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On the F.Smarandache LCM function[1]

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Abstract For any positive integer n, the famous F.Smarandache LCM function $SL(n)$ is defined as the smallest positive integer k such that $n \mid [1,\ 2,\ \cdots,\ k]$, where $[1,\ 2,\ \cdots,\ k]$ denotes the least common multiple of 1, 2, $\cdots$, k. The main purpose of this paper is using the elementary methods to study the value distribution properties of the function $SL(n)$, and give an interesting asymptotic formula for it.

Keywords F.Smarandache LCM function, value distribution, asymptotic formula.

§1. Introduction

For any positive integer n, the famous F.Smarandache LCM function $SL(n)$ defined as the smallest positive integer k such that $n \mid [1,\ 2,\ \cdots,\ k]$, where $[1,\ 2,\ \cdots,\ k]$ denotes the least common multiple of 1, 2, $\cdots$, k. For example, the first few values of $SL(n)$ are $SL(1) = 1$, $SL(2) = 2$, $SL(3) = 3$, $SL(4) = 4$, $SL(5) = 5$, $SL(6) = 3$, $SL(7) = 7$, $SL(8) = 8$, $SL(9) = 9$, $SL(10) = 5$, $SL(11) = 11$, $SL(12) = 4$, $SL(13) = 13$, $SL(14) = 7$, $SL(15) = 5$, $SL(16) = 16$, $SL(17) = 17$, $SL(18) = 9$, $SL(20) = 5$, $\cdots\cdots$. About the elementary properties of $SL(n)$, some authors had studied it, and obtained many interesting results, see reference [2], [3], [4] and [5]. For example, Murthy [2] showed that if n be a prime, then $SL(n) = S(n)$, where $S(n)$ denotes the Smarandache function, i.e., $S(n) = \min\{m :\ n|m!,\ m \in N\}$. Simultaneously, Murthy [2] also proposed the following problem:

$$SL(n) = S(n), \quad S(n) \neq n\ ? \tag{1}$$

Le Maohua [3] completely solved this problem, and proved the following conclusion:

Every positive integer n satisfying (1) can be expressed as

$$n = 12 \ \ \texttt{or} \ \ n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} p,$$

where p_1, p_2, $\cdots$, p_r, p are distinct primes, and α_1, α_2, $\cdots$, α_r are positive integers satisfying $p > p_i^{\alpha_i}$, $i = 1, 2, \cdots, r$.

Lv Zhongtian [4] studied the mean value properties of $SL(n)$, and proved that for any fixed positive integer k and any real number $x > 1$, we have the asymptotic formula

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$$\sum_{n\leq x} SL(n) = \frac{\pi^2}{12}\cdot\frac{x^2}{\ln x} + \sum_{i=2}^{k}\frac{c_i\cdot x^2}{\ln^i x} + O\left(\frac{x^2}{\ln^{k+1} x}\right),$$

where c_i $(i = 2,\ 3, \cdots, k)$ are computable constants.

Ge Jian [5] studied the value distribution of $[SL(n) - S(n)]^2$, and proved that

$$\sum_{n\leq x}[SL(n) - S(n)]^2 = \frac{2}{3}\cdot\zeta\left(\frac{3}{2}\right)\cdot x^{\frac{3}{2}}\cdot\sum_{i=1}^{k}\frac{c_i}{\ln^i x} + O\left(\frac{x^{\frac{3}{2}}}{\ln^{k+1} x}\right),$$

where $\zeta(s)$ is the Riemann zeta-function, c_i $(i = 1,\ 2, \cdots,\ k)$ are constants. The main purpose of this paper is using the elementary methods to study the value distribution properties of $SL(n)$, and prove an interesting asymptotic formula. That is, we shall prove the following conclusion:

Theorem. For any real number $x > 1$, we have the asymptotic formula

$$\sum_{\substack{n\in N\\ SL(n)\leq x}} 1 = 2^{\frac{x}{\ln x}\left[1+O\left(\frac{\ln\ln x}{\ln x}\right)\right]},$$

where N denotes the set of all positive integers.

From this Theorem we may immediately deduce the following:

Corollary. For any real number $x > 1$, let $\pi(x)$ denotes the number of all primes $p \leq x$, then we have the limit formula

$$\lim_{x\longrightarrow\infty}\left[\sum_{\substack{n\in N\\ SL(n)\leq x}} 1\right]^{\frac{1}{\pi(x)}} = 2.$$

§2. Proof of the theorem

In this section, we shall prove our theorem directly. Let x be any real number with $x > 2$, then for any prime $p \leq x$, there exists one and only one positive integer $\alpha(p)$ such that

$$p^{\alpha(p)} \leq x < p^{\alpha(p)+1}.$$

From the properties of $SL(n)$ and [2] we know that if $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ be the factorization of n into primes powers, then

$$SL(n) = \max\{p_1^{\alpha_1},\ p_2^{\alpha_2},\ \cdots,\ p_r^{\alpha_r}\}. \tag{2}$$

Let $m = \prod_{p\leq x} p^{\alpha(p)}$. Then for any integer $d|m$, we have $SL(d) \leq x$. For any positive integers u and v with $(u, v) = 1$, if $SL(u) \leq x$, $SL(v) \leq x$, then $SL(uv) \leq x$. On the other hand, for any $SL(n) \leq x$, from the definition of $SL(n)$ we also have $n|m$. So from these and the properties of the Dirichlet divisor function $d(n)$ we have

$$\sum_{\substack{n\in N\\ SL(n)\le x}} 1 = \sum_{d|m} 1 = \prod_{p\le x}(1+\alpha(p)) = e^{\sum\limits_{p\le x}\ln(1+\alpha(p))}. \quad (3)$$

From the definition of $\alpha(p)$ we have $\alpha(p) \le \frac{\ln x}{\ln p} < \alpha(p)+1$ or

$$\alpha(p) = \left[\frac{\ln x}{\ln p}\right]. \quad (4)$$

Therefore, from (4) we may immediately get

$$\begin{aligned}\sum_{p\le x}\ln(1+\alpha(p)) &= \sum_{p\le x}\ln\left(1+\left[\frac{\ln x}{\ln p}\right]\right)\\ &= \sum_{p\le \frac{x}{\ln^2 x}}\ln\left(1+\left[\frac{\ln x}{\ln p}\right]\right) + \sum_{\frac{x}{\ln^2 x}<p\le x}\ln\left(1+\left[\frac{\ln x}{\ln p}\right]\right). \quad (5)\end{aligned}$$

Now we estimate each term in (5). It is clear that

$$\sum_{p\le \frac{x}{\ln^2 x}}\ln\left(1+\left[\frac{\ln x}{\ln p}\right]\right) \ll \sum_{p\le \frac{x}{\ln^2 x}}\ln\ln x \ll \frac{x\ln\ln x}{\ln^3 x}. \quad (6)$$

If $\frac{x}{\ln^2 x} < p \le x$, then $1 \le \frac{\ln x}{\ln p} < 1 + \frac{2\ln\ln x}{\ln x - 2\ln\ln x}$. So from the famous Prime Theorem

$$\pi(x) = \frac{x}{\ln x} + O\left(\frac{x}{\ln^2 x}\right)$$

and

$$\ln(1+y) \sim y, \quad \texttt{as} \quad y \longrightarrow 0,$$

we have

$$\begin{aligned}\sum_{\frac{x}{\ln^2 x}<p\le x}\ln\left(1+\left[\frac{\ln x}{\ln p}\right]\right) &= \sum_{\frac{x}{\ln^2 x}<p\le x}\ln 2 + O\left(\sum_{\frac{x}{\ln^2 x}<p\le x}\frac{\ln\ln x}{\ln x}\right)\\ &= \ln 2\cdot\frac{x}{\ln x} + O\left(\frac{x\ln\ln x}{\ln^2 x}\right). \quad (7)\end{aligned}$$

Combining (3), (5), (6) and (7) we may immediately obtain

$$\sum_{\substack{n\in N\\ SL(n)\le x}} 1 = 2^{\frac{x}{\ln x}\left[1+O\left(\frac{\ln\ln x}{\ln x}\right)\right]},$$

where N denotes the set of all positive integers. This completes the proof of Theorem.

The corollary follows from

$$\left[\sum_{\substack{n\in N\\SL(n)\leq x}} 1\right]^{\frac{1}{\pi(x)}} = 2^{1+O\left(\frac{\ln\ln x}{\ln x}\right)} = 2 + O\left(\frac{\ln\ln x}{\ln x}\right)$$

as $x \longrightarrow \infty$.

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