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Smarandache cyclic geometric determinant sequences

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Abstract In this paper, the concept of Smarandache cyclic geometric determinant sequence was introduced and a formula for its n^{th} term was obtained using the concept of right and left circulant matrices.

Keywords Smarandache cyclic geometric determinant sequence, determinant, right circulan -t matrix, left circulant matrix.

§1. Introduction and preliminaries

Majumdar [1] gave the formula for n^{th} term of the following sequences: Smarandache cyclic natural determinant sequence, Smarandache cyclic arithmetic determinant sequence, Smarandache bisymmetric natural determinant sequence and Smarandache bisymmetric arithmetic determinant sequence.

Definition 1.1. A Smarandache cyclic geometric determinant sequence $\{SCGDS(n)\}$ is a sequence of the form

$$\{SCGDS(n)\} = \left\{ |a|, \begin{vmatrix} a & ar \\ ar & a \end{vmatrix}, \begin{vmatrix} a & ar & ar^2 \\ ar & ar^2 & a \\ ar^2 & a & ar \end{vmatrix}, \ldots \right\}.$$

Definition 1.2. A matrix $RCIRC_n(\vec{c}) \in M_{nxn}(\mathbb{R})$ is said to be a right circulant matrix if it is of the form

$$RCIRC_n(\vec{c}) = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_2 & c_3 & c_4 & \cdots & c_0 & c_1 \\ c_1 & c_2 & c_3 & \cdots & c_{n-1} & c_0 \end{pmatrix},$$

where $\vec{c} = (c_0, c_1, c_2, ..., c_{n-2}, c_{n-1})$ is the circulant vector.

Definition 1.3. A matrix $LCIRC_n(\vec{c}) \in M_{nxn}(\mathbb{R})$ is said to be a leftt circulant matrix if it is of the form

$$LCIRC_n(\vec{c}) = \begin{pmatrix} c_0 & c_1 & c_2 & ... & c_{n-2} & c_{n-1} \\ c_1 & c_2 & c_3 & ... & c_{n-1} & c_0 \\ c_2 & c_3 & c_4 & ... & c_0 & c_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{n-2} & c_{n-1} & c_0 & ... & c_{n-4} & c_{n-3} \\ c_{n-1} & c_0 & c_1 & ... & c_{n-4} & c_{n-2} \end{pmatrix},$$

where $\vec{c} = (c_0, c_1, c_2, ..., c_{n-2}, c_{n-1})$ is the circulant vector.

Definition 1.4. A right circulant matrix $RCIRC_n(\vec{g})$ with geometric sequence is a matrix of the form

$$RCIRC_n(\vec{g}) = \begin{pmatrix} a & ar & ar^2 & ... & ar^{n-2} & ar^{n-1} \\ ar^{n-1} & a & ar & ... & ar^{n-3} & ar^{n-2} \\ ar^{n-2} & ar^{n-1} & a & ... & ar^{n-4} & ar^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ ar^2 & ar^3 & ar^4 & ... & a & ar \\ ar & ar^2 & ar^3 & ... & ar^{n-1} & a. \end{pmatrix}.$$

Definition 1.5. A left circulant matrix $LCIRC_n(\vec{g})$ with geometric sequence is a matrix of the form

$$LCIRC_n(\vec{g}) = \begin{pmatrix} a & ar & ar^2 & ... & ar^{n-2} & ar^{n-1} \\ ar & ar^2 & ar^3 & ... & ar^{n-1} & a \\ ar^2 & ar^3 & ar^4 & ... & a & ar \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ ar^{n-2} & ar^{n-1} & a & ... & ar^{n-4} & ar^{n-3} \\ ar^{n-1} & a & ar & ... & ar^{n-4} & ar^{n-2}. \end{pmatrix}.$$

The right and left circulant matrices has the following relationship:

$$LCIRC_n(\vec{c}) = \Pi RCIRC_n(\vec{c}).$$

where $\Pi = \begin{pmatrix} 1 & O_1 \\ O_2 & \tilde{I}_{n-1} \end{pmatrix}$ with $\tilde{I}_{n-1} = \begin{pmatrix} 0 & 0 & ... & 0 & 1 \\ 0 & 0 & ... & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & ... & 0 & 0 \\ 1 & 0 & ... & 0 & 0 \end{pmatrix}$, $O_1 = (0\ 0\ 0\ ...\ 0)$ and $O_2 = O_1^T$.

Clearly, the terms of $\{SCGDS(n)\}$ are just the determinants of $LCIRC_n(\vec{g})$. Now, for the rest of this paper, let $|A|$ be the notation for the determinant of a matrix A. Hence

$$\{SCGDS(n)\} = \{|LCIRC_1(\vec{g})|, |LCIRC_2(\vec{g})|, |LCIRC_3(\vec{g})|, \ldots\}.$$

§2. Preliminary results

Lemma 2.1.

$$|RCIRC_n(\vec{g})| = a^n(1-r^n)^{n-1}.$$

Proof.

$$\begin{aligned} RCIRC_n(\vec{g}) &= \begin{pmatrix} a & ar & ar^2 & \ldots & ar^{n-2} & ar^{n-1} \\ ar^{n-1} & a & ar & \ldots & ar^{n-3} & ar^{n-2} \\ ar^{n-2} & ar^{n-1} & a & \ldots & ar^{n-4} & ar^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ ar^2 & ar^3 & ar^4 & \ldots & a & ar \\ ar & ar^2 & ar^3 & \ldots & ar^{n-1} & a \end{pmatrix} \\ &= a\begin{pmatrix} 1 & r & r^2 & \ldots & r^{n-2} & r^{n-1} \\ r^{n-1} & 1 & r & \ldots & r^{n-3} & r^{n-2} \\ r^{n-2} & r^{n-1} & 1 & \ldots & r^{n-4} & r^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ r^2 & r^3 & r^4 & \ldots & 1 & r \\ r & r^2 & r^3 & \ldots & r^{n-1} & 1. \end{pmatrix}. \end{aligned}$$

By applying the row operations $-r^{n-k}R_1 + R_{k+1} \to R_{k+1}$ where $k = 1, 2, 3, \ldots, n-1$,

$$RCIRC_n(\vec{g}) \sim a\begin{pmatrix} 1 & r & r^2 & \ldots & r^{n-2} & r^{n-1} \\ 0 & -(r^n-1) & -r(r^n-1) & \ldots & -r^{n-3}(r^n-1) & -r^{n-2}(r^n-1) \\ 0 & 0 & -(r^n-1) & \ldots & -r^{n-4}(r^n-1) & -r^{n-3}(r^n-1) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & -(r^n-1) & -r(r^n-1) \\ 0 & 0 & 0 & \ldots & 0 & -(r^n-1) \end{pmatrix}.$$

Since $|cA| = c^n|A|$ and its row equivalent matrix is a lower traingular matrix it follows that $|RCIRC_n(\vec{g})| = a^n(1-r^n)^{n-1}$.

Lemma 2.2.

$$|\Pi| = (-1)^{\lfloor \frac{n-1}{2} \rfloor},$$

where $\lfloor x \rfloor$ is the floor function.

Proof. Case 1: $n = 1, 2$,

$$|\Pi| = |I_n| = 1.$$

Case 2: n is even and $n > 2$ If n is even then there will be $n-2$ rows to be inverted because there are two 1's in the main diagonal. Hence there will be $\frac{n-2}{2}$ inversions to bring back Π to I_n so it follows that

$$|\Pi| = (-1)^{\frac{n-2}{2}}.$$

Case 3: n is odd and and $n > 2$ If n is odd then there will be $n-1$ rows to be inverted because of the 1 in the main diagonal of the frist row. Hence there will be $\frac{n-1}{2}$ inversions to bring back Π to I_n so it follows that

$$|\Pi| = (-1)^{\frac{n-1}{2}}.$$

But $\left\lfloor \frac{n-1}{2} \right\rfloor = \left\lfloor \frac{n-2}{2} \right\rfloor$, so the lemma follows.

§3. Main results

Theorem 3.1. The n^{th} term of $\{SCGDS(n)\}$ is given by

$$SCGDS(n) = (-1)^{\left\lfloor \frac{n-1}{2} \right\rfloor} a^n (1 - r^n)^{n-1}$$

via the previous lemmas.

References

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