Smarandache pseudo-CI algebras

L. C. Ciungu
Department of Mathematics, University of Iowa,
14 MacLean Hall, Iowa City, Iowa 52242-1419, USA

lavinia-ciungu@uiowa.edu

A. Rezaei*
Department of Mathematics, Payame Noor University,
P.o.box. 19395-3697, Tehran, Iran
rezaei@pnu.ac.ir
* *Correspondence: Akbar Rezaei (rezaei@pnu.ac.ir)*

A. Radfar
Department of Mathematics, Payame Noor University,
P.o.box. 19395-3697, Tehran, Iran
radfar@pnu.ac.ir

Abstract. In this paper, we define the notion of Smarandache pseudo-CI algebras and we investigate their properties. We also define and study the notions of Smarandache filters, pseudo-CI Smarandache homomorphisms and modal Smarandache operators on pseudo-CI algebras. The classes of Smarandache fantastic, implicative and positive implicative filters of Smarandache pseudo-CI algebras are defined and studied by extending some results regarding Smarandache fantastic, fresh and clean ideals in Smarandache BCI-algebras and Smarandache BCH-algebras to the case of Smarandache pseudo-CI algebras. The notion of Smarandache commutative pseudo-CI algebras is defined and a characterization theorem is given. It is proved that in the case of commutative Q-Smarandache pseudo-CI algebras the notions of Smarandache filters and fantastic filters coincide.

Keywords: pseudo-CI algebra, pseudo-BE algebra, Smarandache pseudo-CI algebra, Q-Smarandache filter.

1. Introduction

Developing algebraic models for non-commutative multiple-valued logics is a central topic in the study of fuzzy systems. Pseudo-BCK algebras were introduced by G. Georgescu and A. Iorgulescu in [13] as algebras with "two differences", a left- and right-difference, and with a constant element 0 as the least element. Pseudo-BCK algebras were intensively studied in [15] (also see [14], [22], [21], [8]). Pseudo-BE algebras were introduced by R. A. Borzooei et al. as a generalization of BE-algebras and properties of these structures have recently been studied in [28] (also see [6]). L. C. Ciungu defined the notion of commutative pseudo-BE algebras and proved that the class of commutative

pseudo-BE algebras is term equivalent to the class of commutative pseudo-BCK algebras ([9]). Recently, A. Rezaei et al. introduced the notion of pseudo-CI algebras as generalizations of CI-algebras and they provided some conditions for a pseudo-CI algebra to be a pseudo-BE algebra ([29]). The class of singular pseudo-CI algebras was defined and it was proved that any singular pseudo-CI algebra is a pseudo-BCI algebra (see [12], [29]). A. Rezaei et al. defined the dual pseudo-Q and dual pseudo-QC algebras, investigated their properties and gave characterizations of these structures ([30]). It was also proved that the class of commutative dual pseudo-QC algebras coincides with the class of commutative pseudo-BCI algebras.
Generally, a *Smarandache structure* on a set A means a weak structure W on A such that there exists a proper subset B which is embedded with a stronger structure S ([16]). Smarandache structures on multiple-valued logic algebras have been studied in [4] (also see [3], [5], [16], [17], [18], [19], [24], [25]). A. Borumand Saeid defined the notion of Smarandache (weak) BE-algebras and proved some of their properties ([2], [3]).

In this paper, we define the notion of Smarandache pseudo-CI algebras and we investigate their properties. We also define and study the notions of Smarandache filters, pseudo-CI Smarandache homomorphisms and modal Smarandache operators on pseudo-CI algebras. The classes of Smarandache fantastic, implicative and positive implicative filters of Smarandache pseudo-CI algebras are defined and studied by extending some results regarding Smarandache fantastic, fresh and clean ideals in Smarandache BCI-algebras and Smarandache BCH-algebras ([19], [18], [4]) to the case of Smarandache pseudo-CI algebras. We give a characterization of Smarandache implicative filters and we present conditions for a Smarandache filter to be a Smarandache implicative filter. For a Q-Smarandache pseudo-CI algebra we prove that any Smarandache implicative filter is a filter and any Smarandache positive implicative filter contained in Q is a Smarandache filter. We also give a characterization of Smarandache positive implicative filters. The notion of Smarandache commutative pseudo-CI algebras is defined and a characterization theorem is given. It is proved that in the case of commutative Q-Smarandache pseudo-CI algebras the notions of Smarandache filters and fantastic filters coincide. Finally, we define and investigate the notion of a Smarandache upper set in a pseudo-CI algebra and we show that every Q-Smarandache filter is a union of Q-Smarandache upper sets.

2. Preliminaries

In this section, we recall some basic notions and results regarding pseudo-CI algebras and pseudo-BE algebras. Pseudo-BE algebras were introduced in [5] as a generalization of BE-algebras (see [20]) and properties of it$'s$ have recently been studied in [30] and [6].

A *CI-algebra* ([23]) is an algebra $(X;\rightarrow,1)$ of type $(2,0)$ satisfying the following axioms, for all $x,y,z\in X$:
(CI_1) $x\rightarrow x=1$;
(CI_2) $1\rightarrow x=x$;
(CI_3) $x\rightarrow(y\rightarrow z)=y\rightarrow(x\rightarrow z)$.

We introduce a binary relation $\leq$ on X by $x\leq y$ if and only if $x\rightarrow y=1$. A CI-algebra $(X;\rightarrow,1)$ is said to be a *BE-algebra* ([20]) if (BE) $x\rightarrow 1=1$, for all $x\in X$.

Definition 2.1. ([29]) An algebra $(X;\rightarrow,\rightsquigarrow,1)$ of type $(2,2,0)$ is called a *pseudo-CI algebra if, for all* $x,y,z\in X$, *it satisfies the following axioms:*
$(psCI_1)$ $x\rightarrow x=x\rightsquigarrow x=1$;
$(psCI_2)$ $1\rightarrow x=1\rightsquigarrow x=x$;
$(psCI_3)$ $x\rightarrow(y\rightsquigarrow z)=y\rightsquigarrow(x\rightarrow z)$;
$(psCI_4)$ $x\rightarrow y=1$ if and only if $x\rightsquigarrow y=1$.

Remark 2.1. *If* $(X;\rightarrow,\rightsquigarrow,1)$ *is a pseudo-CI algebra satisfying* $x\rightarrow y=x\rightsquigarrow y$, *for all* $x,y\in X$, *then* $(X;\rightarrow,1)$ *is a CI-algebra.*
Also, if $(X;\rightarrow,\rightsquigarrow,1)$ *is a pseudo-CI algebra, then* $(X;\rightsquigarrow,\rightarrow,1)$ *is too.*

Remark 2.2. Since every pseudo-*BCI* algebra satisfies $(\mathrm{psCI}_1)-(\mathrm{psCI}_4)$, pseudo-*BCI* algebras are contained in the class of pseudo-*CI* algebras.

In the sequel, we will also refer to the pseudo-CI algebra $(X;\rightarrow,\rightsquigarrow,1)$ by $\mathfrak{X}$.

Any pseudo-CI algebra $\mathfrak{X}$ verifying condition $(psBE)$ $x\rightarrow 1=x\rightsquigarrow 1=1$, for all $x,y\in X$, is said to be a *pseudo-BE algebra* ([6]). A pseudo-CI algebra which is not a pseudo-BE algebra, pseudo-BCI algebra and pseudo-BCH algebra will be called *proper*. A pseudo-CI algebra with condition (A) or a pseudo-CI(A) algebra for short, is a pseudo-CI algebra $\mathfrak{X}$ satisfying the condition (A):

(A) for all $x,y,z\in X$, if $x\preceq y$, then $y\rightarrow z\preceq x\rightarrow z$ and $y\rightsquigarrow z\preceq x\rightsquigarrow z$.

In a pseudo-CI algebra $\mathfrak{X}$ we can introduce a binary relation $\preceq$ by:

$x\preceq y$ if and only if $x\rightarrow y=1$ if and only if $x\rightsquigarrow y=1$, for all $x,y\in X$.

Note that $\preceq$ is reflexive by (psCI_1).

Example 2.1. ([29]) (1) Let $X=\{1,a,b,c,d,e\}$. Define the binary operations $\rightarrow$ and $\rightsquigarrow$ on X as follows:

$\rightarrow$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	a	1	c	b	e	d
b	b	d	1	e	a	c
c	d	b	e	1	c	a
d	c	e	a	d	1	b
e	e	c	d	a	b	1

$\rightsquigarrow$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	a	1	d	e	b	c
b	b	c	1	a	e	d
c	d	e	a	1	c	b
d	c	b	e	d	1	a
e	e	d	c	b	a	1

Then $(X;\rightarrow,\rightsquigarrow,1)$ is a pseudo-CI algebra, but not a pseudo-BE algebra, since $a\rightarrow 1=a\neq 1$ and $a\rightsquigarrow 1=a\neq 1$.

(2) Let $X = \{1, a, b, c, d, e, f, g, h\}$. Define the binary operations $\to$ and $\rightsquigarrow$ on X as follows:

$\to$	1	a	b	c	d	e	f	g	h
1	1	a	b	c	d	e	f	g	h
a	1	1	1	1	d	e	f	g	h
b	1	c	1	1	d	e	f	g	h
c	1	c	b	1	d	e	f	g	h
d	d	d	d	d	1	g	h	e	f
e	e	e	e	e	h	1	g	f	d
f	f	f	f	f	g	h	1	d	e
g	h	h	h	h	e	f	d	1	g
h	g	g	g	g	f	d	e	h	1

$\rightsquigarrow$	1	a	b	c	d	e	f	g	h
1	1	a	b	c	d	e	f	g	h
a	1	1	1	1	d	e	f	g	h
b	1	c	1	1	d	e	f	g	h
c	1	c	b	1	d	e	f	g	h
d	d	d	d	d	1	h	g	f	e
e	e	e	e	e	g	1	h	d	f
f	f	f	f	f	h	g	1	e	d
g	h	h	h	h	f	d	e	1	g
h	g	g	g	g	e	f	d	h	1

Then $(X; \to, \rightsquigarrow, 1)$ is a proper pseudo-CI algebra.

Definition 2.2. ([6]) Let $\mathfrak{X}$ be a pseudo-BE algebra. A subset F of X is called a *filter* of $\mathfrak{X}$ if for all $x, y \in X$:
(F_1) $1 \in F$;
(F_2) $x \to y \in F$ and $x \in F$ imply $y \in F$.

Denote by $\mathcal{F}(X)$ set of all filters of $\mathfrak{X}$. Obviously, $\{1\}, X \in \mathcal{F}(X)$.

Definition 2.3. ([11]) Let $\mathfrak{X}$ be a pseudo-BE algebra. A mapping $f : X \longrightarrow X$ is called a *modal operator* on X if it satisfies the following conditions for all $x, y \in X$:
(M_1) $x \le f(x)$;
(M_2) $f(f(x)) = f(x)$;
(M_3) $f(x \to y) \le f(x) \to f(y)$ and $f(x \rightsquigarrow y) \le f(x) \rightsquigarrow f(y)$.
The pair (X, f) is called a *modal pseudo-BE algebra*.

Denote by $\mathcal{MOD}(X)$ set of all modal operators on X.

3. Smarandache pseudo-CI algebras

In this section, we define the notion of a Smarandache pseudo-CI algebra and investigate these properties. We also define and study the notions of Smarandache filters, pseudo-CI Smarandache homomorphisms and modal Smarandache operators on pseudo-CI algebras.

Definition 3.1. A pseudo-CI algebra $\mathfrak{X}$ is said to be a *Q-Smarandache pseudo-CI algebra* if there is a proper subset Q of X such that:
(S_1) $1 \in Q$ and $|Q| \geq 2$;
(S_2) $\mathfrak{Q} = (Q; \rightarrow, \rightsquigarrow, 1)$ is a pseudo-BE algebra under the operations of $\mathfrak{X}$.
Q is called the *heart of* $\mathfrak{X}$*, if it satisfies* (S_1)*,* (S_2) *and axiom:*
(S_3) If there is $\emptyset \neq S \subseteq X$ satisfies (S_1) and (S_2), then $S \subseteq Q$
(i.e. $Q = \{x \in X : x \rightarrow 1 = 1\}$).

Remark 3.1. *Using* (S_3)*, the heart of* $\mathfrak{X}$ *is unique and* $Q = X$ *if and only if* $\mathfrak{X}$ *is a pseudo-BE algebra.*

Example 3.1. (1) Every pseudo-BE algebra is a Smarandache pseudo-CI algebra.

(2) Consider the pseudo-CI algebra given in Example 2.1 (2), let $Q_1 = \{1, a, b, c\}$, $Q_2 = \{1, a\}$, $Q_3 = \{1, b\}$, $Q_4 = \{1, a, c\}$, and let $Q_5 = \{1, b, c\}$. Then $\mathfrak{X}$ is a Q_1, Q_2, Q_3, Q_4 and Q_5 Smarandache pseudo-CI algebra. Moreover, Q_1 satisfies (S_3), hence it is the heart of $\mathfrak{X}$.

Proposition 3.1. *In any Q-Smarandache pseudo-CI algebra* $\mathfrak{X}$ *the following hold, for all* $x, y \in X$*:*
(1) *if* $x \notin Q$*, then* $x \rightarrow 1 \notin Q$ *and* $x \rightsquigarrow 1 \notin Q$*;*
(2) $x \rightarrow 1 = 1$ *or* $x \rightarrow 1 \notin Q$*;*
(3) *if* $x \rightarrow 1 \notin Q$*, then* $x \notin Q$*;*
(4) *if* $x \rightarrow 1 = y \rightarrow 1$*, then* $x \rightarrow y \in Q$ *and* $y \rightarrow x \in Q$*;*
(5) *if* $x \rightsquigarrow 1 = y \rightsquigarrow 1$*, then* $x \rightsquigarrow y \in Q$ *and* $y \rightsquigarrow x \in Q$*;*
(6) *if* $x \in Q$ *and* $y \notin Q$*, then* $x \rightarrow y \notin Q$*,* $x \rightsquigarrow y \notin Q$ *and* $y \rightarrow x \notin Q$*,* $y \rightsquigarrow x \notin Q$*.*

Theorem 3.1. *Let* $\mathfrak{X}$ *be a proper pseudo-CI algebra. Then* $\mathfrak{X}$ *is a Q-Smarandache pseudo-CI algebra if and only if there exists* $Q \subseteq X$ *such that* $|Q| \geq 2$ *and* $x \rightarrow 1 = 1$*, for all* $x \in Q$*.*

Proof. Let $\mathfrak{X}$ be a Q-Smarandache pseudo-CI algebra. Then by definition we get there exists $Q \subseteq X$ such that $x \rightarrow 1 = 1$, for all $x \in Q$.
Conversely, consider $Q = \{x \in X \mid x \rightarrow 1 = 1\}$. It is suffice to prove that Q is a subalgebra of X. If $x, y \in Q$, then $x \rightarrow 1 = y \rightarrow 1 = 1$. By (a_4), we get

$$(x \rightarrow y) \rightarrow 1 = (x \rightarrow 1) \rightsquigarrow (y \rightarrow 1) = 1 \rightsquigarrow 1 = 1.$$

Thus $x \rightarrow y \in Q$. Similarly, $x \rightsquigarrow y \in Q$. Hence Q is a subalgebra of $\mathfrak{X}$. □

Definition 3.2. A subset F of a pseudo-CI algebra $\mathfrak{X}$ is called a *Smarandache filter* of $\mathfrak{X}$ related to $\mathfrak{Q}$ (or briefly, Q-Smarandache filter of $\mathfrak{X}$) if it satisfies, for all $y \in Q$ and $x \in F$:
(SF_1) $1 \in F$;
(SF_2) $x \rightarrow y \in F$ implies $y \in F$;
(SF_3) $x \rightsquigarrow y \in F$ implies $y \in F$.

Denote by $\mathcal{F}_Q(X)$ set of all Q-Smarandache filters of $\mathfrak{X}$.

Example 3.2. Consider the pseudo-CI algebra given in Example 2.1 (2). We can see that $\mathfrak{X}$ is a Q-Smarandache pseudo-CI algebra where $Q = \{1, a, b, c\}$. Note that $F_1 = \{1, a, b, c, d\}$, $F_2 = \{1, h\}$ and $F_3 = \{1, g, h\}$ are Q-Smarandache filters of $\mathfrak{X}$.

The following we provide some conditions for a subalgebra to be a Q-Smarandache filter.

Theorem 3.2. *Let F be a subalgebra of $\mathfrak{X}$. Then F is a Q-Smarandache filter of $\mathfrak{X}$ if and only if for all $x, y \in X$,*
$x \in F,\ y \in Q \setminus F$ implies $x \to y \in Q \setminus F$ *and* $x \rightsquigarrow y \in Q \setminus F$.

Proof. Assume that $F \in \mathcal{F}_Q(X)$ and $x, y \in X$, such that $x \in F$ and $y \in Q \setminus F$. If $x \to y \notin Q \setminus F$, then $x \to y \in F$ (i.e. $y \in F$), which is a contradiction. Hence $x \to y \in Q \setminus F$. Now, if $x \rightsquigarrow y \notin Q \setminus F$, then $x \rightsquigarrow y \in F$ (i.e. $y \in F$), which is a contradiction. Hence $x \rightsquigarrow y \in Q \setminus F$.

Conversely, assume that the hypothesis is valid. Since F is a subalgebra, we have $1 \in F$. For every $x \in F$, let $x \to y \in F$. If $y \notin F$, then $x \to y \in Q \setminus F$ by assumption, which is a contradiction. Hence $y \in F$. Now, let $x \rightsquigarrow y \in F$. Then by hypothesis we have $y \in F$. Therefore, F is a Q-Smarandache filter of $\mathfrak{X}$. □

Theorem 3.3. *Let $\mathfrak{X}$ be a Q-Smarandache pseudo-CI algebra, and let F be a subset of X such that $Q \subseteq F$. Then F is a Smarandache filter of $\mathfrak{X}$.*

The next example shows that the converse of Theorem 3.3 is not valid in general.

Example 3.3. Let $\mathfrak{X}$ be the pseudo-CI algebra from Example 2.1 (2).
(1) If $Q = \{1, a, b, c\}$ and $F = \{1, b, c, g\}$, then F is a Q-Smarandache filter.
(2) If $Q = \{1, a, b, c\}$ we can easily see that, every filter F of $\mathfrak{X}$ containing Q is a Q-Smarandache filter of $\mathfrak{X}$. For example $F_1 = \{1, a, b, c, d, e, \}$ is a Q-Smarandache filter of $\mathfrak{X}$.

Proposition 3.2. *Any filter of a pseudo-CI algebra $\mathfrak{X}$ is a Q-Smarandache filter.*

The following example shows that the converse of above proposition is not valid in general.

Example 3.4. Consider the pseudo-CI algebra from Example 2.1 (2) and let $Q := \{1, a, b, c\}$. Then $\mathfrak{X}$ is a Q-Smarandache pseudo-CI. Also, $F = \{1, h\}$ is a Q-Smarandache filter of $\mathfrak{X}$, but it is not a filter of $\mathfrak{X}$, since $h \to g = h \rightsquigarrow g = h \in F$ and $h \in F$, but $g \notin F$.

In [7], R. A. Borzooei et al. introduced the notion of distributive pseudo-BE algebras and got some useful results. The following we define the notion of *weak distributive Q-Smarandach pseudo-CI algebras.*

Definition 3.3. A Q-Smarandache pseudo-CI algebra $\mathfrak{X}$, where Q is the heart of $\mathfrak{X}$, is said to be *weak distributive* if it satisfies only one of the following conditions, for all $x, y, z \in Q$:
(WD_1) $x \to (y \rightsquigarrow z) = (x \to y) \rightsquigarrow (x \to z)$;
(WD_2) $x \rightsquigarrow (y \to z) = (x \rightsquigarrow y) \to (x \rightsquigarrow z)$.

Remark 3.2. *Take $x = y$ in (WD_1) and (WD_2) and applying $(psCI_2)$, we get:*
$x \to (x \rightsquigarrow z) = (x \to x) \rightsquigarrow (x \to z) = 1 \rightsquigarrow (x \to z) = x \to z$ *and*
$x \rightsquigarrow (x \to z) = (x \rightsquigarrow x) \to (x \rightsquigarrow z) = 1 \to (x \rightsquigarrow z) = x \rightsquigarrow z$.
Now, using $(psCI_4)$, we have $x \to z = x \to (x \rightsquigarrow z) = x \rightsquigarrow (x \to z) = x \rightsquigarrow z$, for all $x, z \in Q$. Consequently, $\to = \rightsquigarrow$, and so Q is a BE-algebra.

In this paper, weak distributive pseudo-CI algebra satisfies (WD_1).

Example 3.5. (1) Let $X = \{1, a, b, c, d\}$. Define the binary operations $\to$ and $\rightsquigarrow$ on X by the following tables:

$\to$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	1	d
b	1	1	1	1	d
c	1	a	a	1	d
d	d	d	d	d	1

$\rightsquigarrow$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	1	d
b	1	1	1	1	d
c	1	a	b	1	d
d	d	d	d	d	1

Then $(X; \to, \rightsquigarrow, 1)$ is a weak distributive pseudo-CI algebra, where $Q = \{1, a, b, c\}$.

(2) Consider the Q-Smarandache pseudo-CI algebra given in Example 2.1 (2), where $Q := \{1, a, b, c\}$. Then $\mathfrak{X}$ is not a weak distributive, since

$$b \to (b \rightsquigarrow a) = b \to c = 1 \neq c = 1 \rightsquigarrow c = (b \to b) \rightsquigarrow (b \to a).$$

Remark 3.3. Singular pseudo-CI algebras were introduced and studied by Rezaei et al. in [29]. Now, if $\mathfrak{X}$ is a singular pseudo-CI algebra, then $Q = \{1\}$, and so $\mathfrak{X}$ is a weak distributive pseudo-CI algebra.

Proposition 3.3. *If F is a Q-Smarandache filter of weak distributive pseudo-CI algebra $\mathfrak{X}$, then for all $x, y, z \in Q$:*
(1) $z \rightsquigarrow (y \to x) \in F$ *and* $z \rightsquigarrow y \in F$ *imply* $z \rightsquigarrow x \in F$;
(2) $z \to (y \rightsquigarrow x) \in F$ *and* $z \to y \in F$ *imply* $z \to x \in F$.

Corollary 3.1. *If F is a Q-Smarandache filter of weak distributive pseudo-CI algebra $\mathfrak{X}$, then for all $x, y \in Q$:*
(1) $y \rightsquigarrow (y \to x) \in F$ *implies* $y \rightsquigarrow x \in F$;
(2) $y \to (y \rightsquigarrow x) \in F$ *implies* $y \to x \in F$.

Proposition 3.4. *Let F be a Q-Smarandache filter of a pseudo-CI algebra $\mathfrak{X}$ and $x, y \in Q$. Then*
(1) *if $x \in F$, $y \in Q$ and $x \preceq y$, then $y \in F$;*
(2) *if $\mathfrak{X}$ is weak distributive pseudo-CI algebra and $x, y \in F$, then $x \to y \in F$;*
(3) *if $\mathfrak{X}$ is weak distributive pseudo-CI algebra and $x, y \in F$, then $x \rightsquigarrow y \in F$.*

Theorem 3.4. *Any Q-Smarandache filter is a subalgebra of $\mathfrak{Q}$.*

The converse of Theorem 3.4 is not valid in general. Indeed, in Example 2.1 (1), $S = \{1, a\}$ is a subalgebra, but it is not a Q-Smarandache filter.

Theorem 3.5. *Let Q_1 and Q_2 be pseudo-BE algebras which are properly contained in a pseudo-CI algebra $\mathfrak{X}$ and $Q_1 \subseteq Q_2$. Then every Q_2-Smarandache filter is a Q_1-Smarandache filter of $\mathfrak{X}$.*

The following example shows that the converse of Theorem 3.5 is not valid in general.

Example 3.6. Let $X = \{1, a, b, c, d, e, f, g, h\}$, $Q_1 = \{1, a\}$, $Q_2 = \{1, a, b, c\}$ and $F = \{1, a, b\}$. According to Example 2.1 (2), we can see that $\mathfrak{X}$ is a Q_1-Smrandache pseudo-CI algebra and Q_2-Smrandache pseudo-CI algebra. Also, F is a Q_1-Smarandache filter of $\mathfrak{X}$, but F is not Q_2-Smrandache filter of $\mathfrak{X}$. Indeed, $b \to c = 1 \in F$, $b \in F$, $c \in Q_2$, but $c \notin F$.

Definition 3.4. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Q_X and Q_Y-Smarandache pseudo-CI algebras, respectively. A mapping $f : X \longrightarrow Y$ is called a *Smarandache pseudo-CI homomorphism* if $f_s = f_{|Q} : Q_X \longrightarrow Q_Y$ is a pseudo-BE homomorphism.

Theorem 3.6. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Q_X and Q_Y Smarandache pseudo-CI algebras and $f : X \longrightarrow Y$ be a Smarandache pseudo-CI homomorphism. Then:*
(1) *if $G \in \mathcal{F}_{Q_Y}(Y)$, then $f^{-1}(G) \in \mathcal{F}_{f^{-1}(Q_Y)}(X)$;*
(2) *if f is injective and $F \in \mathcal{F}_{Q_X}(X)$, then $f(F) \in \mathcal{F}_{f(Q_X)}(Y)$.*

Proof. (1) Assume that $G \in \mathcal{F}_{Q_Y}(Y)$ and $y \in f^{-1}(G)$. Obviously, $1_X \in f^{-1}(G)$. Let x, $x \to y \in f^{-1}(G)$ and $x \rightsquigarrow y \in f^{-1}(G)$. It follows that $f(x) \to f(y) = f(x \to y) \in G$ and $f(x) \rightsquigarrow f(y) = f(x \rightsquigarrow y) \in G$. Then $f(y) \in Q_Y$, since $f(x) \in G$ and $G \in \mathcal{F}_{Q_Y}(Y)$, we have $f(y) \in G$. Therefore, $y \in f^{-1}(G)$, and so $f^{-1}(G) \in \mathcal{F}_{f^{-1}(Q_Y)}(X)$.

(2) Assume that f is injective and $F \in \mathcal{F}_{Q_X}(X)$. Obviously, $1_Y \in f(F)$. Let a, $a \to b \in f(F)$ and $b \in f(Q_X)$. It follows that there exist x_a, $x_{a\to b} \in F$ and $x_b \in Q_X$ such that $f(x_a) = a$, $f(x_{a\to b}) = a \to b$ and $f(x_b) = b$. Now, we have

$$f(x_{a\to b}) = a \to b = f(x_a) \to f(x_b) = f(x_a \to x_b).$$

Since f is injective, we have $x_{a\to b} = x_a \to x_b \in F$, and so $x_b \in F$. Hence $b = f(x_b) \in f(F)$. Therefore, $f(F) \in \mathcal{F}_{f(Q_X)}(Y)$. □

Definition 3.5. Let $\mathfrak{X}$ be a Q-Smarandache pseudo-CI algebra. A mapping $f : X \to X$ is called a *modal Q-Smarandache operator* if $f_s = f_{|Q} : Q \to Q$ is a modal pseudo-BE algebra.

Denote by $\mathcal{SMOD}_{\mathrm{Q}}(X)$ set of all modal Q-Smarandache operators on X.

Proposition 3.5. *Let Q_1 and Q_2 be pseudo-BE algebras such that $Q_1 \subseteq Q_2 \subseteq X$. Then $\mathcal{SMOD}_{\mathrm{Q_1}}(X) \subseteq \mathcal{SMOD}_{\mathrm{Q_2}}(X)$.*

4. Commutative Smarandache pseudo-CI algebras

The commutative pseudo-BE algebras were defined and investigated in [10], while the commutative Smarandache CI-algebras have been defined and studied in [5]. In this section we introduce the notion of commutative Smarandache pseudo-CI algebras, we give characterizations of these structures and investigate some of their properties.

Let $\mathfrak{X}$ be a pseudo-CI algebra. For all $x, y \in X$, denote:

$$x \vee_1 y = (x \to y) \rightsquigarrow y \text{ and } x \vee_2 y = (x \rightsquigarrow y) \to y.$$

If $\to = \rightsquigarrow$, then the pseudo-CI algebra $\mathfrak{X}$ is a CI-algebra and

$$x \vee y = (x \to y) \to y.$$

Definition 4.1. A Q-Smarandache pseudo-CI algebra $\mathfrak{X}$ is said to be *commutative* if Q is a commutative pseudo-BE algebra, that is, it satisfies the following conditions, for all $x, y \in Q$, $x \vee_1 y = y \vee_1 x$ and $x \vee_2 y = y \vee_2 x$.

Example 4.1. Let $X = \{1, a, b, c, d, e, f, g\}$. Define the binary operations $\to$ and $\rightsquigarrow$ on X by the following tables:

$\to$	1	a	b	c	d	e	f	g
1	1	a	b	c	d	e	f	g
a	1	1	b	c	d	e	f	g
b	1	a	1	c	d	e	f	g
c	c	c	c	1	f	g	d	e
d	d	d	d	g	1	f	e	c
e	e	e	e	f	g	1	c	d
f	g	g	g	d	e	c	1	f
g	f	f	f	e	c	d	g	1

$\rightsquigarrow$	1	a	b	c	d	e	f	g
1	1	a	b	c	d	e	f	g
a	1	1	b	c	d	e	f	g
b	1	a	1	c	d	e	f	g
c	c	c	c	1	g	f	e	d
d	d	d	d	f	1	g	c	e
e	e	e	e	g	f	1	d	c
f	g	g	g	e	c	d	1	f
g	f	f	f	d	e	c	g	1

Then $(X; \to, \rightsquigarrow, 1)$ is a Q-Smarandache commutative pseudo-CI algebra, where $Q = \{1, a, b\}$.

Proposition 4.1. *Let $\mathfrak{X}$ be a Q-Smarandache commutative pseudo-CI algebra, and let $x, y \in Q$ such that $x \to y = y \to x = 1$ or $x \rightsquigarrow y = y \rightsquigarrow x = 1$. Then $x = y$.*

Proof. Consider $x, y \in Q$ such that $x \to y = y \to x = 1$. Since $\mathfrak{X}$ is commutative and applying (psCI$_2$), we get:

$$x = 1 \rightsquigarrow x = (y \to x) \rightsquigarrow x = (x \to y) \rightsquigarrow y = 1 \rightsquigarrow y = y.$$

Similarly, $x \rightsquigarrow y = y \rightsquigarrow x = 1$ implies $x = y$. □

Proposition 4.2. *In any Q-Smarandache commutative pseudo-CI algebra* $\mathfrak{X}$ *the following hold, for all* $x, y \in Q$*:*
(1) $x \to y = y \vee_1 x \to y$ *and* $x \rightsquigarrow y = y \vee_2 x \rightsquigarrow y$*;*
(2) $x \vee_1 y = (x \vee_1 y) \vee_1 x$ *and* $x \vee_2 y = (x \vee_2 y) \vee_2 x$*;*
(3) $x \leq y$ *implies* $y \vee_1 x = y \vee_2 x = y$.

Proof. It follows by [10, Prop. 4.9]. □

Theorem 4.1. *An algebra* $\mathfrak{X}$ *of the type* $(2, 2, 0)$ *is a Q-Smarandache commutative pseudo-CI algebra if and only if the following hold, for all* $x, y, z \in Q$*:*
(P_1) $1 \to x = 1 \rightsquigarrow x = x$*;*
(P_2) $x \to 1 = x \rightsquigarrow 1 = 1$*;*
(P_3) $(x \to z) \rightsquigarrow (y \to z) = (z \to x) \rightsquigarrow (y \to x)$ *and*
$(x \rightsquigarrow z) \to (y \rightsquigarrow z) = (z \rightsquigarrow x) \to (y \rightsquigarrow x)$*;*
(P_4) $x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z)$*;*
(P_5) $x \to y = 1$ *if and only if* $x \rightsquigarrow y = 1$.

Proof. It follows by [10, Th. 4.13]. □

Theorem 4.2. *An algebra* $\mathfrak{X}$ *of the type* $(2, 2, 0)$ *is a Q-Smarandache commutative pseudo-CI algebra if and only if the following hold, for all* $x, y, z \in Q$*:*
(Q_1) $(x \to 1) \rightsquigarrow y = (x \rightsquigarrow 1) \to y = y$*;*
(Q_2) $(x \to z) \rightsquigarrow (y \to z) = (z \to x) \rightsquigarrow (y \to x)$ *and*
$(x \rightsquigarrow z) \to (y \rightsquigarrow z) = (z \rightsquigarrow x) \to (y \rightsquigarrow x)$*;*
(Q_3) $x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z)$*;*
(Q_4) $x \to y = 1$ *if and only if* $x \rightsquigarrow y = 1$.

Proof. It follows by [10, Th. 4.14]. □

Remark 4.1. *According to [9] the following hold:*
– *Any pseudo BCK-algebra is a pseudo-BE algebra;*
– *The class of commutative pseudo-BE algebras is term equivalent to the class of commutative pseudo-BCK algebras.*
It follows that in the definition of commutative Q-Smarandache pseudo-CI algebras, the pseudo-BE algebra can be replaced with a pseudo-BCK algebra.

5. Classes of Smarandache filters of Smarandache pseudo-CI algebras

Developing filter theory of multiple-valued logic algebras is a central topic in the study of fuzzy systems (see, e.g., [1, 26, 27]).

In this section we define and study the classes of Smarandache fantastic, implicative and positive implicative filters of Smarandache pseudo-CI algebras. We generalize some results regarding Smarandache fantastic, fresh and clean ideals proved in [19], [18] and [4] for Smarandache BCI-algebras and Smarandache BCH-algebras. It is proved that in the case of commutative Q-Smarandache pseudo-CI algebras the notions of Smarandache filters and fantastic filters coincide. We give a characterization of Smarandache implicative filters and we present conditions for a Smarandache filter to be a Smarandache implicative filter. For a Q-Smarandache pseudo-CI algebra we prove that any Smarandache implicative filter is a filter and any Smarandache positive implicative filter contained in Q is a Smarandache filter. Finally, we give a characterization of Smarandache positive implicative filters.

Definition 5.1. Let $\mathfrak{X}$ be a Q-Smarandache pseudo-CI algebra. A filter F of $\mathfrak{X}$ is said to be *Q-Smarandache fantastic filter* of $\mathfrak{X}$ if it satisfies the following conditions, for all $x, y \in Q$:
(FF_1) $y \to x \in F$ implies $x \vee_1 y \to x \in F$;
(FF_2) $y \rightsquigarrow x \in F$ implies $x \vee_2 y \rightsquigarrow x \in F$.

Denote by $\mathcal{F}_Q^F(X)$ set of all Q-Smarandache fantastic filters of $\mathfrak{X}$.

Theorem 5.1. *Let $\mathfrak{X}$ be a Q-Smarandache pseudo-CI algebra and let $F \subseteq X$. Then $F \in \mathcal{F}_Q^F(X)$ if and only if it satisfies the following conditions, for all $x, y \in Q$ and $z \in X$:*
(1) $1 \in F$;
(2) $z \to (y \to x) \in F$ and $z \in F$ imply $x \vee_1 y \to x \in F$;
(3) $z \to (y \rightsquigarrow x) \in F$ and $z \in F$ imply $x \vee_2 y \rightsquigarrow x \in F$.

Proof. Consider $F \in \mathcal{F}_Q(X)$. Since $1 \in F$, condition (1) is satisfied. Let $x, y \in Q$ and $z \in F$ such that $z \to (y \to x) \in F$. Obviously, $y \to x \in Q$. Since $F \in \mathcal{F}(X)$, we have $y \to x \in F$, hence $x \vee_1 y \to x \in F$, that is, condition (2). Similarly, from $z \rightsquigarrow (y \rightsquigarrow x) \in F$ and $z \in F$, we get $x \vee_2 y \rightsquigarrow x \in F$, that is, condition (3).

Conversely, let $F \subseteq X$ satisfying conditions (1), (2) and (3). Obviously, $1 \in F$. Let $x, y \in Q$ such that $x \to y$, $x \in F$. Since $x \to (1 \to y) = x \to y \in F$, using (2), we have $y = y \vee_1 1 \to y \in F$. It follows that $F \in \mathcal{F}_Q(X)$. Let $x, y \in Q$ such that $y \to x \in F$. Since $1 \to (y \to x) \in F$ and $1 \in F$, by (2), we get $x \vee_1 y \to x \in F$. Similarly, from $y \rightsquigarrow x \in F$, we get $x \vee_2 y \rightsquigarrow x \in F$. We conclude that $F \in \mathcal{F}_Q^F(X)$. $\square$

Proposition 5.1. *Let $\mathfrak{X}$ be a pseudo-CI algebra and Q_1, Q_2 be proper subsets of $\mathfrak{X}$ such that $Q_1 \subseteq Q_2$. Then $\mathcal{F}_{Q_2}^F(X) \subseteq \mathcal{F}_{Q_1}^F(X)$.*

Proposition 5.2. *Let $\mathfrak{X}$ be a Q-Smarandache pseudo-CI(A) algebra and $F_1 \in \mathcal{F}_Q^F(X)$, $F_2 \in \mathcal{F}_Q(X)$ such that $F_1 \subseteq F_2$. Then $F_2 \in \mathcal{F}_Q^F(X)$.*

Proof. Consider $x, y \in Q$ such that $u = y \to x \in F_2$. It follows that
$$y \to (u \rightsquigarrow x) = y \to ((y \to x) \rightsquigarrow x) = 1 \in F_1.$$
Since F_1 is fantastic, we have $(u \rightsquigarrow x) \vee_1 y \to (u \rightsquigarrow x) \in F_1$.
From $F_1 \subseteq F_2$, we get $(u \rightsquigarrow x) \vee_1 y \to (u \rightsquigarrow x) \in F_2$.
Applying $(psCI_3)$, it follows that $u \rightsquigarrow ((u \rightsquigarrow x) \vee_1 y \to x) \in F_2$.
Since $u \in F_2$ and $(u \rightsquigarrow x) \vee_1 y \to x \in Q$, we get $(u \rightsquigarrow x) \vee_1 y \to x \in F_2$.
In the pseudo-BE algebra Q, $x \preceq (y \to x) \rightsquigarrow x = u \rightsquigarrow x$, hence by (A), we have $(u \rightsquigarrow x) \to y \preceq x \to y$, and $(x \to y) \rightsquigarrow y \preceq ((u \rightsquigarrow x) \to y) \rightsquigarrow y$, that is, $x \vee_1 y \preceq (u \rightsquigarrow x) \vee_1 y$.
Finally, applying again (A), $(u \rightsquigarrow x) \vee_1 y \to x \preceq x \vee_1 y \to x$.
Hence $x \vee_1 y \to x \in F_2$. Similarly, from $y \rightsquigarrow x \in F_2$, we get $x \vee_2 y \rightsquigarrow x \in F_2$.
We conclude that $F_2 \in \mathcal{F}^F_Q(X)$. □

Corollary 5.1. *Let $\mathfrak{X}$ be a Q-Smarandache pseudo-CI(A) algebra. Then $\{1\} \in \mathcal{F}^F_Q(X)$ if and only if $\mathcal{F}_Q(X) = \mathcal{F}^F_Q(X)$.*

Theorem 5.2. *If $\mathfrak{X}$ is a commutative Q-Smarandache pseudo-CI algebra, then $\mathcal{F}_Q(X) = \mathcal{F}^F_Q(X)$.*

Proof. Let $F \in \mathcal{F}_Q(X)$, and let $x, y \in Q$ such that $y \to x \in F$.
Obviously, $((y \to x) \rightsquigarrow x) \to x \in Q$ and by (a_6), $y \to x \preceq ((y \to x) \rightsquigarrow x) \to x$, hence $((y \to x) \rightsquigarrow x) \to x = y \vee_1 x \to x \in F$. Since X is commutative, we get $x \vee_1 y \to x \in X$.
Similarly, $x, y \in Q$ and $y \rightsquigarrow x \in F$ imply $x \vee_2 y \rightsquigarrow x \in F$, hence $F \in \mathcal{F}^F_Q(X)$.
We conclude that $\mathcal{F}_Q(X) \subseteq \mathcal{F}^F_Q(X)$, that is, $\mathcal{F}_Q(X) = \mathcal{F}^F_Q(X)$. □

Definition 5.2. *Let $\mathfrak{X}$ be a Q-Smarandache pseudo-CI algebra. A subset F of X is said to be a* Q-Smarandache implicative filter *of $\mathfrak{X}$ if it satisfies the following conditions, for all $x, y \in Q$ and $z \in F$:*
(IF_1) $1 \in F$;
(IF_2) $z \to ((x \to y) \rightsquigarrow x) \in F$ *implies* $x \in F$;
(IF_3) $z \rightsquigarrow ((x \rightsquigarrow y) \to x) \in F$ *implies* $x \in F$.

Denote by $\mathcal{F}^i_Q(X)$ set of all Q-Smarandache implicative filters of $\mathfrak{X}$.

Proposition 5.3. *In any Q-Smarandache pseudo-CI algebra $\mathfrak{X}$, $\mathcal{F}^I_Q(X) \subseteq \mathcal{F}_Q(X)$.*

Proof. Let $F \in \mathcal{F}^I_Q(X)$. Obviously, (SF_1) is (IF_1). Let $x \in F$ and $y \in Q$ such that $x \to y \in F$. Since $y \to ((x \to x) \rightsquigarrow x) = y \to x \in F$, by (IF_2), we get $x \in F$, that is, (SF_2) is verified. Similarly, (SF_3) follows from (IF_3), hence $F \in \mathcal{F}_Q(X)$. We conclude that $\mathcal{F}^I_Q(X) \subseteq \mathcal{F}_Q(X)$. □

Theorem 5.3. *Let $\mathfrak{X}$ be a Q-Smarandache pseudo-CI algebra, and let $F \in \mathcal{F}_Q(X)$. Then the following are equivalent; for all $x, y \in Q$,* (1) *$F \in \mathcal{F}^I_Q(X)$;*
(2) *$(x \to y) \rightsquigarrow x \in F$ implies $x \in F$ and $(x \rightsquigarrow y) \to x \in F$ implies $x \in F$.*

Proof. (1) $\Rightarrow$ (2) Let $F \in \mathcal{F}_Q^I(X)$, and let $x, y \in Q$ such that $(x \to y) \rightsquigarrow x \in F$. Since $1 \to ((x \to y) \rightsquigarrow x) = (x \to y) \rightsquigarrow x \in F$ and $1 \in F$, by (IF_2), we get $x \in F$. Similarly, $(x \rightsquigarrow y) \to x \in F$ implies $x \in F$.
(2) $\Rightarrow$ (1) Let $x, y \in Q$ such that $z \to ((x \to y) \rightsquigarrow x) \in F$, and let $z \in F$.
Since $F \in \mathcal{F}_Q(X)$, we get $(x \to y) \rightsquigarrow x \in F$, and applying (2), we get $x \in F$.
Similarly, $z \to ((x \rightsquigarrow y) \to x) \in F$ and $z \in F$ imply $x \in F$. Therefore, $F \in \mathcal{F}_Q^I(X)$. $\square$

Proposition 5.4. *Let $\mathfrak{X}$ be a Q-Smarandache pseudo-CI algebra, and let $F \in \mathcal{F}_Q(X)$ such that $((x \to y) \rightsquigarrow x) \rightsquigarrow x \in F$ and $((x \rightsquigarrow y) \to x) \to x \in F$, for all $x, y \in Q$. Then $F \in \mathcal{F}_Q^I(X)$.*

Proof. Let $F \in \mathcal{F}_Q(X)$ and let $x, y \in Q$ such that $z \to ((x \to y) \rightsquigarrow x) \in F$ and $z \in F$. Since $F \in \mathcal{F}_Q(X)$, by (SF_2), we have $(x \to y) \rightsquigarrow x \in F$. Moreover, from $((x \to y) \rightsquigarrow x) \rightsquigarrow x \in F$, applying again (SF_2), we get $x \in F$. Similarly, $z \rightsquigarrow ((x \rightsquigarrow y) \to x) \in F$ and $z \in F$ imply $x \in F$. Therefore, $F \in \mathcal{F}_Q^I(X)$. $\square$

Definition 5.3. Let $\mathfrak{X}$ be a Q-Smarandache pseudo-CI algebra. A subset F of X is said to be a *Q-Smarandache positive implicative filter* of $\mathfrak{X}$ if it satisfies the following conditions, for all $x, y, z \in Q$:
(PIF_1) $1 \in F$;
(PIF_2) $z \to (x \to y) \in F$ and $z \rightsquigarrow x \in F$ imply $z \to y \in F$;
(PIF_3) $z \rightsquigarrow (x \rightsquigarrow y) \in F$ and $z \to x \in F$ imply $z \rightsquigarrow y \in F$.

Denote by $\mathcal{F}_Q^{PI}(X)$ set of all Q-Smarandache implicative filters of $\mathfrak{X}$.

Proposition 5.5. *In any Q-Smarandache pseudo-CI algebra $\mathfrak{X}$, $\{F \in \mathcal{F}_Q^{pi}(X) \mid F \subseteq Q\} \subseteq \mathcal{F}_Q(X)$.*

Proof. Let $F \in \mathcal{F}_Q^{PI}(X)$. Obviously, (SF_1) is (PIF_1). Let $x \in F$ and $y \in Q$ such that $x \to y \in F$. Since $1 \to (x \to y) = x \to y \in F$ and $1 \rightsquigarrow x = x \in F \subseteq Q$, applying (PIF_2), we get $1 \to y = y \in F$. Thus (SF_2) is verified. Similarly, applying (PIF_3), we get (SF_3), hence $F \in \mathcal{F}_Q(X)$. We conclude that $\mathcal{F}_Q^{PI}(X) \subseteq \mathcal{F}_Q(X)$. $\square$

Proposition 5.6. *Let $\mathfrak{X}$ be a Q-Smarandache pseudo-CI algebra, and let $F \in \mathcal{F}_Q(X)$ such that the following conditions are satisfied, for all $x, y, z \in Q$:*
(PIF_4) $z \to (x \to y) \in F$ implies $(z \rightsquigarrow x) \to (z \to y) \in F$;
(PIF_5) $z \rightsquigarrow (x \rightsquigarrow y) \in F$ implies $(z \to x) \rightsquigarrow (z \rightsquigarrow y) \in F$.
Then $F \in \mathcal{F}_Q^{PI}(X)$.

Proof. Let $F \in \mathcal{F}_Q(X)$, and let $x, y, z \in Q$ such that $z \to (x \to y) \in F$ and $z \rightsquigarrow x \in F$. By (PIF_4), we have $(z \rightsquigarrow x) \to (z \to y) \in F$ and by (SF_2), we get $z \to y \in F$. Similarly, applying (PIF_5), from $z \rightsquigarrow (x \rightsquigarrow y) \in F$ and $z \to x \in F$, we get $z \rightsquigarrow y \in F$. It follows that $F \in \mathcal{F}_Q^{PI}(X)$. $\square$

Corollary 5.2. *Let $\mathfrak{X}$ be a Q-Smarandache pseudo-CI algebra, and let $F \in \mathcal{F}_Q(X)$ such that the following conditions are satisfied, for all $x, y \in Q$:*
$(PIF_4)'$ $x \to (x \to y) \in F$ *implies* $x \to y \in F$;
$(PIF_5)'$ $x \rightsquigarrow (x \rightsquigarrow y) \in F$ *implies* $x \rightsquigarrow y \in F$.
Then $F \in \mathcal{F}_Q^{PI}(X)$.

Lemma 5.1. *Let $\mathfrak{X}$ be a Q-Smarandache pseudo-CI algebra, and let $F \in \mathcal{F}_Q^{PI}(X)$. Then F satisfies $(PIF_4)'$ and $(PIF_5)'$, for all $x, y \in Q$.*

Proof. Let $F \in \mathcal{F}_Q(X)$, and let $x, y \in Q$ such that $x \to (x \to y) \in F$. Since $x \rightsquigarrow x = 1 \in F$, applying (PIF_2) we get $x \to y \in F$. Similarly, from $x \rightsquigarrow (x \rightsquigarrow y) \in F$, applying (PIF_3), we get $x \rightsquigarrow y \in F$. □

Theorem 5.4. *Let $\mathfrak{X}$ be a Q-Smarandache pseudo-CI algebra, and let $F \in \mathcal{F}_Q(X)$. Then $F \in \mathcal{F}_Q^{PI}(X)$ if and only if it satisfies $(PIF_4)'$ and $(PIF_5)'$.*

Proof. It follows by Lemma 5.1 and Corollary 5.2. □

Proposition 5.7. *Let $\mathfrak{X}$ be a Q-Smarandache pseudo-CI algebra, and let $F \in \mathcal{F}_Q^{PI}(X)$ such that $F \subseteq Q$. Then the following hold, for all $x, y \in Q$, $z \in F$:*
(PIF_6) $z \to (x \to (x \to y)) \in F$ *implies* $x \to y \in F$;
(PIF_7) $z \rightsquigarrow (x \rightsquigarrow (x \rightsquigarrow y)) \in F$ *implies* $x \rightsquigarrow y \in F$.

Proof. Let $F \in \mathcal{F}_Q^{PI}(X)$, $F \subseteq Q$, and let $x, y \in Q$, $z \in F$ such that $z \to (x \to (x \to y)) \in F$. Since $F \subseteq Q$ we have $z \in Q$. By Proposition 5.5, $F \in \mathcal{F}_Q(X)$ and applying (SF_2), we have $x \to (x \to y) \in F$. Hence by Lemma 5.1, we get $x \to y \in F$, thus (PIF_6) is verified. Similarly, for (PIF_7). □

Theorem 5.5. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Q_X and Q_Y-Smarandache pseudo-CI algebras and $f : X \longrightarrow Y$ be a Smarandache pseudo-CI homomorphism. Then:*
(1) *if* $G \in \mathcal{F}_{Q_Y}^F(Y)$ $(\mathcal{F}_{Q_Y}^I(Y), \mathcal{F}_{Q_Y}^{PI}(Y))$, *then*
$f^{-1}(G) \in \mathcal{F}_{f^{-1}(Q_Y)}^F(X)$ $(\mathcal{F}_{f^{-1}(Q_Y)}^I(X), \mathcal{F}_{f^{-1}(Q_Y)}^I(X))$;
(2) *if f is injective and* $F \in \mathcal{F}_{Q_X}^F(X)$ $(\mathcal{F}_{Q_X}^I(X), \mathcal{F}_{Q_X}^{PI}(X))$, *then*
$f(F) \in \mathcal{F}_{f(Q_X)}^F(Y)$ $(\mathcal{F}_{f(Q_X)}^I(Y), \mathcal{F}_{f(Q_X)}^{PI}(Y))$.

Proof. (1) Let $G \in \mathcal{F}_{Q_Y}^F(Y)$, and let $x, y \in Q_X$ such that $y \to x \in f_{Q_Y}^{-1}(G)$, that is, $f(y \to x) \in G$, so $f(y) \to f(x) \in G$. Since $G \in \mathcal{F}_{Q_Y}^F(Y)$, we have $f(x) \vee_1 f(y) \to f(x) \in G$. It follows that $f(x \vee_1 y \to x) \in G$, hence $x \vee_1 y \to x \in f^{-1}(G)$. Similarly, $y \rightsquigarrow x \in f_{Q_Y}^{-1}(G)$ implies $x \vee_2 y \rightsquigarrow x \in f^{-1}(G)$. We conclude that $f^{-1}(G) \in \mathcal{F}_{f^{-1}(Q_Y)}^F(X)$. Similarly, for $G \in \mathcal{F}_{Q_Y}^I(Y)$ and $G \in \mathcal{F}_{Q_Y}^{PI}(Y)$.
(2) Let $F \in \mathcal{F}_{Q_X}^F(X)$ and $x, y \in f(Q)$ such that $y \to x \in f(F)$. There exist $x_1, y_1, z_1 \in Q$ such that $x = f(x_1)$, $y = f(y_1)$, $y \to x = f(z_1)$. Therefore, $f(y_1) \to f(x_1) = f(z_1)$, that is, $f(y_1 \to x_1) = f(z_1)$. Since f is injective and F is fantastic, we get $y_1 \to x_1 = z_1 \in F$, hence $x_1 \vee_1 y_1 \to x_1 \in F$. It follows that $f(x_1 \vee_1 y_1 \to x_1) \in f(F)$, so $f(x_1) \vee_1 f(y_1) \to f(x_1) \in f(F)$, that is, $x \vee_1 y \to x \in f(F)$. Similarly, $y \rightsquigarrow x \in f(F)$ implies $x \vee_2 y \rightsquigarrow x \in f(F)$.
Hence $f(F) \in \mathcal{F}_{f(Q_X)}^F(Y)$. Similarly, for $F \in \mathcal{F}_{Q_X}^I(X)$ and $F \in \mathcal{F}_{Q_X}^{PI}(X)$. □

6. Q-Smarandache upper sets

In this section, we define and investigate the notion of Smarandache upper sets in a pseudo-CI algebra and we investigate some of their properties. We prove that every Q-Smarandache filter is a union of Q-Smarandache upper sets.

Let $x, y \in Q$ and $Q \subseteq X$ be a pseudo-BE algebra. Denote:

$$A(x, y) := \{z \in Q : x \to (y \rightsquigarrow z) = 1\}.$$

We call $A(x, y)$ a Q-Smarandache upper set of x and y.

Remark 6.1. *It is easy to see that, $1, x, y \in A(x, y)$. The set $A(x, y)$, where $x, y \in Q$, is not a filter of $\mathfrak{X}$, in general. Also, using (psCI$_3$) and (psCI$_4$) we have*

$$\begin{aligned} A(x, y) &= \{z \in Q : x \to (y \rightsquigarrow z) = 1\} \\ &= \{z \in Q : x \rightsquigarrow (y \rightsquigarrow z) = 1\} \\ &= \{z \in Q : y \rightsquigarrow (x \to z) = 1\} \\ &= \{z \in Q : y \to (x \to z) = 1\}. \end{aligned}$$

Example 6.1. (1) Consider the pseudo-CI algebras from Example 2.1 (2) and let $Q := \{1, a, c\}$. Then $A(a, c) = \{1, a, c\}$.

(2) Consider the Q-Smarandache pseudo-CI algebras from Example 4.1. Then $A(a, 1) = \{1, a, b\} \neq A(1, a) = \{1, a\}$, and so $A(x, y) \neq A(y, x)$, for some $x, y \in Q$.

Proposition 6.1. *Let $x, y \in Q$. Then*
(1) *$A(x, 1) \subseteq A(x, y)$;*
(2) *if $A(x, 1) \in F_Q(\mathfrak{X})$ and $y \in A(x, 1)$, then $A(x, y) \subseteq A(x, 1)$;*
(3) *if there is $y \in Q$, such that $y \to z = 1$ or $y \rightsquigarrow z = 1$, for all $z \in Q$, then $Q = A(x, y)$;*
(4) *$A(x, 1) = \bigcap_{y \in Q} A(x, y)$.*

Theorem 6.1. *Let $\emptyset \neq F \subseteq Q$. Then $F \in F_Q(\mathfrak{X})$ if and only if $A(x, y) \subseteq F$, for all $x, y \in F$.*

Proof. Assume that $F \in F_Q(\mathfrak{X})$ and $x, y \in F$. If $z \in A(x, y)$, then $x \to (y \rightsquigarrow z) = 1 \in F$. Since $F \in F_Q(\mathfrak{X})$ and $x, y \in F$, by (SF$_2$), $y \rightsquigarrow z \in F$, and so by (SF$_3$), $z \in F$. Hence $A(x, y) \subseteq F$.

Conversely, suppose $A(x, y) \subseteq F$, for all $x, y \in F$.
Since $x \to (y \rightsquigarrow 1) = x \to 1 = 1$, we get $1 \in A(x, y) \subseteq F$. Let a, $a \to b \in F$ and $a \rightsquigarrow c \in F$. Since $1 = (a \to b) \rightsquigarrow (a \to b) = a \to ((a \to b) \rightsquigarrow b)$ and
$(a \rightsquigarrow c) \to (a \rightsquigarrow c) = 1$, we have $b \in A \subseteq F$ and $c \in A \subseteq F$. Hence $b, c \in F$. Thus, $F \in F_Q(\mathfrak{X})$. □

Theorem 6.2. *Let $a \in Q$. Then the set $A(a, 1) \in F_Q(\mathfrak{X})$ if and only if the following hold, for all $x, y, z \in Q$:*

(1) $z \preceq x \to y$ and z $\preceq$ x *imply* $z \preceq y$;
(2) $z \preceq x \rightsquigarrow y$ and z $\preceq$ x *imply* $z \preceq y$.

Proof. Assume that for each $a \in Q$, $A(a,1) \in F_Q(\mathfrak{X})$. Let $x, y, z \in Q$ be such that $z \preceq x \to y$, $z \preceq x \rightsquigarrow y$, and $z \preceq x$. Then $x \to y \in A(z,1)$, $x \rightsquigarrow y \in A(z,1)$, and $x \in A(z,1)$. Since $A(z,1) \in F_Q(\mathfrak{X})$, we have $y \in A(z,1)$. Therefore, $z \preceq y$.

Conversely, consider $A(z,1)$, for $z \in Q$. Obviously, $1 \in A(z,1)$.
Let $x \to y \in A(z,1)$, and $x \rightsquigarrow b \in A(z,1)$, for all $x \in A(z,1)$ (i.e. $z \preceq x \to y$, $z \preceq x \rightsquigarrow b$ and $z \preceq x$). Then from hypothesis, $z \preceq y$ and $z \preceq b$ (i.e. $y \in A(z,1)$ and $b \in A(z,1)$). Hence $A(z,1) \in F_Q(\mathfrak{X})$, for all $z \in Q$. □

Theorem 6.3. *Let* $F \in F_Q(\mathfrak{X})$ *and* $F \subseteq Q$*, then* $F = \bigcup_{x \in F} A(x,1)$.

Proof. Assume that $F \in F_Q(\mathfrak{X})$, $F \subseteq Q$ and $z \in F$. Since $z \in A(z,1)$, we have $F \subseteq \bigcup_{z \in F} A(z,1)$. Let $z \in \bigcup_{x \in F} A(x,1)$. Then there exists $a \in F$ such that $z \in A(a,1)$, and so $a \to z = a \to (1 \rightsquigarrow z) = 1 \in F$. Since $F \in F_Q(\mathfrak{X})$ and $a \in F$, we have $z \in F$. Thus, $\bigcup_{x \in F} A(x,1) \subseteq F$. □

7. Conclusions and future work

In this paper we introduced the notion of Smarandache pseudo-CI algebras and we defined and studied some classes of Smarandache filters of Smarandache pseudo-CI algebras. This study could potentially lead to more results on Smarandache pseudo-CI algebras.

A. Borumand Saeid studied in [2] the notion of a *Smarandache weak BE-algebra*, as a BE-algebra X in which there exists a proper subset Q of X such that:

(S_1) $1 \in Q$ and $\mid Q \mid \geq 2$;
(S_2) Q is a CI-algebra under the operation of X.

Another topic of research could be to define and investigate the notion of a Smarandache weak pseudo-BE algebra.
A *Smarandache strong n-structure* on a set S means a structure W_0 on a set S such that there exists a chain of proper subsets $P_{n-1} < P_{n-2} < \cdots < P_2 < P_1 < S$, where $<$ means P_i included P_{i-1} in whose corresponding structures verify the inverse chain $W_{n-1} > W_{n-2} > \cdots > W_2 > W_1 > W_0$, where $>$ signifies strictly stronger (i.e. a structure satisfying more axioms) (see [5]).

A. Borumand Saeid and A. Rezaei introduced in [5] the notion of a Smarandache strong 3-structure of a CI-algebra X as a chain $X_1 > X_2 > X_3 > X_4$, where X_1 is a CI-algebra, X_2 is a BE-algebra, X_3 is a dual BCK-algebra, and X_4 is an implication algebra.
One could define and investigate the notion of a strong n-structure of a pseudo-CI algebra.

References

[1] S. Z. Alavi, R. A. Borzooei, M. A. Kologani, *Filter theory of pseudo hoop-algebras*, Ital. J. Pure Appl. Math. 37(2017), 619–632.

[2] A. Borumand Saeid, *Smarandache BE-algebras*, Education Publisher, Columbus, Ohio, USA, 2013.

[3] A. Borumand Saeid, *Smarandache weak BE-algebras*, Commun. Korean Math. Soc. 27(3)(2012), 489–496.

[4] A. Borumand Saeid, A. Namdar, *Smarandache BCH-algebras*, World Applied Sciences Journal 7(2009), 77–83.

[5] A. Borumand Saeid, A. Rezaei, *Smarandache n-structure on CI-algebras*, Results. Math. 63(2013), 209–219.

[6] R. A. Borzooei, A. Borumand Saeid, A. Rezaei, A. Radfar, R. Ameri, *On pseudo-BE algebras*, Discussiones Mathematicae, General Algebra and Applications 33(2013), 95–108.

[7] R. A. Borzooei, A. Borumand Saeid, A. Rezaei, A. Radfar, R. Ameri, *Distributive pseudo-BE algebras*, Fasciculi Mathematici 54(2015), 21–39.

[8] L. C. Ciungu, *Non-commutative Multiple-Valued Logic Algebras*, Springer, Cham, Heidelberg, New York, Dordrecht, London, 2014.

[9] L. C. Ciungu, *Commutative pseudo BE-algebras,* Iran. J. Fuzzy Syst. 13(1)(2016), 131–144.

[10] L. C. Ciungu, *Fantastic deductive systems in probability theory on generalizations of fuzzy structures,* Fuzzy Sets. Syst. 36(3)(2019), 113–137.

[11] L. C. Ciungu, A Borumand Saeid, A. Rezaei, *Modal operators on pseudo-BE algebras,* Iran. J. Fuzzy Syst. 17(6)(2020), 175-191.

[12] W. A. Dudek, Y. B. Jun, *Pseudo-BCI algebras*, East Asian Math. J. 24(2008), 187–190.

[13] G. Georgescu, A. Iorgulescu, *Pseudo-BCK algebras: an extension of BCK-algebras*, DMTCS01: Combinatorics, Computability and Logic, Springer, London, 2001, 97–114.

[14] A. Iorgulescu, *Classes of pseudo-BCK algebras - Part I*, J. Mult.-Valued Logic Soft Comput. 12(2006), 71–130.

[15] A. Iorgulescu, *Algebras of logic as BCK-algebras*, ASE Ed., Bucharest, 2008.

[16] Y. B. Jun, *Smarandache BCI-algebras*, Sci. Math. Jpn. 62(2005), 137–142.

[17] Y. B. Jun, *Smarandache structures on generalized BCK-algebras*, Sci. Math. Jpn. 64(2006), 607–612.

[18] Y. B. Jun, *Smarandache fantastic ideals of Smarandache BCI-algebras*, Scientia Magna 2(4)(2006), 40–44.

[19] Y. B. Jun, *Smarandache fresh and clean ideals of Smarandache BCI-algebras*, Kyungpook Math. J. 46(2006), 409–416.

[20] H. S. Kim, Y. H. Kim, *On BE-algebras*, Sci. Math. Jpn. 66(1)(2007), 113–117.

[21] J. Kühr, *Pseudo-BCK semilattices*, Demonstr. Math. 40(2007), 495–516.

[22] J. Kühr, *Pseudo-BCK algebras and related structures*, Habilitation thesis, Palacký University in Olomouc, 2007.

[23] B. L. Meng, *CI-algebras*, Sci. Math. Jpn. 71(1)(2010), 11–17.

[24] S. Motamed, M. Sadeghi KosarKhizi, *n-Fold filters in Smarandache residuated lattices, part (I)*, U.P.B. Sci. Bull., Series A, 79(1)(2017), 119–130.

[25] S. Motamed, M. Sadeghi KosarKhizi, *n-Fold filters in Smarandache residuated lattices, part (II)*, U.P.B. Sci. Bull., Series A, 79(2)(2017), 21–30.

[26] J. Rachůnek, D. Šalounová, *Filter theory of bounded residuated lattice ordered monoids*, J. Mult.-Valued Logic Soft Comput. 16(2010), 449–465.

[27] J. Rachůnek, D. Šalounová, *Ideals and involutive filters in generalizations of fuzzy structures*, Fuzzy Sets Syst. 311(2017), 70–85.

[28] A. Rezaei, A. Borumand Saeid, A. Radfar, R. A. Borzooei, *Congruence relations on pseudo-BE algebras*, An. Univ. Craiova Math. Comp. Sci. Ser. 41(2014), 166–176.

[29] A. Rezaei, A. Borumand Saeid, K. Yousefi Sikari Saber, *On pseudo-CI algebras,* Soft Comput. 23(13)(2019), 4643–4654.

[30] A. Rezaei, A. Borumand Saeid, A. Walendziak, *Some results on pseudo-Q algebras,* Ann. Univ. Paedagog. Crac. Stud. Math. 16(2017), 61–72.

.

Accepted: .....