacan investigated special Smarandache curves according to Bishop frame in Euclidean 3-Space and they gave some differential geometric properties of Smarandache curves, also they found the centers of the osculating spheres and curvature spheres of Smarandache curves ([5]). Şenyurt and Çalışkan investigated special Smarandache curves in terms of Sabban frame of spherical indicatrix curves and they gave some characterization of Smarandache curves ([4]). Özcan Bektaş and Salim Yüce studied some special Smarandache curves according to Darboux Frame in  $E^3$  ([2]). Nurten Bayrak, Özcan Bektaş and Salim Yüce studied some special Smarandache curves in  $E_1^3$  [3]. Kemal Taşköprü, Murat Tosun studied special Smarandache curves according to Sabban frame on  $S^2$  ([11]).

In this paper, special Smarandache curve belonging to  $\alpha^*$  Mannheim partner curve such as  $N^*C^*$  drawn by Frenet frame are defined and some related results are given.

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## §2. Preliminaries

The Euclidean 3-space  $E^3$  be inner product given by

$$\langle , \rangle = x_1^2 + x_2^2 + x_3^2$$

where  $(x_1, x_2, x_3) \in E^3$ . Let  $\alpha : I \rightarrow E^3$  be a unit speed curve denote by  $\{T, N, B\}$  the moving Frenet frame . For an arbitrary curve  $\alpha \in E^3$ , with first and second curvature,  $\kappa$  and  $\tau$  respectively, the Frenet formulae is given by ([6], [9])

$$\begin{cases} T' = \kappa N \\ N' = -\kappa T + \tau B \\ B' = -\tau N. \end{cases} \quad (2.1)$$

For any unit speed  $\alpha : I \rightarrow \mathbb{E}^3$ , the vector  $W$  is called Darboux vector defined by

$$W = \tau(s)T(s) + \kappa(s) + B(s).$$

If consider the normalization of the Darboux  $C = \frac{1}{\|W\|}W$ , we have

$$\begin{aligned} \cos \varphi &= \frac{\kappa(s)}{\|W\|}, \quad \sin \varphi = \frac{\tau(s)}{\|W\|}, \\ C &= \sin \varphi T(s) + \cos \varphi B(s) \end{aligned} \quad (2.2)$$

where  $\angle(W, B) = \varphi$ . Let  $\alpha : I \rightarrow \mathbb{E}^3$  and  $\alpha^* : I \rightarrow \mathbb{E}^3$  be the  $C^2$ - class differentiable unit speed two curves and let  $\{T(s), N(s), B(s)\}$  and  $\{T^*(s), N^*(s), B^*(s)\}$  be the Frenet frames of the curves  $\alpha$  and  $\alpha^*$ , respectively. If the principal normal vector  $N$  of the curve  $\alpha$  is linearly dependent on the binormal vector  $B$  of the curve  $\alpha^*$ , then  $(\alpha)$  is called a Mannheim curve and  $(\alpha^*)$  a Mannheim partner curve of  $(\alpha)$ . The pair  $(\alpha, \alpha^*)$  is said to be Mannheim pair ([7], [8]). The relations between the Frenet frames  $\{T(s), N(s), B(s)\}$  and  $\{T^*(s), N^*(s), B^*(s)\}$  are as follows:

$$\begin{cases} T^* = \cos \theta T - \sin \theta B \\ N^* = \sin \theta T + \cos \theta B \\ B^* = N \end{cases} \quad (2.3)$$

$$\begin{cases} \cos \theta = \frac{ds^*}{ds} \\ \sin \theta = \lambda \tau^* \frac{ds^*}{ds} \end{cases} . \quad (2.4)$$

where  $\angle(T, T^*) = \theta$  ([8]).

**Theorem 2.1**([7]) *The distance between corresponding points of the Mannheim partner curves in  $\mathbb{E}^3$  is constant.*

**Theorem 2.2** Let  $(\alpha, \alpha^*)$  be a Mannheim pair curves in  $\mathbb{E}^3$ . For the curvatures and the torsions of the Mannheim curve pair  $(\alpha, \alpha^*)$  we have,

$$\begin{cases} \kappa = \tau^* \sin \theta \frac{ds^*}{ds} \\ \tau = -\tau^* \cos \theta \frac{ds^*}{ds} \end{cases} \quad (2.5)$$

and

$$\begin{cases} \kappa^* = \frac{d\theta}{ds^*} = \theta' \frac{\kappa}{\lambda\tau\sqrt{\kappa^2 + \tau^2}} \\ \tau^* = (\kappa \sin \theta - \tau \cos \theta) \frac{ds^*}{ds} \end{cases} \quad (2.6)$$

**Theorem 2.3** Let  $(\alpha, \alpha^*)$  be a Mannheim pair curves in  $\mathbb{E}^3$ . For the torsions  $\tau^*$  of the Mannheim partner curve  $\alpha^*$  we have

$$\tau^* = \frac{\kappa}{\lambda\tau}$$

**Theorem 2.4**([10]) Let  $(\alpha, \alpha^*)$  be a Mannheim pair curves in  $\mathbb{E}^3$ . For the vector  $C^*$  is the direction of the Mannheim partner curve  $\alpha^*$  we have

$$C^* = \frac{1}{\sqrt{1 + \left(\frac{\theta'}{\|W\|}\right)^2}} C + \frac{\frac{\theta'}{\|W\|}}{\sqrt{1 + \left(\frac{\theta'}{\|W\|}\right)^2}} N \quad (2.7)$$

where the vector  $C$  is the direction of the Darboux vector  $W$  of the Mannheim curve  $\alpha$ .

### §3. $N^*C^*$ – Smarandache Curves of Mannheim Curve Couple According to Frenet Frame

Let  $(\alpha, \alpha^*)$  be a Mannheim pair curves in  $E^3$  and  $\{T^*N^*B^*\}$  be the Frenet frame of the Mannheim partner curve  $\alpha^*$  at  $\alpha^*(s)$ . In this case,  $N^*C^*$  - Smarandache curve can be defined by

$$\beta_1(s) = \frac{1}{\sqrt{2}}(N^* + C^*). \quad (3.1)$$

Solving the above equation by substitution of  $N^*$  and  $C^*$  from (2.3) and (2.7), we obtain

$$\beta_1(s) = \frac{(\cos \theta \|W\| + \sin \theta \sqrt{\theta'^2 + \|W\|^2})T + \theta' N + (\cos \theta \sqrt{\theta'^2 + \|W\|^2} - \sin \theta \|W\|)B}{\sqrt{\theta'^2 + \|W\|^2}}. \quad (3.2)$$

The derivative of this equation with respect to  $s$  is as follows,

$$\begin{aligned}
T_{\beta_1}(s) = & \frac{\left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \cos \theta - \frac{\theta' \kappa \cos \theta}{\lambda \tau \|W\|} \right] T + \left[ \frac{\kappa}{\lambda \tau} - \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\|W\|}{\theta'} \right] N}{\sqrt{\left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]^2 + \frac{\kappa(\theta'^2 + \|W\|^2)}{\lambda \tau \|W\|} \left[ \frac{\kappa}{\lambda \tau \|W\|} - 2 \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{1}{\theta'} \right]}} \\
& + \frac{\left[ \frac{\theta' \kappa \sin \theta}{\lambda \tau \|W\|} - \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \sin \theta \right] B}{\sqrt{\left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]^2 + \frac{\kappa(\theta'^2 + \|W\|^2)}{\lambda \tau \|W\|} \left[ \frac{\kappa}{\lambda \tau \|W\|} - 2 \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{1}{\theta'} \right]}} \quad (3.3)
\end{aligned}$$

In order to determine the first curvature and the principal normal of the curve  $\beta_1(s)$ , we formalize

$$\sqrt{2} \left[ (\bar{r}_1 \cos \theta + \bar{r}_2 \sin \theta) T + \bar{r}_3 N + (-\bar{r}_1 \sin \theta + \bar{r}_2 \cos \theta) B \right]$$

$$T'_{\beta_1}(s) = \frac{\left( \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]^2 + \frac{\kappa(\theta'^2 + \|W\|^2)}{\lambda \tau \|W\|} \left[ \frac{\kappa}{\lambda \tau \|W\|} - 2 \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{1}{\theta'} \right] \right)^2}{\sqrt{2}}$$

where

$$\begin{aligned}
\bar{r}_1 = & 2 \left( \frac{\kappa}{\lambda \tau} \right)^2 \left( \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]' \right) \left( \frac{\theta'}{\sqrt{\theta'^2 + \|W\|^2}} \right) \\
& - \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right) \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right] \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]' \\
& \left( \frac{\theta'}{\sqrt{\theta'^2 + \|W\|^2}} \right)^2 - \left( \frac{\kappa}{\lambda \tau} \right) \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right] \left( \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]' \right) \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right) \left( \frac{\theta'}{\sqrt{\theta'^2 + \|W\|^2}} \right) - \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]' \\
& \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]^4 \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right) - \left( \frac{\kappa}{\lambda \tau} \right)^2 \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right)' - \left( \frac{\kappa}{\lambda \tau} \right)^2 \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]'
\end{aligned}$$

$$\begin{aligned}
 & \times \left[ \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]^2 \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right) + 2 \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right) \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]^3 \\
 & \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right) \left( \frac{\theta'}{\sqrt{\theta'^2 + \|W\|^2}} \right) + 2 \left( \frac{\kappa}{\lambda \tau} \right) \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]^3 \\
 & \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)^2 - \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right)^2 \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]^2 \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right) \\
 & - 2 \kappa^* \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right)' \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right] \left( \frac{\theta'}{\sqrt{\theta'^2 + \|W\|^2}} \right) - 2 \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right)' \\
 & \left( \frac{\kappa}{\lambda \tau} \right) \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right] \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right) - \tau^* \left( \frac{\kappa}{\lambda \tau} \right)' \\
 & \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right] \left( \frac{\theta'}{\sqrt{\theta'^2 + \|W\|^2}} \right) + \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right)' \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \right. \\
 & \left. \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right] \left( \frac{\theta'}{\sqrt{\theta'^2 + \|W\|^2}} \right)^2 - \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right)' \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]^2 \\
 & + \left( \frac{\kappa}{\lambda \tau} \right)' \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]^2 \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right) \left( \frac{\theta'}{\sqrt{\theta'^2 + \|W\|^2}} \right) \\
 & \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right) \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right] \left( \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]' \right) \\
 & + \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right) \left( \frac{\kappa}{\lambda \tau} \right) \left( \frac{\kappa}{\lambda \tau} \right)' - \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right) \left( \frac{\kappa}{\lambda \tau} \right) \left( \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]' \right) \\
 & \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right) - \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right] \left( \frac{\kappa}{\lambda \tau} \right)' \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right) \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right), \\
 \\
 \bar{r}_2 & = \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right) \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]^3 \left( \frac{\theta'}{\sqrt{\theta'^2 + \|W\|^2}} \right) + 3 \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right)^3 \\
 & \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right] \left( \frac{\theta'}{\sqrt{\theta'^2 + \|W\|^2}} \right) + 3 \left( \frac{\kappa}{\lambda \tau} \right)^2 \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right)
 \end{aligned}$$

$$\begin{aligned}
& \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right] \left( \frac{\theta'}{\sqrt{\theta'^2 + \|W\|^2}} \right) - 2 \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right)^2 \\
& \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]^2 \left( \frac{\theta'}{\sqrt{\theta'^2 + \|W\|^2}} \right)^2 - \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right)^2 \\
& \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]^2 - \left( \frac{\kappa}{\lambda \tau} \right)^2 \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]^2 \\
& + 3 \left( \frac{\kappa}{\lambda \tau} \right)^3 \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right] \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right) + \left( \frac{\kappa}{\lambda \tau} \right) \\
& \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]^3 \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right) - 2 \left( \frac{\kappa}{\lambda \tau} \right)^2 \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \right. \\
& \left. \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]^2 \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)^2 - 4 \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right) \left( \frac{\kappa}{\lambda \tau} \right) \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \right. \\
& \left. \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]^2 \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right) \left( \frac{\theta'}{\sqrt{\theta'^2 + \|W\|^2}} \right) - \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right)^4 - 2 \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right)^2 \\
& \left( \frac{\kappa}{\lambda \tau} \right)^2 + 3 \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right)^2 \left( \frac{\kappa}{\lambda \tau} \right) \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right] \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right) \\
\bar{r}_3 = & 2 \left( \frac{\kappa}{\lambda \tau} \right)' \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]^2 + \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right)^2 \left( \frac{\kappa}{\lambda \tau} \right)' - 2 \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right) \\
& \left( \frac{\kappa}{\lambda \tau} \right)' \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right] \left( \frac{\theta'}{\sqrt{\theta'^2 + \|W\|^2}} \right) - \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right)^2 \\
& \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]' \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right) + \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right) \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \right. \\
& \left. \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right] \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]' \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right) \left( \frac{\theta'}{\sqrt{\theta'^2 + \|W\|^2}} \right) \\
& + \left( \frac{\kappa}{\lambda \tau} \right) \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right] \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]'
\end{aligned}$$

$$\begin{aligned}
 & \times \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)^2 - \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]^4 \left( \frac{\theta'}{\sqrt{\theta'^2 + \|W\|^2}} \right) \\
 & - \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right)^2 \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]^2 \left( \frac{\theta'}{\sqrt{\theta'^2 + \|W\|^2}} \right) - \left( \frac{\kappa}{\lambda \tau} \right)' \\
 & \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]^2 \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)^2 - \left( \frac{\kappa}{\lambda \tau} \right)^2 \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \right. \\
 & \left. \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]^2 \left( \frac{\theta'}{\sqrt{\theta'^2 + \|W\|^2}} \right) + 2 \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right) \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]^3 \\
 & \left( \frac{\theta'}{\sqrt{\theta'^2 + \|W\|^2}} \right)^2 + 2 \left( \frac{\kappa}{\lambda \tau} \right) \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]^3 \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right) \\
 & \left( \frac{\theta'}{\sqrt{\theta'^2 + \|W\|^2}} \right) - \left( \frac{\kappa}{\lambda \tau} \right) \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right] \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \right. \\
 & \left. \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]' - \left( \frac{\kappa}{\lambda \tau} \right) \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right) \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right)' + \left( \frac{\kappa}{\lambda \tau} \right) \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right) \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \right. \\
 & \left. \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]' \left( \frac{\theta'}{\sqrt{\theta'^2 + \|W\|^2}} \right) + \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right)' \left( \frac{\kappa}{\lambda \tau} \right) \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \right. \\
 & \left. \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right] \left( \frac{\theta'}{\sqrt{\theta'^2 + \|W\|^2}} \right) + \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right) \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right)' \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \right. \\
 & \left. \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right] \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right) - \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right)' \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]^2 \\
 & \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right) \left( \frac{\theta'}{\sqrt{\theta'^2 + \|W\|^2}} \right).
 \end{aligned}$$

The first curvature is

$$\kappa_{\beta_1} = \frac{\sqrt{2}(\sqrt{\bar{r}_1^2 + \bar{r}_2^2 + \bar{r}_3^2})}{\left( \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]^2 + \frac{\kappa(\theta'^2 + \|W\|^2)}{\lambda \tau \|W\|} \left[ \frac{\kappa}{\lambda \tau \|W\|} - 2 \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{1}{\theta'} \right] \right)^2}.$$

The principal normal vector field and the binormal vector field are respectively given by

$$N_{\beta_1} = \frac{(\bar{r}_1 \cos \theta + \bar{r}_2 \sin \theta)T + \bar{r}_3 N + (-\bar{r}_1 \sin \theta + \bar{r}_2 \cos \theta)B}{\sqrt{\bar{r}_1^2 + \bar{r}_2^2 + \bar{r}_3^2}}, \quad (3.4)$$

$$B_{\beta_1}(s) = \frac{\xi_1}{\xi_4}T + \frac{\xi_2}{\xi_4}N + \frac{\xi_3}{\xi_4}B, \quad (3.5)$$

where

$$\left\{ \begin{array}{l} \xi_1 = \bar{r}_2 \cos \theta \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\|W\|}{\theta'} - \bar{r}_2 \cos \theta \frac{\kappa}{\lambda\tau} - \left[ \bar{r}_1 \frac{\kappa}{\lambda\tau} - \bar{r}_1 \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\|W\|}{\theta'} \right. \\ \quad \left. - \bar{r}_3 \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' + \bar{r}_3 \left( \frac{\theta' \kappa}{\lambda\tau \|W\|} \right) \right] \sin \theta \\ \xi_2 = \bar{r}_1 \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' - \frac{\theta' \kappa}{\lambda\tau \|W\|} \right] \\ \xi_3 = \bar{r}_2 \sin \theta \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\|W\|}{\theta'} - \frac{\kappa}{\lambda\tau} \right] + \left[ \bar{r}_1 \frac{\kappa}{\lambda\tau} - \bar{r}_1 \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\|W\|}{\theta'} \right. \\ \quad \left. - \bar{r}_3 \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' + \bar{r}_3 \frac{\theta' \kappa}{\lambda\tau \|W\|} \right] \cos \theta \\ \xi_4 = \sqrt{\left( (\bar{r}_1^2 + \bar{r}_2^2 + \bar{r}_3^2) \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]^2 + (\bar{r}_1^2 + \bar{r}_2^2 + \bar{r}_3^2) \frac{\kappa(\theta'^2 + \|W\|^2)}{\lambda\tau \|W\|} \right.} \\ \quad \left. \left[ \frac{\kappa}{\lambda\tau \|W\|} - 2 \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{1}{\theta'} \right] \right)} . \end{array} \right.$$

In order to calculate the torsion of the curve  $\beta_1$ , we differentiate

$$\begin{aligned} \beta_1'' &= \frac{1}{\sqrt{2}} \left( \left[ \cos \theta \left( \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]' \frac{\theta'}{\sqrt{\theta'^2 + \|W\|^2}} \right. \right. \right. \\ &\quad \left. \left. - \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]^2 \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} - \left( \frac{\theta' \kappa}{\lambda\tau \|W\|} \right)' \right] + \right. \\ &\quad \left. + \sin \theta \left( \left( \frac{\theta' \kappa}{\lambda\tau \|W\|} \right) \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right] \frac{\theta'}{\sqrt{\theta'^2 + \|W\|^2}} + \left( \frac{\kappa}{\lambda\tau} \right) \right. \right. \\ &\quad \left. \left. \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right] \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} - \left( \frac{\theta' \kappa}{\lambda\tau \|W\|} \right)^2 - \left( \frac{\kappa}{\lambda\tau} \right)^2 \right] \right) \mathbf{T} \\ &\quad \left. + \left[ \left( \frac{\kappa}{\lambda\tau} \right) - \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]' \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right. \right. \end{aligned}$$



$$\begin{aligned}
 & - \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right] \frac{\theta'}{\sqrt{\theta'^2 + \|W\|^2}} \mathbf{N} \\
 & \left[ -\sin \theta \left( \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]' \frac{\theta'}{\sqrt{\theta'^2 + \|W\|^2}} \right. \right. \\
 & - \left. \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]^2 \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} - \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right)' \right) + \\
 & + \cos \theta \left( \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right) \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right] \frac{\theta'}{\sqrt{\theta'^2 + \|W\|^2}} + \left( \frac{\kappa}{\lambda \tau} \right) \right. \\
 & \left. \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right] \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} - \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right)^2 - \left( \frac{\kappa}{\lambda \tau} \right)^2 \right] \right) \mathbf{B}.
 \end{aligned}$$

and thus

$$\beta_1''' = \frac{(t_1 \cos \theta + t_2 \sin \theta + t_3)T + t_3N + (t_2 \cos \theta - t_1 \sin \theta + t_3)T}{\sqrt{2}},$$

where

$$\begin{aligned}
 t_1 &= \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]'' \frac{\theta'}{\sqrt{\theta'^2 + \|W\|^2}} - 3 \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \right. \\
 & \left. \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right] \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]' \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \\
 & - \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]^3 \frac{\theta'}{\sqrt{\theta'^2 + \|W\|^2}} - \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right)'' \\
 & - \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right)^2 \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right] \frac{\theta'}{\sqrt{\theta'^2 + \|W\|^2}} - \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right) \left( \frac{\kappa}{\lambda \tau} \right) \\
 & \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right] \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} + \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right)^3 + \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right) \left( \frac{\kappa}{\lambda \tau} \right)^2 \\
 t_2 &= 2 \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right) \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]' \frac{\theta'}{\sqrt{\theta'^2 + \|W\|^2}} - 2 \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right) \\
 & \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]^2 \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} - 3 \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right) \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right)'
 \end{aligned}$$

$$\begin{aligned}
& + \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right) \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]' \frac{\theta'}{\sqrt{\theta'^2 + \|W\|^2}} + \left( \frac{\kappa}{\lambda \tau} \right)' \\
& \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right] \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} + 2 \left( \frac{\kappa}{\lambda \tau} \right) \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \right. \\
& \left. \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]' \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} + 2 \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]^2 \\
& \frac{\theta'}{\sqrt{\theta'^2 + \|W\|^2}} - 3 \left( \frac{\kappa}{\lambda \tau} \right) \left( \frac{\kappa}{\lambda \tau} \right)' \\
t_3 = & \left( \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right) \left( \frac{\kappa}{\lambda \tau} \right) \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right] - 3 \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \right. \right. \\
& \left. \left. \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right] \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]' \right) \frac{\theta'}{\sqrt{\theta'^2 + \|W\|^2}} + \left( \frac{\kappa}{\lambda \tau} \right)^2 \\
& \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right] - \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]'' \\
& + \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]^3 \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} - \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right)^2 \left( \frac{\kappa}{\lambda \tau} \right) \\
& - \left( \frac{\kappa}{\lambda \tau} \right)^3 + \left( \frac{\kappa}{\lambda \tau} \right)''
\end{aligned}$$

The torsion is then given by

$$\begin{aligned}
\tau_{\beta_1} &= \frac{\det(\beta_1', \beta_1'', \beta_1''')}{\|\beta_1' \wedge \beta_1''\|^2}, \\
\tau_{\beta_1} &= \sqrt{2} \frac{\Omega_1}{\Omega_2}
\end{aligned}$$

where

$$\begin{aligned}
\Omega_1 &= -2t_1 \left( \frac{\kappa}{\lambda \tau} \right)^2 \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\|W\|}{\theta'} + t_1 \frac{\kappa}{\lambda \tau} \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \right]^2 \frac{\|W\|^2}{\theta'^2} - t_1 \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right)^2 \\
& \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\|W\|}{\theta'} + \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right) t_2 \left( \frac{\kappa}{\lambda \tau} \right) + \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]^3 \frac{\theta'^2}{\theta'^2 + \|W\|^2} t_2
\end{aligned}$$

$$\begin{aligned}
 & + \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \right]^2 t_3 \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right) - 2 \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' t_3 \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right)^2 - \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' t_3 \left( \frac{\kappa}{\lambda \tau} \right)^2 \\
 & - \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' t_2 \left( \frac{\kappa}{\lambda \tau} \right) - \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right) t_2 \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]' \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} - \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right) \\
 & t_2 \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]^2 \frac{\theta'}{\sqrt{\theta'^2 + \|W\|^2}} + t_2 \frac{\kappa}{\lambda \tau} \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]' \frac{\theta'}{\sqrt{\theta'^2 + \|W\|^2}} \\
 & - t_2 \frac{\kappa}{\lambda \tau} \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]^2 \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} + \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \right]^2 t_3 \frac{\kappa}{\lambda \tau} \frac{\|W\|}{\theta'} + t_2 \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right)' \\
 & \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\|W\|}{\theta'} - \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right) t_3 \frac{\kappa}{\lambda \tau} \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\|W\|}{\theta'} - t_1 \frac{\kappa}{\lambda \tau} \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right) \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \\
 & + t_1 \frac{\kappa}{\lambda \tau} \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \right]^2 + t_1 \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right)^2 \left( \frac{\kappa}{\lambda \tau} \right) + t_2 \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]^3 \frac{\|W\|^2}{\theta'^2 + \|W\|^2} \\
 & - t_2 \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right)' \left( \frac{\kappa}{\lambda \tau} \right) + \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right) t_3 \left( \frac{\kappa}{\lambda \tau} \right)^2 + t_3 \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right)^3 + t_1 \left( \frac{\kappa}{\lambda \tau} \right)^3,
 \end{aligned}$$

$$\begin{aligned}
 \Omega_2 = & \left( \frac{\kappa}{\lambda \tau} \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \right]^2 + \frac{\kappa}{\lambda \tau} \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]^2 \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} - \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right)^2 \right. \\
 & \left. \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right] - 2 \left( \frac{\kappa}{\lambda \tau} \right)^2 \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\|W\|}{\theta'} + \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right)^2 \left( \frac{\kappa}{\lambda \tau} \right) + \left( \frac{\kappa}{\lambda \tau} \right)^3 \right)^2 \\
 & + \left( \frac{\kappa}{\lambda \tau} \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]' - \left( \frac{\kappa}{\lambda \tau} \right)' \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right] \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \right. \right. \\
 & \left. \left. \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]^2 - \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right) \frac{\theta'}{\sqrt{\theta'^2 + \|W\|^2}} + \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right)' \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right] \right. \\
 & \left. - \frac{\kappa}{\lambda \tau} \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]^2 - \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right) \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]' \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right. \\
 & \left. - \frac{\kappa}{\lambda \tau} \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right)' + \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]^3 + \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right) \left( \frac{\kappa}{\lambda \tau} \right)' \right)^2 + \left( \left[ \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right) \right. \right. \\
 & \left. \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]^2 \frac{\theta'}{\sqrt{\theta'^2 + \|W\|^2}} + \left( \frac{\kappa}{\lambda \tau} \right) \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right]^2 \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right. \right. \\
 & \left. \left. - 2 \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right)^2 \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right] - \left( \frac{\kappa}{\lambda \tau} \right)^2 \left[ \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \frac{\sqrt{\theta'^2 + \|W\|^2}}{\theta'} \right] \right] \right. \\
 & \left. \frac{\theta'}{\sqrt{\theta'^2 + \|W\|^2}} + \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right)^3 + \left( \frac{\theta' \kappa}{\lambda \tau \|W\|} \right) \left( \frac{\kappa}{\lambda \tau} \right)^2 - \left( \frac{\kappa}{\lambda \tau} \right) \frac{\kappa}{\lambda \tau} \left( \frac{\|W\|}{\sqrt{\theta'^2 + \|W\|^2}} \right)' \right)^2. \quad \square
 \end{aligned}$$

**Example 3.1** Let us consider the unit speed Mannheim curve and Mannheim partner curve:

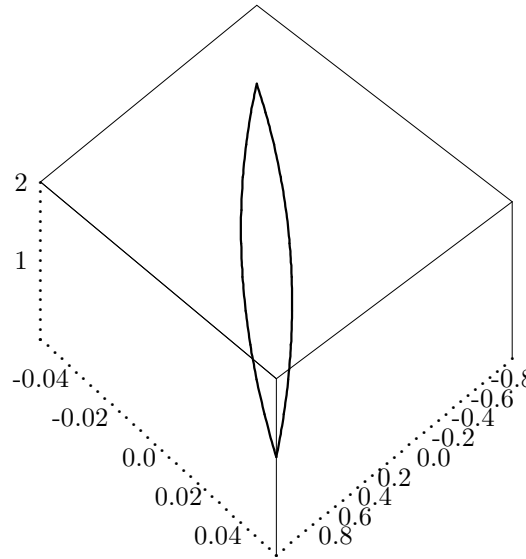
$$\alpha(s) = \frac{1}{\sqrt{2}}(-\cos s, -\sin s, s), \quad \alpha^*(s) = \frac{1}{\sqrt{2}}(-2\cos s, -2\sin s, s).$$

The Frenet invariants of the partner curve,  $\alpha^*(s)$  are given as following

$$T^*(s) = \frac{1}{\sqrt{5}}(2\sin s, -2\cos s, 1),$$

$$\begin{aligned}
N^*(s) &= \frac{1}{\sqrt{5}}(\sin s, \cos s, -2) \\
B^*(s) &= (\cos s, \sin s, 0) \\
C^*(s) &= \left(\frac{2}{5}\sin s + \frac{2}{\sqrt{5}}\cos s, -\frac{2}{5}\cos s + \frac{2}{\sqrt{5}}\sin s, \frac{1}{5}\right) \\
\kappa^*(s) &= \frac{2\sqrt{2}}{5} \\
\tau^*(s) &= \frac{\sqrt{2}}{5}.
\end{aligned}$$

In terms of definitions, we obtain special Smarandache curve, see Figure 1.



**Figure 1**  $\beta_1 = \frac{1}{5\sqrt{5}}((5 + 2\sqrt{5})\sin s + 10\cos s, (5 - 2\sqrt{5})\cos s + 10\sin s, -9\sqrt{5})$

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