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On Smarandache Bryant Schneider Group of A Smarandache Loop

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Abstract: The concept of Smarandache Bryant Schneider Group of a Smarandache loop is introduced. Relationship(s) between the Bryant Schneider Group and the Smarandache Bryant Schneider Group of an S-loop are discovered and the later is found to be useful in finding Smarandache isotopy-isomorphy condition(s) in S-loops just like the formal is useful in finding isotopy-isomorphy condition(s) in loops. Some properties of the Bryant Schneider Group of a loop are shown to be true for the Smarandache Bryant Schneider Group of a Smarandache loop. Some interesting and useful cardinality formulas are also established for a type of finite Smarandache loop.

Key Words: Smarandache Bryant Schneider group, Smarandache loops, Smarandache f, g-principal isotopes.

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§1. Introduction

The study of Smarandache loops was initiated by W. B. Vasantha Kandasamy in 2002. In her book [16], she defined a Smarandache loop (S-loop) as a loop with at least a subloop which forms a subgroup under the binary operation of the loop. For more on loops and their properties, readers should check [14], [3], [5], [7], [6] and [16]. In her book, she introduced over 75 Smarandache concepts in loops but the concept Smarandache Bryant Schneider Group which is to be studied here for the first time is not among. In her first paper [17], she introduced some types of Smarandache loops. The present author has contributed to the study of S-quasigroups and S-loops in [9], [10] and [11] while Muktibodh [13] did a study on the first.

Robinson [15] introduced the idea of Bryant-Schneider group of a loop because its importance and motivation stem from the work of Bryant and Schneider [4]. Since the advent of the Bryant-Schneider group, some studies by Adeniran [1], [2] and Chiboka [6] have been done on it relative to CC-loops, C-loops and extra loops after Robinson [15] studied the Bryant-Schneider group of a Bol loop. The judicious use of it was earlier predicted by Robinson [15]. As mentioned in [Section 5, Robinson [15]], the Bryant-Schneider group of a loop is extremely useful in investigating isotopy-isomorphy condition(s) in loops.

In this study, the concept of Smarandache Bryant Schneider Group of a Smarandache

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loop is introduced. Relationship(s) between the Bryant Schneider Group and the Smarandache Bryant Schneider Group of an S-loop are discovered and the later is found to be useful in finding Smarandache isotopy-isomorphy condition(s) in S-loops just like the formal is useful in finding isotopy-isomorphy condition(s) in loops. Some properties of the Bryant Schneider Group of a loop are shown to be true for the Smarandache Bryant Schneider Group of a Smarandache loop. Some interesting and useful cardinality formulas are also established for a type of finite Smarandache loop. But first, we state some important definitions.

§2. Definitions and Notations

Definition 2.1 *Let L be a non-empty set. Define a binary operation $(\cdot)$ on L : If $x\cdot y\in L$ for $\forall\ x,y\in L$, $(L,\cdot)$ is called a groupoid. If the system of equations ; $a\cdot x=b$ and $y\cdot a=b$ have unique solutions for x and y respectively, then $(L,\cdot)$ is called a quasigroup. Furthermore, if there exists a unique element $e\in L$ called the identity element such that for $\forall\ x\in L$, $x\cdot e=e\cdot x=x$, $(L,\cdot)$ is called a loop.*

Furthermore, if there exist at least a non-empty subset M of L such that $(M,\cdot)$ is a non-trivial subgroup of $(L,\cdot)$, then L is called a Smarandache loop(S-loop) with Smarandache subgroup(S-subgroup) M.

The set $SYM(L,\cdot)=SYM(L)$ of all bijections in a loop $(L,\cdot)$ forms a group called the permutation(symmetric) group of the loop $(L,\cdot)$. The triple (U,V,W) such that $U,V,W\in SYM(L,\cdot)$ is called an autotopism of L if and only if $xU\cdot yV=(x\cdot y)W\ \forall\ x,y\in L$. The group of autotopisms(under componentwise multiplication [14]) of L is denoted by $AUT(L,\cdot)$. If $U=V=W$, then the group $AUM(L,\cdot)=AUM(L)$ formed by such U's is called the automorphism group of $(L,\cdot)$. If L is an S-loop with an arbitrary S-subgroup H, then the group $SSYM(L,\cdot)=SSYM(L)$ formed by all $\theta\in SYM(L)$ such that $h\theta\in H\ \forall\ h\in H$ is called the Smarandache permutation(symmetric) group of L. Hence, the group $SA(L,\cdot)=SA(L)$ formed by all $\theta\in SSYM(L)\cap AUM(L)$ is called the *Smarandache automorphism group* of L.

Let $(G,\cdot)$ be a loop. The bijection $L_x:G\longrightarrow G$ defined as $yL_x=x\cdot y,\forall\ x,y\in G$ is called a left translation(multiplication) of G while the bijection $R_x:G\longrightarrow G$ defined as $yR_x=y\cdot x,\forall\ x,y\in G$ is called a right translation(multiplication) of G.

Definition 2.2(Robinson [15]) *Let $(G,\cdot)$ be a loop. A mapping $\theta\in SYM(G,\cdot)$ is a special map for G means that there exist $f,g\in G$ so that $(\theta R_g^{-1},\theta L_f^{-1},\theta)\in AUT(G,\cdot)$.*

Definition 2.3 *Let $(G,\cdot)$ be a Smarandache loop with S-subgroup $(H,\cdot)$. A mapping $\theta\in SSYM(G,\cdot)$ is a Smarandache special map(S-special map) for G if and only if there exist $f,g\in H$ such that $(\theta R_g^{-1},\theta L_f^{-1},\theta)\in AUT(G,\cdot)$.*

Definition 2.4(Robinson [15]) *Let the set*

$$BS(G,\cdot)=\{\theta\in SYM(G,\cdot)\ :\ \exists\ f,g\in G\ \ni\ (\theta R_g^{-1},\theta L_f^{-1},\theta)\in AUT(G,\cdot)\}$$

i.e the set of all special maps in a loop, then $BS(G,\cdot)\leqslant SYM(G,\cdot)$ is called the Bryant-Schneider group of the loop $(G,\cdot)$.

Definition 2.5 *Let the set*

$$SBS(G,\cdot)=\{\theta\in SSYM(G,\cdot)\ :\ \textit{there exist } f,g\in H\ \ni\ (\theta R_g^{-1},\theta L_f^{-1},\theta)\in AUT(G,\cdot)\}$$

i.e the set of all S-special maps in a S-loop, then $SBS(G,\cdot)$ *is called the Smarandache Bryant-Schneider group(SBS group) of the S-loop* $(G,\cdot)$ *with S-subgroup* H *if* $SBS(G,\cdot)\leqslant SYM(G,\cdot)$.

Definition 2.6 *The triple* $\phi=(R_g,L_f,I)$ *is called an* f,g*-principal isotopism of a loop* $(G,\cdot)$ *onto a loop* $(G,\circ)$ *if and only if*

$$x\cdot y=xR_g\circ yL_f,\forall\ x,y\in G\ \textit{ or }\ x\circ y=xR_g^{-1}\cdot yL_f^{-1},\forall\ x,y\in G.$$

f *and* g *are called translation elements of* G *or at times written in the pair form* (g,f)*, while* $(G,\circ)$ *is called an* f,g*-principal isotope of* $(G,\cdot)$.

On the other hand, $(G,\otimes)$ *is called a Smarandache* f,g*-principal isotope of* $(G,\oplus)$ *if for some* $f,g\in S$,

$$xR_g\otimes yL_f=(x\oplus y)\ \forall\ x,y\in G$$

where $(S,\oplus)$ *is a S-subgroup of* $(G,\oplus)$*. In these cases,* f *and* g *are called Smarandache elements(S-elements).*

Let $(L,\cdot)$ *and* $(G,\circ)$ *be S-loops with S-subgroups* L' *and* G' *respectively such that* $xA\in G',\forall\ x\in L'$*, where* $A\ :\ (L,\cdot)\longrightarrow(G,\circ)$*. Then the mapping* A *is called a Smarandache isomorphism if* $(L,\cdot)\cong(G,\circ)$*, hence we write* $(L,\cdot)\succsim(G,\circ)$*. An S-loop* $(L,\cdot)$ *is called a G-Smarandache loop(GS-loop) if and only if* $(L,\cdot)\succsim(G,\circ)$ *for all S-loop isotopes* $(G,\circ)$ *of* $(L,\cdot)$.

Definition 2.7 *Let* $(G,\cdot)$ *be a Smarandache loop with an S-subgroup* H.

$$\Omega(G,\cdot)=\Big\{(\theta R_g^{-1},\theta L_f^{-1},\theta)\in AUT(G,\cdot)\textit{ for some } f,g\in H\ :\ h\theta\in H,\forall\ h\in H\Big\}.$$

§3. Main Results

3.1 Smarandache Bryant Schneider Group

Theorem 3.1 *Let* $(G,\cdot)$ *be a Smarandache loop.* $SBS(G,\cdot)\leqslant BS(G,\cdot)$.

Proof Let $(G,\cdot)$ be an S-loop with S-subgroup H. Comparing Definitions 2.4 and 2.5, it can easily be observed that $SBS(G,\cdot)\subset BS(G,\cdot)$. The case $SBS(G,\cdot)\subseteq BS(G,\cdot)$ is possible when $G=H$ where H is the S-subgroup of G but this will be a contradiction since G is an S-loop.

Identity. If I is the identity mapping on G, then $hI=h\in H,\ \forall\ h\in H$ and there exists $e\in H$ where e is the identity element in G such that $(IR_e^{-1},IL_e^{-1},I)=(I,I,I)\in AUT(G,\cdot)$. So, $I\in SBS(G,\cdot)$. Thus $SBS(G,\cdot)$ is non-empty.

Closure and Inverse. Let $\alpha,\beta\in SBS(G,\cdot)$. Then there exist $f_1,g_1,f_2,g_2\in H$ such that

$$A=(\alpha R_{g_1}^{-1},\alpha L_{f_1}^{-1},\alpha),\ B=(\beta R_{g_2}^{-1},\beta L_{f_2}^{-1},\beta)\in AUT(G,\cdot).$$

$$AB^{-1}=(\alpha R_{g_1}^{-1},\alpha L_{f_1}^{-1},\alpha)(R_{g_2}\beta^{-1},L_{f_2}\beta^{-1},\beta^{-1})$$

$$=(\alpha R_{g_1}^{-1}R_{g_2}\beta^{-1},\alpha L_{f_1}^{-1}L_{f_2}\beta^{-1},\alpha\beta^{-1})\in AUT(G,\cdot).$$

Let $\delta=\beta R_{g_1}^{-1}R_{g_2}\beta^{-1}$ and $\gamma=\beta L_{f_1}^{-1}L_{f_2}\beta^{-1}$. Then,

$$(\alpha\beta^{-1}\delta,\alpha\beta^{-1}\gamma,\alpha\beta^{-1})\in AUT(G,\cdot)\Leftrightarrow(x\alpha\beta^{-1}\delta)\cdot(y\alpha\beta^{-1}\gamma)=(x\cdot y)\alpha\beta^{-1}\ \forall\ x,y\in G.$$

Putting $y=e$ and replacing x by $x\beta\alpha^{-1}$, we have $(x\delta)\cdot(e\alpha\beta^{-1}\gamma)=x$ for all $x\in G$. Similarly, putting $x=e$ and replacing y by $y\beta\alpha^{-1}$, we have $(e\alpha\beta^{-1}\delta)\cdot(y\gamma)=y$ for all $y\in G$. Thence, $x\delta R_{(e\alpha\beta^{-1}\gamma)}=x$ and $y\gamma L_{(e\alpha\beta^{-1}\delta)}=y$ which implies that

$$\delta=R_{(e\alpha\beta^{-1}\gamma)}^{-1}\text{ and }\gamma=L_{(e\alpha\beta^{-1}\delta)}^{-1}.$$

Thus, since $g=e\alpha\beta^{-1}\gamma,\ f=e\alpha\beta^{-1}\delta\in H$ then

$$AB^{-1}=(\alpha\beta^{-1}R_g^{-1},\alpha\beta^{-1}L_f^{-1},\alpha\beta^{-1})\in AUT(G,\cdot)\Leftrightarrow\alpha\beta^{-1}\in SBS(G,\cdot).$$

Therefore, $SBS(G,\cdot)\leqslant BS(G,\cdot)$. □

Corollary 3.1 *Let $(G,\cdot)$ be a Smarandache loop. Then, $SBS(G,\cdot)\leqslant SSYM(G,\cdot)\leqslant SYM(G,\cdot)$. Hence, $SBS(G,\cdot)$ is the Smarandache Bryant-Schneider group(SBS group) of the S-loop $(G,\cdot)$.*

Proof Although the fact that $SBS(G,\cdot)\leqslant SYM(G,\cdot)$ follows from Theorem 3.1 and the fact in [Theorem 1, [15]] that $BS(G,\cdot)\leqslant SYM(G,\cdot)$. Nevertheless, it can also be traced from the facts that $SBS(G,\cdot)\leqslant SSYM(G,\cdot)$ and $SSYM(G,\cdot)\leqslant SYM(G,\cdot)$.

It is easy to see that $SSYM(G,\cdot)\subset SYM(G,\cdot)$ and that $SBS(G,\cdot)\subset SSYM(G,\cdot)$ while the trivial cases $SSYM(G,\cdot)\subseteq SYM(G,\cdot)$ and $SBS(G,\cdot)\subseteq SSYM(G,\cdot)$ will contradict the fact that G is an S-loop because these two are possible if the S-subgroup H is G. Reasoning through the axioms of a group, it is easy to show that $SSYM(G,\cdot)\leqslant SYM(G,\cdot)$. By using the same steps in Theorem 3.1, it will be seen that $SBS(G,\cdot)\leqslant SSYM(G,\cdot)$. □

3.2 The SBS Group of a Smarandache f,g-principal isotope

Theorem 3.2 *Let $(G,\cdot)$ be a S-loop with a Smarandache f,g-principal isotope $(G,\circ)$. Then, $(G,\circ)$ is an S-loop.*

Proof Let $(G,\cdot)$ be an S-loop, then there exist an S-subgroup $(H,\cdot)$ of G. If $(G,\circ)$ is a Smarandache f,g-principal isotope of $(G,\cdot)$, then

$$x\cdot y=xR_g\circ yL_f,\forall\ x,y\in G\text{ which implies }x\circ y=xR_g^{-1}\cdot yL_f^{-1},\forall\ x,y\in G$$

where $f, g \in H$. So

$$h_1 \circ h_2 = h_1 R_g^{-1} \cdot h_2 L_f^{-1}, \forall\ h_1, h_2 \in H \text{ for some } f, g \in H.$$

Let us now consider the set H under the operation "$\circ$". That is the pair $(H, \circ)$.

Groupoid. Since $f, g \in H$, then by the definition $h_1 \circ h_2 = h_1 R_g^{-1} \cdot h_2 L_f^{-1}$, $h_1 \circ h_2 \in H, \forall\ h_1, h_2 \in H$ since $(H, \cdot)$ is a groupoid. Thus, $(H, \circ)$ is a groupoid.

Quasigroup. With the definition $h_1 \circ h_2 = h_1 R_g^{-1} \cdot h_2 L_f^{-1}, \forall\ h_1, h_2 \in H$, it is clear that $(H, \circ)$ is a quasigroup since $(H, \cdot)$ is a quasigroup.

Loop. It can easily be seen that $f \cdot g$ is an identity element in $(H, \circ)$. So, $(H, \circ)$ is a loop.

Group. Since $(H, \cdot)$ is a associative, it is easy to show that $(H, \circ)$ is associative.

Hence, $(H, \circ)$ is an S-subgroup in $(G, \circ)$ since the latter is a loop(a quasigroup with identity element $f \cdot g$). Therefore, $(G, \circ)$ is an S-loop. □

Theorem 3.3 *Let $(G, \cdot)$ be a Smarandache loop with an S-subgroup $(H, \cdot)$. A mapping $\theta \in SYM(G, \cdot)$ is a S-special map if and only if θ is an S-isomorphism of $(G, \cdot)$ onto some Smarandache f, g-principal isotopes $(G, \circ)$ where $f, g \in H$.*

Proof By Definition 2.3, a mapping $\theta \in SSYM(G)$ is a S-special map implies there exist $f, g \in H$ such that $(\theta R_g^{-1}, \theta L_f^{-1}, \theta) \in AUT(G, \cdot)$. It can be observed that

$$(\theta R_g^{-1}, \theta L_f^{-1}, \theta) = (\theta, \theta, \theta)(R_g^{-1}, L_f^{-1}, I) \in AUT(G, \cdot).$$

But since $(R_g^{-1}, L_f^{-1}, I)\ :\ (G, \circ) \longrightarrow (G, \cdot)$ then for $(\theta R_g^{-1}, \theta L_f^{-1}, \theta) \in AUT(G, \cdot)$ we must have $(\theta, \theta, \theta)\ :\ (G, \cdot) \longrightarrow (G, \circ)$ which means $(G, \cdot) \overset{\theta}{\cong} (G, \circ)$, hence $(G, \cdot) \overset{\theta}{\succsim} (G, \circ)$ because $(H, \cdot)\theta = (H, \circ)$. $(R_g, L_f, I)\ :\ (G, \cdot) \longrightarrow (G, \circ)$ is an f, g-principal isotopism so $(G, \circ)$ is a Smarandache f, g-principal isotope of $(G, \cdot)$ by Theorem 3.2.

Conversely, if θ is an S-isomorphism of $(G, \cdot)$ onto some Smarandache f, g-principal isotopes $(G, \circ)$ where $f, g \in H$ such that $(H, \cdot)$ is a S-subgroup of $(G, \cdot)$ means $(\theta, \theta, \theta)\ :\ (G, \cdot) \longrightarrow (G, \circ)$, $(R_g, L_f, I)\ :\ (G, \cdot) \longrightarrow (G, \circ)$ which implies $(R_g^{-1}, L_f^{-1}, I)\ :\ (G, \circ) \longrightarrow (G, \cdot)$ and $(H, \cdot)\theta = (H, \circ)$. Thus, $(\theta R_g^{-1}, \theta L_f^{-1}, \theta) \in AUT(G, \cdot)$. Therefore, θ is a S-special map because $f, g \in H$. □

Corollary 3.2 *Let $(G, \cdot)$ be a Smarandache loop with an S-subgroup $(H, \cdot)$. A mapping $\theta \in SBS(G, \cdot)$ if and only if θ is an S-isomorphism of $(G, \cdot)$ onto some Smarandache f, g-principal isotopes $(G, \circ)$ such that $f, g \in H$ where $(H, \cdot)$ is an S-subgroup of $(G, \cdot)$.*

Proof This follows from Definition 2.5 and Theorem 3.3. □

Theorem 3.4 *Let $(G, \cdot)$ and $(G, \circ)$ be S-loops. $(G, \circ)$ is a Smarandache f, g-principal isotope of $(G, \cdot)$ if and only if $(G, \cdot)$ is a Smarandache g, f-principal isotope of $(G, \circ)$.*

Proof Let $(G, \cdot)$ and $(G, \circ)$ be S-loops such that if $(H, \cdot)$ is an S-subgroup in $(G, \cdot)$, then $(H, \circ)$ is an S-subgroup of $(G, \circ)$. The left and right translation maps relative to an element x

in $(G,\circ)$ shall be denoted by $\mathcal{L}_x$ and $\mathcal{R}_x$ respectively.

If $(G,\circ)$ is a Smarandache f,g-principal isotope of $(G,\cdot)$ then, $x\cdot y = xR_g\circ yL_f, \forall\ x,y\in G$ for some $f,g\in H$. Thus, $xR_y = xR_g\mathcal{R}_{yL_f}$ and $yL_x = yL_f\mathcal{L}_{xR_g}$ $x,y\in G$ and we have $R_y = R_g\mathcal{R}_{yL_f}$ and $L_x = L_f\mathcal{L}_{xR_g}, x,y\in G$. So, $\mathcal{R}_y = R_g^{-1}R_{yL_f^{-1}}$ and $\mathcal{L}_x = L_f^{-1}L_{xR_g^{-1}} =\ x,y\in G$. Putting $y=f$ and $x=g$ respectively, we now get $\mathcal{R}_f = R_g^{-1}R_{fL_f^{-1}} = R_g^{-1}$ and $\mathcal{L}_g = L_f^{-1}L_{gR_g^{-1}} = L_f^{-1}$. That is, $\mathcal{R}_f = R_g^{-1}$ and $\mathcal{L}_g = L_f^{-1}$ for some $f,g\in H$.

Recall that

$$x\cdot y = xR_g\circ yL_f, \forall\ x,y\in G\ \Leftrightarrow\ x\circ y = xR_g^{-1}\cdot yL_f^{-1}, \forall\ x,y\in G.$$

So using the last two translation equations,

$$x\circ y = x\mathcal{R}_f\cdot y\mathcal{L}_g, \forall\ x,y\in G\ \Leftrightarrow\ \text{the triple } (\mathcal{R}_f,\mathcal{L}_g,I)\ :\ (G,\circ)\longrightarrow(G,\cdot)$$

is a Smarandache g,f-principal isotopism. Therefore, $(G,\cdot)$ is a Smarandache g,f-principal isotope of $(G,\circ)$.

The converse is achieved by doing the reverse of the procedure described above. □

Theorem 3.5 *If $(G,\cdot)$ is an S-loop with a Smarandache f,g-principal isotope $(G,\circ)$, then $SBS(G,\cdot) = SBS(G,\circ)$.*

Proof Let $(G,\circ)$ be the Smarandache f,g-principal isotope of the S-loop $(G,\cdot)$ with S-subgroup $(H,\cdot)$. By Theorem 3.2, $(G,\circ)$ is an S-loop with S-subgroup $(H,\circ)$. The left and right translation maps relative to an element x in $(G,\circ)$ shall be denoted by $\mathcal{L}_x$ and $\mathcal{R}_x$ respectively.

Let $\alpha\in SBS(G,\cdot)$, then there exist $f_1,g_1\in H$ so that $(\alpha R_{g_1}^{-1},\alpha L_{f_1}^{-1},\alpha)\in AUT(G,\cdot)$. Recall that the triple $(R_{g_1},L_{f_1},I)\ :\ (G,\cdot)\longrightarrow(G,\circ)$ is a Smarandache f,g-principal isotopism, so $x\cdot y = xR_g\circ yL_f, \forall\ x,y\in G$ and this implies

$$R_x = R_g\mathcal{R}_{xL_f} \text{ and } L_x = L_f\mathcal{L}_{xR_g}, \forall\ x\in G \text{ which also implies that}$$

$$\mathcal{R}_{xL_f} = R_g^{-1}R_x \text{ and } \mathcal{L}_{xR_g} = L_f^{-1}L_x, \forall\ x\in G \text{ which finally gives}$$

$$\mathcal{R}_x = R_g^{-1}R_{xL_f^{-1}} \text{ and } \mathcal{L}_x = L_f^{-1}L_{xR_g^{-1}}, \forall\ x\in G.$$

Set $f_2 = f\alpha R_{g_1}^{-1}R_g$ and $g_2 = g\alpha L_{f_1}^{-1}L_f$. Then

$$\mathcal{R}_{g_2} = R_g^{-1}R_{g\alpha L_{f_1}^{-1}L_fL_f^{-1}} = R_g^{-1}R_{g\alpha L_{f_1}^{-1}}, \tag{1}$$

$$\mathcal{L}_{f_2} = L_f^{-1}L_{f\alpha R_{g_1}^{-1}R_gR_g^{-1}} = L_f^{-1}L_{f\alpha R_{g_1}^{-1}}, \forall\ x\in G. \tag{2}$$

Since, $(\alpha R_{g_1}^{-1},\alpha L_{f_1}^{-1},\alpha)\in AUT(G,\cdot)$, then

$$(x\alpha R_{g_1}^{-1})\cdot(y\alpha L_{f_1}^{-1}) = (x\cdot y)\alpha, \forall\ x,y\in G. \tag{3}$$

Putting $y = g$ and $x = f$ separately in the last equation,

$$x\alpha R_{g_1}^{-1} R_{(g\alpha L_{f_1}^{-1})} = xR_g\alpha \text{ and } y\alpha L_{f_1}^{-1} L_{(f\alpha R_{g_1}^{-1})} = yL_f\alpha, \forall\ x, y \in G.$$

Thus by applying (1) and (2), we now have

$$\alpha R_{g_1}^{-1} = R_g\alpha R_{(g\alpha L_{f_1}^{-1})}^{-1} = R_g\alpha \mathcal{R}_{g_2}^{-1} R_g^{-1} \text{ and } \alpha L_{f_1}^{-1} = L_f\alpha L_{(f\alpha R_{g_1}^{-1})}^{-1} = L_f\alpha \mathcal{L}_{f_2}^{-1} L_f^{-1}. \qquad (4)$$

We shall now compute $(x \circ y)\alpha$ by (2) and (3) and then see the outcome.

$(x\circ y)\alpha = (xR_g^{-1}\cdot yL_f^{-1})\alpha = xR_g^{-1}\alpha R_{g_1}^{-1}\cdot yL_f^{-1}\alpha L_{f_1}^{-1} = xR_g^{-1}R_g\alpha\mathcal{R}_{g_2}^{-1}R_g^{-1}\cdot yL_f^{-1}L_f\alpha\mathcal{L}_{f_2}^{-1}L_f^{-1} = x\alpha\mathcal{R}_{g_2}^{-1}R_g^{-1}\cdot y\alpha\mathcal{L}_{f_2}^{-1}L_f^{-1} = x\alpha\mathcal{R}_{g_2}^{-1}\circ y\alpha\mathcal{L}_{f_2}^{-1}, \forall\ x, y \in G$.

Thus,

$$(x \circ y)\alpha = x\alpha\mathcal{R}_{g_2}^{-1} \circ y\alpha\mathcal{L}_{f_2}^{-1}, \forall\ x, y \in G \Leftrightarrow (\alpha\mathcal{R}_{g_2}^{-1}, \alpha\mathcal{L}_{f_2}^{-1}, \alpha) \in AUT(G, \circ) \Leftrightarrow \alpha \in SBS(G, \circ).$$

Whence, $SBS(G, \cdot) \subseteq SBS(G, \circ)$.

Since $(G, \circ)$ is the Smarandache f, g-principal isotope of the S-loop $(G, \cdot)$, then by Theorem 3.4, $(G, \cdot)$ is the Smarandache g, f-principal isotope of $(G, \circ)$. So following the steps above, it can similarly be shown that $SBS(G, \circ) \subseteq SBS(G, \cdot)$. Therefore, the conclusion that $SBS(G, \cdot) = SBS(G, \circ)$ follows. □

3.3 Cardinality Formulas

Theorem 3.6 *Let $(G, \cdot)$ be a finite Smarandache loop with n distinct S-subgroups. If the SBS group of $(G, \cdot)$ relative to an S-subgroup $(H_i, \cdot)$ is denoted by $SBS_i(G, \cdot)$, then*

$$|BS(G, \cdot)| = \frac{1}{n}\sum_{i=1}^{n} |SBS_i(G, \cdot)|\ [BS(G, \cdot) : SBS_i(G, \cdot)].$$

Proof Let the n distinct S-subgroups of G be denoted by H_i, $i = 1, 2, \cdots n$. Note here that $H_i \neq H_j, i, j = 1, 2, \cdots n$. By Theorem 3.1, $SBS_i(G, \cdot) \leqslant BS(G, \cdot), i = 1, 2, \cdots n$. Hence, by the Lagrange's theorem of classical group theory,

$$|BS(G, \cdot)| = |SBS_i(G, \cdot)|\ [BS(G, \cdot) : SBS_i(G, \cdot)], i = 1, 2, \cdots n.$$

Thus, adding the equation above for all $i = 1, 2, \cdots n$, we get

$$n|BS(G, \cdot)| = \sum_{i=1}^{n} |SBS_i(G, \cdot)|\ [BS(G, \cdot) : SBS_i(G, \cdot)], i = 1, 2, \cdots n, \text{ thence,}$$

$$|BS(G, \cdot)| = \frac{1}{n}\sum_{i=1}^{n} |SBS_i(G, \cdot)|\ [BS(G, \cdot) : SBS_i(G, \cdot)].$$

□

Theorem 3.7 *Let $(G, \cdot)$ be a Smarandache loop. Then, $\Omega(G, \cdot) \leqslant AUT(G, \cdot)$.*

Proof Let $(G, \cdot)$ be an S-loop with S-subgroup H. By Definition 2.7, it can easily be observed that $\Omega(G, \cdot) \subseteq AUT(G, \cdot)$.

Identity. If I is the identity mapping on G, then $hI = h \in H, \forall\ h \in H$ and there exists $e \in H$ where e is the identity element in G such that $(IR_e^{-1}, IL_e^{-1}, I) = (I, I, I) \in AUT(G, \cdot)$. So, $(I, I, I) \in \Omega(G, \cdot)$. Thus $\Omega(G, \cdot)$ is non-empty.

Closure and Inverse. Let $A, B \in \Omega(G, \cdot)$. Then there exist $\alpha, \beta \in SSYM(G, \cdot)$ and some $f_1, g_1, f_2, g_2 \in H$ such that

$$A = (\alpha R_{g_1}^{-1}, \alpha L_{f_1}^{-1}, \alpha),\ B = (\beta R_{g_2}^{-1}, \beta L_{f_2}^{-1}, \beta) \in AUT(G, \cdot).$$

$$AB^{-1} = (\alpha R_{g_1}^{-1}, \alpha L_{f_1}^{-1}, \alpha)(R_{g_2}\beta^{-1}, L_{f_2}\beta^{-1}, \beta^{-1})$$

$$= (\alpha R_{g_1}^{-1} R_{g_2}\beta^{-1}, \alpha L_{f_1}^{-1} L_{f_2}\beta^{-1}, \alpha\beta^{-1}) \in AUT(G, \cdot).$$

Using the same techniques for the proof of closure and inverse in Theorem 3.1 here and by letting $\delta = \beta R_{g_1}^{-1} R_{g_2}\beta^{-1}$ and $\gamma = \beta L_{f_1}^{-1} L_{f_2}\beta^{-1}$, it can be shown that,

$$AB^{-1} = (\alpha\beta^{-1} R_g^{-1}, \alpha\beta^{-1} L_f^{-1}, \alpha\beta^{-1}) \in AUT(G, \cdot) \text{ where } g = e\alpha\beta^{-1}\gamma,\ f = e\alpha\beta^{-1}\delta \in H$$

$$\text{such that } \alpha\beta^{-1} \in SSYM(G, \cdot) \Leftrightarrow AB^{-1} \in \Omega(G, \cdot).$$

Therefore, $\Omega(G, \cdot) \leqslant AUT(G, \cdot)$. □

Theorem 3.8 *Let $(G, \cdot)$ be a Smarandache loop with an S-subgroup H such that $f, g \in H$ and $\alpha \in SBS(G, \cdot)$. If the mapping*

$$\Phi\ :\ \Omega(G, \cdot) \longrightarrow SBS(G, \cdot)\ \text{ is defined as }\ \Phi\ :\ (\alpha R_g^{-1}, \alpha L_f^{-1}, \alpha) \mapsto \alpha,$$

then Φ is an homomorphism.

Proof Let $A, B \in \Omega(G, \cdot)$. Then there exist $\alpha, \beta \in SSYM(G, \cdot)$ and some $f_1, g_1, f_2, g_2 \in H$ such that

$$A = (\alpha R_{g_1}^{-1}, \alpha L_{f_1}^{-1}, \alpha),\ B = (\beta R_{g_2}^{-1}, \beta L_{f_2}^{-1}, \beta) \in AUT(G, \cdot).$$

$\Phi(AB) = \Phi[(\alpha R_{g_1}^{-1}, \alpha L_{f_1}^{-1}, \alpha)(\beta R_{g_2}^{-1}, \beta L_{f_2}^{-1}, \beta)] = \Phi(\alpha R_{g_1}^{-1}\beta R_{g_2}^{-1}, \alpha L_{f_1}^{-1}\beta L_{f_2}^{-1}, \alpha\beta)$. It will be good if this can be written as; $\Phi(AB) = \Phi(\alpha\beta\delta, \alpha\beta\gamma, \alpha\beta)$ such that $h\alpha\beta \in H\ \forall\ h \in H$ and $\delta = R_g^{-1}$, $\gamma = L_f^{-1}$ for some $g, f \in H$.

This is done as follows. If

$$(\alpha R_{g_1}^{-1}\beta R_{g_2}^{-1}, \alpha L_{f_1}^{-1}\beta L_{f_2}^{-1}, \alpha\beta) = (\alpha\beta\delta, \alpha\beta\gamma, \alpha\beta) \in AUT(G, \cdot), \text{then,}$$

$$x\alpha\beta\delta \cdot y\alpha\beta\gamma = (x \cdot y)\alpha\beta, \forall\ x, y \in G.$$

Put $y = e$ and replace x by $x\beta^{-1}\alpha^{-1}$ then $x\delta \cdot e\alpha\beta\gamma = x \Leftrightarrow \delta = R_{e\alpha\beta\gamma}^{-1}$.

Similarly, put $x = e$ and replace y by $y\beta^{-1}\alpha^{-1}$. Then, $e\alpha\beta\delta \cdot y\gamma = y \Leftrightarrow \gamma = L_{e\alpha\beta\delta}^{-1}$. So,

$$\Phi(AB) = (\alpha\beta R_{e\alpha\beta\gamma}^{-1}, \alpha\beta L_{e\alpha\beta\delta}^{-1}, \alpha\beta) = \alpha\beta = \Phi(\alpha R_{g_1}^{-1}, \alpha L_{f_1}^{-1}, \alpha)\Phi(\beta R_{g_2}^{-1}, \beta L_{f_2}^{-1}, \beta) = \Phi(A)\Phi(B).$$

Therefore, Φ is an homomorphism. □

Theorem 3.9 *Let* $(G,\cdot)$ *be a Smarandache loop with an S-subgroup* H *such that* $f,g\in H$ *and* $\alpha\in SSYM(G,\cdot)$. *If the mapping*

$$\Phi\ :\ \Omega(G,\cdot)\longrightarrow SBS(G,\cdot)\ \ \textit{is defined as}\ \ \Phi\ :\ (\alpha R_g^{-1},\alpha L_f^{-1},\alpha)\mapsto\alpha$$

then,

$$A=(\alpha R_g^{-1},\alpha L_f^{-1},\alpha)\in\ker\Phi\ \textit{if and only if}\ \alpha$$

is the identity map on G, $g\cdot f$ *is the identity element of* $(G,\cdot)$ *and* $g\in N_\mu(G,\cdot)$ *the middle nucleus of* $(G,\cdot)$.

Proof For the necessity, $\ker\Phi=\{A\in\Omega(G,\cdot):\Phi(A)=I\}$. So, if $A=(\alpha R_{g_1}^{-1},\alpha L_{f_1}^{-1},\alpha)\in\ker\Phi$, then $\Phi(A)=\alpha=I$. Thus, $A=(R_{g_1}^{-1},L_{f_1}^{-1},I)\in AUT(G,\cdot)\Leftrightarrow$

$$x\cdot y=xR_g^{-1}\cdot yL_f^{-1},\forall\ x,y\in G. \qquad (5)$$

Replace x by xR_g and y by yL_f in (5) to get

$$x\cdot y=xg\cdot fy,\forall\ x,y\in G. \qquad (6)$$

Putting $x=y=e$ in (6), we get $g\cdot f=e$. Replace y by yL_f^{-1} in (6) to get

$$x\cdot yL_f^{-1}=xg\cdot y,\forall\ x,y\in G. \qquad (7)$$

Put $x=e$ in (7), then we have $yL_f^{-1}=g\cdot y,\forall\ y\in G$ and so (7) now becomes

$$x\cdot(gy)=xg\cdot y,\forall\ x,y\in G\Leftrightarrow g\in N_\mu(G,\cdot).$$

For the sufficiency, let α be the identity map on G, $g\cdot f$ the identity element of $(G,\cdot)$ and $g\in N_\mu(G,\cdot)$. Thus, $fg\cdot f=f\cdot gf=fe=f$. Thus, $f\cdot g=e$. Then also, $y=fg\cdot y=f\cdot gy\ \forall\ y\in G$ which results into $yL_f^{-1}=gy\ \forall\ y\in G$. Thus, it can be seen that $x\alpha R_g^{-1}\cdot y\alpha L_f^{-1}=xR_g^{-1}\cdot yL_f^{-1}=xR_g^{-1}\alpha\cdot yL_f^{-1}\alpha=xR_g^{-1}\cdot yL_f^{-1}=xR_g^{-1}\cdot gy=(xR_g^{-1}\cdot g)y=xR_g^{-1}R_g\cdot y=x\cdot y=(x\cdot y)\alpha,\forall\ x,y\in G$. Thus, $\Phi(A)=\Phi(\alpha R_g^{-1},\alpha L_f^{-1},\alpha)=\Phi(R_g^{-1},L_f^{-1},I)=I\Rightarrow A\in\ker\Phi$. □

Theorem 3.10 *Let* $(G,\cdot)$ *be a Smarandache loop with an S-subgroup* H *such that* $f,g\in H$ *and* $\alpha\in SSYM(G,\cdot)$. *If the mapping*

$$\Phi\ :\ \Omega(G,\cdot)\longrightarrow SBS(G,\cdot)\ \ \textit{is defined as}\ \ \Phi\ :\ (\alpha R_g^{-1},\alpha L_f^{-1},\alpha)\mapsto\alpha$$

then,

$$|N_\mu(G,\cdot)|=|\ker\Phi|\ \textit{and}\ |\Omega(G,\cdot)|=|SBS(G,\cdot)||N_\mu(G,\cdot)|.$$

Proof Let the identity map on G be I. Using Theorem 3.9, if

$$g\theta=(R_g^{-1},L_{g^{-1}}^{-1},I),\forall\ g\in N_\mu(G,\cdot)\ \text{then,}\ \theta\ :\ N_\mu(G,\cdot)\longrightarrow\ker\Phi.$$

θ is easily seen to be a bijection, hence $|N_\mu(G,\cdot)|=|\ker\Phi|$.

Since Φ is an homomorphism by Theorem 3.8, then by the first isomorphism theorem in classical group theory, $\Omega(G,\cdot)/\ker\Phi \cong \mathrm{Im}\Phi$. Φ is clearly onto, so $\mathrm{Im}\Phi = SBS(G,\cdot)$, so that $\Omega(G,\cdot)/\ker\Phi \cong SBS(G,\cdot)$. Thus, $|\Omega(G,\cdot)/\ker\Phi| = |SBS(G,\cdot)|$. By Lagrange's theorem, $|\Omega(G,\cdot)| = |\ker\Phi||\Omega(G,\cdot)/\ker\Phi|$, so, $|\Omega(G,\cdot)| = |\ker\Phi||SBS(G,\cdot)|$, $\therefore$ $|\Omega(G,\cdot)| = |N_\mu(G,\cdot)||SBS(G,\cdot)|$. □

Theorem 3.11 *Let $(G,\cdot)$ be a Smarandache loop with an S-subgroup H. If*

$$\begin{aligned}\Theta(G,\cdot) &= \{(f,g)\in H\times H\ :\ (G,\circ)\succsim(G,\cdot)\\ &\quad for\ (G,\circ)\ the\ Smarandache\ principal\ f,g-isotope\ of\ (G,\cdot)\},\end{aligned}$$

then

$$|\Omega(G,\cdot)| = |\Theta(G,\cdot)||SA(G,\cdot)|.$$

Proof Let $A,B\in\Omega(G,\cdot)$. Then there exist $\alpha,\beta\in SSYM(G,\cdot)$ and some $f_1,g_1,f_2,g_2\in H$ such that

$$A = (\alpha R_{g_1}^{-1},\alpha L_{f_1}^{-1},\alpha),\ B = (\beta R_{g_2}^{-1},\beta L_{f_2}^{-1},\beta)\in AUT(G,\cdot).$$

Define a relation $\sim$ on $\Omega(G,\cdot)$ such that

$$A\sim B \Longleftrightarrow f_1 = f_2 \text{ and } g_1 = g_2.$$

It is very easy to show that $\sim$ is an equivalence relation on $\Omega(G,\cdot)$. It can easily be seen that the equivalence class $[A]$ of $A\in\Omega(G,\cdot)$ is the inverse image of the mapping

$$\Psi\ :\ \Omega(G,\cdot)\longrightarrow\Theta(G,\cdot) \text{ defined as } \Psi\ :\ (\alpha R_{g_1}^{-1},\alpha L_{f_1}^{-1},\alpha)\mapsto(f,g).$$

If $A,B\in\Omega(G,\cdot)$ then $\Psi(A)=\Psi(B)$ if and only if $(f_1,g_1)=(f_2,g_2)$ so, $f_1=f_2$ and $g_1=g_2$. Since $\Omega(G,\cdot)\leqslant AUT(G,\cdot)$ by Theorem 3.7, then $AB^{-1} = (\alpha R_{g_1}^{-1},\alpha L_{f_1}^{-1},\alpha)(\beta R_{g_2}^{-1},\beta L_{f_2}^{-1},\beta)^{-1}$ $= (\alpha R_{g_1}^{-1}R_{g_2}\beta^{-1},\alpha L_{f_1}^{-1}L_{f_2}\beta^{-1},\alpha\beta^{-1}) = (\alpha\beta^{-1},\alpha\beta^{-1},\alpha\beta^{-1})\in AUT(G,\cdot)\Leftrightarrow\alpha\beta^{-1}\in SA(G,\cdot)$. So,

$$A\sim B\Longleftrightarrow\alpha\beta^{-1}\in SA(G,\cdot) \text{ and } (f_1,g_1)=(f_2,g_2).$$

Whence, $|[A]| = |SA(G,\cdot)|$. But each $A = (\alpha R_g^{-1},\alpha L_f^{-1},\alpha)\in\Omega(G,\cdot)$ is determined by some $f,g\in H$. So since the set $\{[A]\ :\ A\in\Omega(G,\cdot)\}$ of all equivalence classes partitions $\Omega(G,\cdot)$ by the fundamental theorem of equivalence relation,

$$|\Omega(G,\cdot)| = \sum_{f,g\in H}|[A]| = \sum_{f,g\in H}|SA(G,\cdot)| = |\Theta(G,\cdot)||SA(G,\cdot)|.$$

Therefore, $|\Omega(G,\cdot)| = |\Theta(G,\cdot)||SA(G,\cdot)|$. □

Theorem 3.12 *Let $(G,\cdot)$ be a finite Smarandache loop with a finite S-subgroup H. $(G,\cdot)$ is S-isomorphic to all its S-loop S-isotopes if and only if*

$$|(H,\cdot)|^2|SA(G,\cdot)| = |SBS(G,\cdot)||N_\mu(G,\cdot)|.$$

Proof As shown in [Corollary 5.2, [12]], an S-loop is S-isomorphic to all its S-loop S-isotopes if and only if it is S-isomorphic to all its Smarandache f, g principal isotopes. This will happen if and only if $H \times H = \Theta(G, \cdot)$ where $\Theta(G, \cdot)$ is as defined in Theorem 3.11.

Since $\Theta(G, \cdot) \subseteq H \times H$ then it is easy to see that for a finite Smarandache loop with a finite S-subgroup H, $H \times H = \Theta(G, \cdot)$ if and only if $|H|^2 = |\Theta(G, \cdot)|$. So the proof is complete by Theorems $3.10 - 3.11$. □

Corollary 3.3 *Let $(G, \cdot)$ be a finite Smarandache loop with a finite S-subgroup H. $(G, \cdot)$ is a GS-loop if and only if*

$$|(H, \cdot)|^2|SA(G, \cdot)| = |SBS(G, \cdot)||N_\mu(G, \cdot)|.$$

Proof This follows by the definition of a GS-loop and Theorem 3.12. □

Lemma 3.1 *Let $(G, \cdot)$ be a finite GS-loop with a finite S-subgroup H and a middle nucleus $N_\mu(G, \cdot)$.*

$$|(H, \cdot)| = |N_\mu(G, \cdot)| \Longleftrightarrow |(H, \cdot)| = \frac{|SBS(G, \cdot)|}{|SA(G, \cdot)|}.$$

Proof From Corollary 3.3,

$$|(H, \cdot)|^2|SA(G, \cdot)| = |SBS(G, \cdot)||N_\mu(G, \cdot)|.$$

(1)If $|(H, \cdot)| = |N_\mu(G, \cdot)|$, then

$$|(H, \cdot)||SA(G, \cdot)| = |SBS(G, \cdot)| \Longrightarrow |(H, \cdot)| = \frac{|SBS(G, \cdot)|}{|SA(G, \cdot)|}.$$

(2)If $|(H, \cdot)| = \frac{|SBS(G, \cdot)|}{|SA(G, \cdot)|}$, then $|(H, \cdot)||SA(G, \cdot)| = |SBS(G, \cdot)|$. Hence, multiplying both sides by $|(H, \cdot)|$,

$$|(H, \cdot)|^2|SA(G, \cdot)| = |SBS(G, \cdot)||(H, \cdot)|.$$

So that

$$|SBS(G, \cdot)||N_\mu(G, \cdot)| = |SBS(G, \cdot)||(H, \cdot)| \Longrightarrow |(H, \cdot)| = |N_\mu(G, \cdot)|.$$

□

Corollary 3.4 *Let $(G, \cdot)$ be a finite GS-loop with a finite S-subgroup H. If $|N_\mu(G, \cdot)| \gneqq 1$, then,*

$$|(H, \cdot)| = \frac{|SBS(G, \cdot)|}{|SA(G, \cdot)|}. \textit{ Hence, } |(G, \cdot)| = \frac{n|SBS(G, \cdot)|}{|SA(G, \cdot)|} \textit{ for some } n \gneqq 1.$$

Proof By hypothesis, $\{e\} \neq H \neq G$. In a loop, $N_\mu(G, \cdot)$ is a subgroup, hence if $|N_\mu(G, \cdot)| \gneqq 1$, then, we can take $(H, \cdot) = N_\mu(G, \cdot)$. So that $|(H, \cdot)| = |N_\mu(G, \cdot)|$. Thus by Lemma 3.1, $|(H, \cdot)| = \frac{|SBS(G, \cdot)|}{|SA(G, \cdot)|}$.

As shown in [Section 1.3, [8]], a loop L obeys the Lagrange's theorem relative to a subloop H if and only if $H(hx) = Hx$ for all $x \in L$ and for all $h \in H$. This condition is obeyed by $N_\mu(G, \cdot)$, hence

$$|(H, \cdot)|\Big||(G, \cdot)| \Longrightarrow \frac{|SBS(G, \cdot)|}{|SA(G, \cdot)|}\Bigg||(G, \cdot)| \Longrightarrow$$

there exists $n \in \mathbb{N}$ such that

$$|(G, \cdot)| = \frac{n|SBS(G, \cdot)|}{|SA(G, \cdot)|}.$$

But if $n = 1$, then $|(G, \cdot)| = |(H, \cdot)| \Longrightarrow (G, \cdot) = (H, \cdot)$ hence $(G, \cdot)$ is a group which is a contradiction to the fact that $(G, \cdot)$ is an S-loop. Therefore,

$$|(G, \cdot)| = \frac{n|SBS(G, \cdot)|}{|SA(G, \cdot)|}$$

for some natural numbers $n \gneqq 1$. $\square$

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