THE SMARANDACHE MULTIPLICATIVE FUNCTION

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Abstract For any positive integer n, we define $f(n)$ as a Smarandache multiplicative function, if $f(ab) = \max(f(a), f(b))$, $(a, b) = 1$. Now for any prime p and any positive integer α, we take $f(p^\alpha) = \alpha p$. It is clear that $f(n)$ is a Smarandache multiplicative function. In this paper, we study the mean value properties of $f(n)$, and give an interesting mean value formula for it.

Keywords: Smarandache multiplicative function; Mean Value; Asymptotic formula.

§1 Introduction and results

For any positive integer n, we define $f(n)$ as a Smarandache multiplicative function, if $f(ab) = \max(f(a), f(b))$, $(a, b) = 1$. Now for any prime p and any positive integer α, we take $f(p^\alpha) = \alpha p$. It is clear that $f(n)$ is a new Smarandache multiplicative function, and if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ is the prime powers factorization of n, then

$$f(n) = \max_{1 \le i \le k} \{f(p_i^{\alpha_i})\} = \max_{1 \le i \le k} \{\alpha_i p_i\}. \quad (1)$$

About the arithmetical properties of $f(n)$, it seems that none had studied it before. This function is very important, because it has many similar properties with the Smarandache function $S(n)$ (see reference [1][2]). The main purpose of this paper is to study the mean value properties of $f(n)$, and obtain an interesting mean value formula for it. That is, we shall prove the following:

Theorem. For any real number $x \ge 2$, we have the asymptotic formula

$$\sum_{n \le x} f(n) = \frac{\pi^2}{12} \cdot \frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).$$

§2 Proof of the theorem

In this section, we shall complete the proof of the theorem. First we need following one simple Lemma. For convenience, let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the prime powers factorization of n, and $P(n)$ be the greatest prime factor of n, that is, $P(n) = \max_{1 \le i \le k} \{p_i\}$. Then we have

Lemma. For any positive integer n, if there exists $P(n)$ such that $P(n) > \sqrt{n}$, then we have the identity

$$f(n) = P(n).$$

Proof. From the definition of $P(n)$ and the condition $P(n) > \sqrt{n}$, we get

$$f(P(n)) = P(n). \tag{2}$$

For other prime divisors p_i of n $(1 \leq i \leq k$ and $p_i \neq P(n))$, we have

$$f(p_i^{\alpha_i}) = \alpha_i p_i.$$

Now we will debate the upper bound of $f(p_i^{\alpha_i})$ in three cases:

(I) If $\alpha_i = 1$, then $f(p_i) = p_i \leq \sqrt{n}$.

(II) If $\alpha_i = 2$, then $f(p_i^2) = 2p_i \leq 2 \cdot n^{\frac{1}{4}} \leq \sqrt{n}$.

(III) If $\alpha_i \geq 3$, then $f(p_i^{\alpha_i}) = \alpha_i \cdot p_i \leq \alpha_i \cdot n^{\frac{1}{2\alpha_i}} \leq n^{\frac{1}{2\alpha_i}} \cdot \frac{\ln n}{\ln p_i} \leq \sqrt{n}$, where we use the fact that $\alpha \leq \frac{\ln n}{\ln p}$ if $p^\alpha | n$.

Combining (I)-(III), we can easily obtain

$$f(p_i^{\alpha_i}) \leq \sqrt{n}. \tag{3}$$

From (2) and (3), we deduce that

$$f(n) = \max_{1 \leq i \leq k} \{f(p_i^{\alpha_i})\} = f(P(n)) = P(n).$$

This completes the proof of Lemma.

Now we use the above Lemma to complete the proof of the theorem. First we define two sets $\mathcal{A}$ and $\mathcal{B}$ as following:

$$\mathcal{A} = \{n | n \leq x, P(n) \leq \sqrt{n}\}, \qquad \mathcal{B} = \{n | n \leq x, P(n) > \sqrt{n}\}.$$

Using the Euler summation formula (see reference [3]), we may get

$$\begin{aligned}
&\sum_{n \in \mathcal{A}} f(n) \ll \sum_{n \leq x} \sqrt{n} \ln n \\
&= \int_1^x \sqrt{t} \ln t dt + \int_1^x (t - [t])(\sqrt{t} \ln t)' dt + \sqrt{x} \ln x (x - [x]) \\
&\ll x^{\frac{3}{2}} \ln x.
\end{aligned} \tag{4}$$

Similarly, from the Abel's identity we also have

$$\begin{aligned}
&\sum_{n \in \mathcal{B}} f(n) = \sum_{\substack{n \leq x \\ P(n) > \sqrt{n}}} P(n) = \sum_{n \leq \sqrt{x}} \sum_{n \leq p \leq \frac{x}{n}} p \\
= \quad & \sum_{n \leq \sqrt{x}} \sum_{\sqrt{x} \leq p \leq \frac{x}{n}} p + O\left(\sum_{n \leq \sqrt{x}} \sum_{n \leq p \leq \frac{x}{n}} \sqrt{x} \right) \\
= \quad & \sum_{n \leq \sqrt{x}} \left(\frac{x}{n} \pi\left(\frac{x}{n}\right) - \sqrt{x} \pi(\sqrt{x}) - \int_{\sqrt{x}}^{\frac{x}{n}} \pi(s) ds \right) + O\left(x^{\frac{3}{2}} \ln x\right),
\end{aligned} \tag{5}$$

where $\pi(x)$ denotes all the numbers of prime which is not exceeding x. Note that

$$\pi(x) = \frac{x}{\ln x} + O\left(\frac{x}{\ln^2 x}\right),$$

from (5) we have

$$\begin{aligned}\sum_{\sqrt{x}\le p\le \frac{x}{n}} &= \frac{x}{n}\pi\left(\frac{x}{n}\right) - \sqrt{x}\pi(\sqrt{x}) - \int_{\sqrt{x}}^{\frac{x}{n}} \pi(s)ds \\ &= \frac{1}{2}\cdot\frac{x^2}{n^2\ln x/n} - \frac{1}{2}\cdot\frac{x}{\ln\sqrt{x}} + O\left(\frac{x^2}{n^2\ln^2 x/n}\right) \\ &\quad + O\left(\frac{x}{\ln^2\sqrt{x}}\right) + O\left(\frac{x^2}{n^2\ln^2 x/n} - \frac{x}{\ln^2\sqrt{x}}\right). \qquad (6)\end{aligned}$$

Hence

$$\begin{aligned}\sum_{n\le\sqrt{x}} \frac{x^2}{n^2\ln x/n} &= \sum_{n\le \ln^2 x} \frac{x^2}{n^2\ln x/n} + O\left(\sum_{\ln^2 x\le n\le\sqrt{x}} \frac{x^2}{n^2\ln x}\right) \\ &= \frac{\pi^2}{6}\cdot\frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right), \qquad (7)\end{aligned}$$

and

$$\sum_{n\le\sqrt{x}} \frac{x^2}{n^2\ln^2 x/n} = O\left(\frac{x^2}{\ln^2 x}\right). \qquad (8)$$

From (4), (5), (6), (7) and (8) , we may immediately deduce that

$$\begin{aligned}\sum_{n\le x} f(n) &= \sum_{n\in\mathcal{A}} f(n) + \sum_{n\in\mathcal{B}} f(n) \\ &= \frac{\pi^2}{12}\cdot\frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).\end{aligned}$$

This completes the proof of the theorem.

Note. If we use the asymptotic formula

$$\pi(x) = \frac{x}{\ln x} + \frac{c_1 x}{\ln^2 x} + \cdots + \frac{c_m x}{\ln^m x} + O\left(\frac{x}{\ln^{m+1} x}\right)$$

to substitute

$$\pi(x) = \frac{x}{\ln x} + O\left(\frac{x}{\ln^2 x}\right)$$

in (5) and (6), we can get a more accurate asymptotic formula for $\sum_{n\le x} f(n)$.

References

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[3] Tom M. Apostol, Introduction to Analytic Number Theory, New York, 1976.