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Smarandache quasigroup rings

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Abstract In this paper, we have introduced Smarandache quasigroups which are Smarandache non-associative structures. W.B.Kandasamy [2] has studied groupoid ring and loop ring. We have defined Smarandache quasigroup rings which are again non-associative structures having two binary operations. Substructures of quasigroup rings are also studied.
Keywords Non-associative rings; Smarandache non-associative rings; Quasigroups; Smarandache quasigroups; Smarandache quasigroup rings.

§1. Introduction

In the paper [2] W.B.Kandasamy has introduced a new concept of groupoid rings. This structure provides number of examples of SNA-rings (Smarandache non-associative rings). SNA-rings are non-associative structure on which are defined two binary operations one associative and other being non-associative and addition distributes over multiplication both from right and left. We are introducing a new concept of quasigroup rings. These are non associative structures. In our view groupoid rings and quasigroup rings are the rich source of non-associative SNA-rings without unit since all other rings happen to be either associative or non-associative rings with unit. To make this paper self contained we recollect some definitions and results which we will use subsequently.

§2. Preliminaries

Definition 2.1. A groupoid S such that for all $a, b \in S$ there exist unique $x, y \in S$ such that $ax = b$ and $ya = b$ is called a quasigroup.

Thus a quasigroup does not have an identity element and it is also non-associative.
Here is a quasigroup that is not a loop.

$*$	1	2	3	4	5
1	3	1	4	2	5
2	5	2	3	1	4
3	1	4	2	5	3
4	4	5	1	3	2
5	2	3	5	4	1

We note that the definition of quasigroup Q forces it to have a property that every element of Q appears exactly once in every row and column of its operation table. Such a table is called a LATIN SQUARE. Thus, quasigroup is precisely a groupoid whose multiplication table is a LATIN SQUARE.

Definition 2.2. If a quasigroup $(Q, *)$ contains a group $(G, *)$ properly then the quasigroup is said to be Smarandache quasigroup.

Example 2.1. Let Q be a quasigroup defined by the following table:

$*$	a_0	a_1	a_2	a_3	a_4
a_0	a_0	a_1	a_3	a_4	a_2
a_1	a_1	a_0	a_2	a_3	a_4
a_2	a_3	a_4	a_1	a_2	a_0
a_3	a_4	a_2	a_0	a_1	a_3
a_4	a_2	a_3	a_4	a_0	a_1

Clearly, $A = \{a_0, a_1\}$ is a group w.r.t. $*$ which is a proper subset of Q. Therefore Q is a Smarandache quasigroup.

Definition 2.3. A quasigroup Q is idempotent if every element x in Q satisfies $x * x = x$.

Definition 2.4. A ring $(R, +, *)$ is said to be a non-associative ring if $(R, +)$ is an additive abelian group, $(R, *)$ is a non-associative semigroup (i.e. binary operation $*$ is non-associative) such that the distributive laws
$a * (b + c) = a * b + a * c$ and $(a + b) * c = a * c + b * c$ for all a, b, c in R.

Definition 2.5. Let R be a commutative ring with one. G be any group (S any semigroup with unit) RG (RS the semigroup ring of the semigroup S over the ring R) the group ring of the group G over the ring R consists of finite formal sums of the form $\sum_{i=1}^{n} \alpha_i g_i, (n < \infty)$ i.e. i runs over a finite number where $\alpha_i \in R$ and $g_i \in G \quad (g_i \in S)$ satisfying the following conditions:

1. $\sum_{i=1}^{n} \alpha_i m_i = \sum_{i=1}^{n} \beta_i m_i \Leftrightarrow \alpha_i = \beta_i$, for $i = 1, 2, \cdots, n$
2. $\sum_{i=1}^{n} \alpha_i m_i + \sum_{i=1}^{n} \beta_i m_i \Leftrightarrow \sum_{i=1}^{n} (\alpha_i + \beta i) m_i$
3. $(\sum_{i=1}^{n} \alpha_i m_i)(\sum_{i=1}^{n} \beta_i m_i) = \sum_{i=1}^{n} \gamma_k m_k, m_k = m_i m_j$, where $\gamma_k = \sum \alpha_i \beta_i$
4. $r_i m_i = m_i r_i$ for all $r_i \in R$ and $m_i \in G (m_i \in S)$.
5. $r \sum_{i=1}^{n} r_i m_i = \sum_{i=1}^{n} r r_i m_i$ for all $r \in R$ and $\sum_{i=1}^{n} r_i m_i \in RG$. RG is an associative ring with $0 \in R$ acts as its additive identity. Since $I \in R$ we have $G = IG \subseteq RG$ and $R.e = R \subseteq RG$ where e is the identity element of G.

If we replace the group G in the above definition by a quasigroup Q we get RQ the quasigroup ring which will satisfy all the five conditions 1 to 5 given in the definition. But RQ will only be a non-associative ring without identy. As $I \in R$ we have $Q \subseteq RQ$. Thus we define quasigroup rings as follows:

Definition 2.6. For any quasigroup Q the quasigroup ring RQ is the quasigroup Q over the ring R consisting of all finite formal sums of the form $\sum_{i=1}^{n} r_i q_i$, $(n < \infty)$ i.e. i runs over a finite number where $r_i \in R$ and $q_i \in Q$ satisfying conditions 1 to 5 given in the definition of group rings above.

Note that only when Q is a quasigroup with identity (i.e. then Q is a Loop) that the quasigroup ring RQ will be a non-associative ring with unit. Here we give examples of non-associative quasigroup rings.

Example 2.2. Let Z be the ring of integers and $(Q, *)$ be the quasigroup given by the following table:

$*$	1	2	3	4	5
1	3	1	4	2	5
2	5	2	3	1	4
3	1	4	2	5	3
4	4	5	1	3	2
5	2	3	5	4	1

Clearly $(Q, *)$ is a quasigroup and does not posses an identity element. The quasigroup ring ZQ is a non-associative ring without unit element.

Example 2.3. Let R be the ring of reals and $(Q, *)$ be the quasigroup defined by the following table:

$*$	1	2	3	4
1	1	3	4	2
2	4	2	1	3
3	2	4	3	1
4	3	1	2	4

$(Q, *)$ is an idempotent quasigroup. Again RQ is a non-associative quasigroup ring without unit. Note that $R\langle 1\rangle$, $R\langle 2\rangle$, $R\langle 3\rangle$, $R\langle 4\rangle$ are the subrings of RQ which are associative.

Result: All quasigroup rings RQ of a quasigroup Q over the ring R are non-associative rings without unit.

The smallest non-associative ring without unit is quasigroup ring given by the following example. This example was quoted by W.B.Kandasamy [2] as a groupoid ring.

Example 2.4. Let $Z_2 = \{0, 1\}$ be the prime field of characteristic 2. $(Q, *)$ be a quasigroup of order 3 given by the following table:

$*$	q_1	q_2	q_3
q_1	q_1	q_2	q_3
q_2	q_3	q_1	q_2
q_3	q_2	q_3	q_1

Z_2Q is a quasigroup ring having only eight elements given by $\{0, q_1, q_2, q_3, q_1 + q_2, q_2 + q_3, q_1 + q_3, q_1 + q_2 + q_3\}$. Clearly, Z_2Q is a non-associative ring without unit. This happens to be the smallest non-associative ring without unit known to us.

§3. SNA-Quasigroup rings

We introduce Smarandache non-associative quasigroup rings. It is true that quasigroup rings are always non-associative. We write "Smarandache non-associative quasigroup ring" only to emphasize the fact that they are non-associative.

Definition 3.1. Let S be a quasigroup ring. S is said to be SNA-quasigroup ring (Smarandache non-associative quasigroup ring) if S contains a proper subset P such that P is an associative ring under the operations of S.

Example 3.1. Let Z be the ring of integers and Q be a quasigroup defined by the following table;

$*$	a_0	a_1	a_2	a_3	a_4
a_0	a_0	a_1	a_3	a_4	a_2
a_1	a_1	a_0	a_2	a_3	a_4
a_2	a_3	a_4	a_1	a_2	a_0
a_3	a_4	a_2	a_0	a_1	a_3
a_4	a_2	a_3	a_4	a_0	a_1

Clearly, $A = \{a_0, a_1\}$ is group and $ZQ \supset ZA$. Thus the quasigroup ring ZQ contains an associative ring properly. Hence ZQ is an SNA-quasigroup ring. Note that Q is a Smarandache quasigroup.

Example 3.2. Let R be the reals, $(Q, *)$ be the quasigroup defined by the following table;

$*$	0	1	2	3
0	0	1	3	2
1	1	0	2	3
2	3	2	1	0
3	2	3	0	1

Then clearly RQ is an SNA-quasigroup ring as $RQ \supset R\langle 0,1\rangle$ and $R\langle 0,1\rangle$ is an associative ring.

Theorem 3.1. *Let Q be a quasigroup and R be any ring. Then the quasigroup ring RQ is not always an SNA-quasigroup ring.*

Proof. Since Q does not have an identity element, there is no guarantee that R is contained in RQ .

Example 3.3. Let R be an arbitrary ring and Q be a quasigroup defined by the table;

$*$	1	2	3	4	5
1	3	1	4	2	5
2	5	2	3	1	4
3	1	4	2	5	3
4	4	5	1	3	2
5	2	3	5	4	1

Then clearly, RQ is not an SNA-quasigroup ring as the quasigroup ring RQ does not contain an associative ring.

Theorem 3.2. *If Q is a quasigroup with identity, then quasigroup ring RQ is SNA-quasigroup ring.*

Proof. Quasigroup with identity is a Loop. So, $RI \subseteq RQ$ and R serves as the associative ring in RQ. Thus RQ is an SNA-quasigroup ring.

Theorem 3.3. *Let R be a ring. If Q is a Smarandache quasigroup, then quasigroup ring RQ is an SNA-quasigroup ring. .*

Proof. Obviously RQ is a non-associative ring. As Q is a Smarandache quasigroup Q contains a group G properly. So $RQ \supset RG$ and RG is an associative ring contained in RQ. Therefore RQ is an SNA-quasigroup ring.

§4. Substructure of SNA-quasigroup rings

Definition 4.1. Let R be a SNA-quasigroup ring. Let S be a non-empty subset of R. Then S is said to be S-quasigroup subring of R if S itself is a ring and contains a proper subset P such that P is an associative ring under the operation of R.

Example 4.1. Let Z be the ring of integers. Let Q be the quasigroup defined by the following table:

$*$	1	2	3	4	5	6	7	8
1	1	2	3	4	6	5	8	7
2	2	1	4	3	5	6	7	8
3	3	4	1	2	7	8	6	5
4	4	3	2	1	8	7	5	6
5	6	5	7	8	1	2	3	4
6	5	6	8	7	2	3	4	1
7	8	7	6	5	3	4	1	2
8	7	8	5	6	4	1	2	3

Clearly the quasigroup ring ZQ is a non-associative ring. Consider the subset $S = \{1, 2, 3, 4\}$ then S is a group and hence ZS is a group ring and hence also a quasigroup ring. Let $P = \{1, 2\}$. Note that ZS also contains ZP where $P = \{1, 2\}$. So, ZS is an S-quasigroup subring of SNA-quasigroup ring ZQ.

We have not yet been able to find a Smarandache non associative quasigroup subring for a given quasigroup ring. We think that it is not possible to obtain a subquasigroup for any quasigroup because for a quasigroup its composition table is a LATIN SQUARE.

Theorem 4.1. *Let R be a quasigroup ring, if R has a SNA-quasigroup subring S, then R itself is SNA-quasigroup ring.*

Proof. As S is an SNA-quasigroup surbring S contains an associative ring. As a result R contains an associtive ring. Thus R is an SNA-quasigroup ring.

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