Smarandache's Orthic Theorem

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Abstract.
We present the Smarandache's Orthic Theorem in the geometry of the triangle.

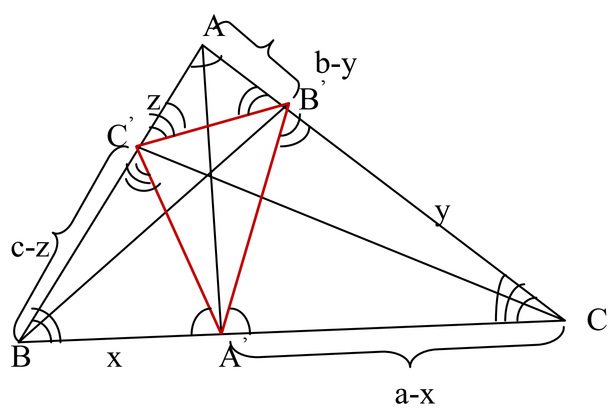
Smarandache's Orthic Theorem.

Given a triangle ABC whose angles are all acute (acute triangle), we consider $A'B'C'$, the triangle formed by the legs of its altitudes.

In which conditions the expression:

$$\|A'B'\|\cdot\|B'C'\|+\|B'C'\|\cdot\|C'A'\|+\|C'A'\|\cdot\|A'B'\|$$

is maximum?



Proof.

We have

$$\Delta ABC \sim \Delta A'B'C' \sim \Delta AB'C \sim \Delta A'BC' \qquad (1)$$

We note

$$\|BA'\| = x,\ \|CB'\| = y,\ \|AC'\| = z\,.$$

It results that

$$\|A'C\| = a - x,\ \|B'A\| = b - y,\ \|C'B\| = c - z$$

$$B\widehat{A}C = B'\widehat{A'}C = B\widehat{A'}C';\ \ \widehat{A}BC = A\widehat{B'}C' = A'\widehat{B'}C;\ \ BC\widehat{A} = B\widehat{C'}A' = B'\widehat{C'}A$$

From these equalities it results the relation (1)

$$\Delta A'BC' \sim \Delta A'B'C \Rightarrow \frac{\|A'C'\|}{a-x} = \frac{x}{\|A'B'\|} \quad (2)$$

$$\Delta A'B'C \sim \Delta AB'C' \Rightarrow \frac{\|A'C'\|}{z} = \frac{c-z}{\|B'C'\|} \quad (3)$$

$$\Delta AB'C' \sim \Delta A'B'C \Rightarrow \frac{\|B'C'\|}{y} = \frac{b-y}{\|A'B'\|} \quad (4)$$

From (2), (3) and (4) we observe that the sum of the products from the problem is equal to:

$$x(a-x)+y(b-y)+z(c-z)=\frac{1}{4}\left(a^2+b^2+c^2\right)-\left(x-\frac{a}{2}\right)^2-\left(y-\frac{b}{2}\right)^2-\left(z-\frac{c}{2}\right)^2$$

which will reach its maximum as long as $x=\frac{a}{2}$, $y=\frac{b}{2}$, $z=\frac{c}{2}$, that is when the altitudes' legs are in the middle of the sides, therefore when the ΔABC is equilateral. The maximum of the expression is $\frac{1}{4}\left(a^2+b^2+c^2\right)$.

Conclusion (Smarandache's Orthic Theorem):
If we note the lengths of the sides of the triangle Δ ABC by ||AB|| = *c*, ||BC|| = *a*, ||CA|| = *b*, and the lengths of the sides of its orthic triangle Δ A\`B\`C\` by ||A\`B\`|| = *c*\`, ||B\`C\`|| = *a*\`, ||C\`A\`|| = *b*\`, then we proved that:

$$4(a`b` + b`c` + c`a`) \le a^2 + b^2 + c^2.$$

Open Problems related to Smarandache's Orthic Theorem:

1. Generalize this problem to polygons. Let $A_1A_2\ldots A_m$ be a polygon and P a point inside it. From P we draw perpendiculars on each side A_iA_{i+1} of the polygon and we note by Ai' the intersection between the perpendicular and the side A_iA_{i+1}. A pedal polygon $A_1'A_2'\ldots A_m'$ is formed. What properties does this pedal polygon have?
2. Generalize this problem to polyhedrons. Let $A_1A_2\ldots A_n$ be a poliyhedron and P a point inside it. From P we draw perpendiculars on each polyhedron face F_i and we note by Ai' the intersection between the perpendicular and the side F_i. A pedal polyhedron $A_1'A_2'\ldots A_p'$ is formed, where p is the number of polyhedron's faces. What properties does this pedal polyhedron have?

References:

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