

## SMARANDACHE STRUCTURES OF GENERALIZED BCK-ALGEBRAS

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**ABSTRACT.** The Smarandache structure of generalized BCK-algebras is considered. Several examples of a  $qS$ -gBCK-algebra are provided. The notion of  $S_\Omega$ -ideals and  $qS_\Omega$ -ideals is introduced, and related properties are investigated.

**1 Introduction.** A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki and was extensively investigated by several researchers. Hong et al. [1] established a new algebra, called a **generalized BCK-algebra**, which is a generalization of a positive implicative BCK-algebra, and gave a method to construct a generalized BCK-algebra from a quasi-ordered set. They studied also ideal theory in a generalized BCK-algebra. Generally, in any human field, a **Smarandache Structure** on a set  $A$  means a weak structure  $W$  on  $A$  such that there exists a proper subset  $B$  of  $A$  which is embedded with a strong structure  $S$ . In [6], W. B. Vasantha Kandasamy studied the concept of Smarandache groupoids, subgroupoids, ideal of groupoids, semi-normal subgroupoids, Smarandache Bol groupoids and strong Bol groupoids and obtained many interesting results about them. Smarandache semigroups are very important for the study of congruences, and it was studied by R. Padilla [5]. It will be very interesting to study the Smarandache structure in BCK/BCI-algebras. In [2], Y. B. Jun discussed the Smarandache structure in BCI-algebras. He introduced the notion of Smarandache (positive implicative, commutative, implicative) BCI-algebras, Smarandache subalgebras and Smarandache ideals, and investigated some related properties. In [3], the author dealt with Smarandache ideal structures in Smarandache BCI-algebras. He introduced the notion of Smarandache fresh ideals and Smarandache clean ideals in Smarandache BCI-algebras, and investigated its useful properties. He gave relations between  $Q$ -Smarandache fresh ideals and  $Q$ -Smarandache clean ideals. He also establish extension properties for  $Q$ -Smarandache fresh ideals and  $Q$ -Smarandache clean ideals. In this paper we discuss a Smarandache structure on generalized BCK-algebras, and introduce the notion of  $S_\Omega$ -ideal and  $qS_\Omega$ -ideal, and investigate its properties.

**2 Preliminaries.** An algebra  $(X; *, 0)$  of type  $(2, 0)$  is called a **BCI-algebra** if it satisfies the following conditions:

- (I)  $(\forall x, y, z \in X) (((x * y) * (x * z)) * (z * y) = 0)$ ,
- (II)  $(\forall x, y \in X) ((x * (x * y)) * y = 0)$ ,
- (III)  $(\forall x \in X) (x * x = 0)$ ,
- (IV)  $(\forall x, y \in X) (x * y = 0, y * x = 0 \Rightarrow x = y)$ .

If a BCI-algebra  $X$  satisfies the following identity:

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$$(V) (\forall x \in X) (0 * x = 0),$$

then  $X$  is called a **BCK-algebra**. In a BCK-algebra  $X$ , the following identity holds.

$$(a1) (\forall x, y, z \in X) ((x * y) * z = (x * z) * y).$$

We refer the reader to the book [4] for further information regarding BCK/BCI-algebras.

### 3 Quasi Smarandache generalized BCK-algebras

**Definition 3.1.** [1] By a **generalized BCK-algebra** (gBCK-algebra, for short) we mean an algebra  $(G, *, 0)$ , where  $G$  is a nonempty set,  $*$  is a binary operation on  $G$  and  $0 \in G$  is a nullary operation, called *zero element*, such that

$$(G1) x * 0 = x,$$

$$(G2) x * x = 0,$$

$$(G3) (x * y) * z = (x * z) * y,$$

$$(G4) (x * y) * z = (x * z) * (y * z).$$

Notice that gBCK-algebras are determined by identities, and thus the class of gBCK-algebras forms a variety.

**Example 3.2.** [1] Let  $G = \{0, a, b, c\}$  be a set with the following Cayley table.

$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	a
b	b	b	0	0
c	c	c	0	0

It is routine to check that  $(G, *, 0)$  is a gBCK-algebras, which is not a BCK-algebra.

**Example 3.3.** Let  $G = \{0, 1, 2, 3, 4\}$  be a set with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	0
2	2	1	0	2	0
3	3	3	3	0	3
4	4	4	4	4	0

It is routine to verify that  $(G, *, 0)$  is a BCK-algebra which is not a gBCK-algebra.

**Proposition 3.4.** [1] Let  $G$  be a gBCK-algebra. Then

$$(i) (\forall x \in G) (0 * x = 0).$$

$$(ii) (\forall x, y \in G) (x * y) * x = 0).$$

$$(iii) (\forall x, y, z \in G) (x * y = 0 \Rightarrow (x * z) * (y * z) = 0).$$

**Proposition 3.5.** Let  $(G, *, 0)$  be a nontrivial gBCK-algebra. For every  $a (\neq 0) \in G$ , the set  $\{0, a\}$  is a BCK-algebra under the operation in  $G$ .

*Proof.* By the conditions (G1) and (G2), it is straightforward. □

Proposition 3.5 shows that every nontrivial gBCK-algebra  $(G, *, 0)$  has a BCK-algebra of order 2. The following example shows that there is a gBCK-algebra in which there are no proper BCK-algebras of order more than 3.

**Example 3.6.** Let  $G = \{0, a, b, c\}$  be a set with the following Cayley table.

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	0
b	b	0	0	0
c	c	0	0	0

Then  $(G, *, 0)$  is a gBCK-algebra which is not a BCK-algebra, and the sets  $\{0, a, b\}$ ,  $\{0, a, c\}$ ,  $\{0, b, c\}$  are not BCK-algebras.

In [1], Hong et al. showed how to construct a gBCK-algebra from any given quasi-ordered set.

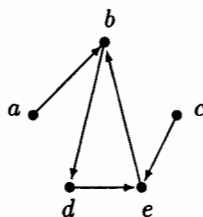
**Proposition 3.7.** [1] Let  $(G, R)$  be a quasi-ordered set. Suppose  $0 \notin G$  and let  $G_0 = G \cup \{0\}$ . Define a binary operation  $*$  on  $G_0$  as follows:

$$x * y = \begin{cases} 0 & \text{if } (x, y) \in R \\ x & \text{otherwise.} \end{cases}$$

Then  $(G_0, *, 0)$  is a gBCK-algebra.

Using Proposition 3.7, we construct a gBCK-algebra.

**Example 3.8.** Let  $G = \{a, b, c, d, e\}$  be a quasi-ordered set with the following directed graph:



Then  $(G_0 = G \cup \{0\}; *, 0)$  is a gBCK-algebra with the following Cayley table:

$*$	0	a	b	c	d	e
0	0	0	0	0	0	0
a	a	0	0	a	0	0
b	b	b	0	b	0	0
c	c	c	0	0	0	0
d	d	d	0	d	0	0
e	e	e	0	e	0	0

Note that  $(G_0 = G \cup \{0\}; *, 0)$  is not a BCK-algebra.

Based on the results above, we give the following definition.

**Definition 3.9.** A quasi Smarandache gBCK-algebra (briefly, **qS-gBCK-algebra**) is defined to be a gBCK-algebra in which there exists a proper subset  $\Omega$  of  $G$  such that

- (s1)  $0 \in \Omega$  and  $|\Omega| \geq 3$ ,
- (s2)  $\Omega$  is a BCK-algebra with respect to the same operation on  $G$ .

Note that any gBCK-algebra of order 3 cannot be a qS-gBCK-algebra. Hence, if  $G$  is a qS-gBCK-algebra, then  $|G| \geq 4$ . Notice that the gBCK-algebra  $G$  in Example 3.6 is not a qS-gBCK-algebra.

**Example 3.10.** (1) The gBCK-algebra  $G$  in Example 3.2 is a qS-gBCK-algebra since  $\Omega = \{0, a, b\}$  is a BCK-algebra which is properly contained in  $G$ .

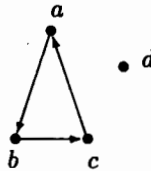
(2) Let  $G = \{0, a, b, c, d\}$  be a set with the following Cayley table.

$*$	$0$	$a$	$b$	$c$	$d$
$0$	$0$	$0$	$0$	$0$	$0$
$a$	$a$	$0$	$a$	$0$	$a$
$b$	$b$	$b$	$0$	$0$	$b$
$c$	$c$	$b$	$a$	$0$	$c$
$d$	$d$	$d$	$d$	$d$	$0$

It is routine to check that  $(G, *, 0)$  is a qS-gBCK-algebra.

(3) The gBCK-algebra  $G_0$  in Example 3.8 is a qS-gBCK-algebra in which  $\Omega_1 = \{0, a, b\}$ ,  $\Omega_2 = \{0, a, c\}$ ,  $\Omega_3 = \{0, a, d\}$ ,  $\Omega_4 = \{0, a, c, e\}$ , and  $\Omega_5 = \{0, a, d, e\}$ , are BCK-algebras.

(4) Consider a quasi-ordered set  $G = \{a, b, c, d\}$  with the following directed graph:



Then  $(G_0 = G \cup \{0\}; *, 0)$  is a qS-gBCK-algebra, where  $*$  is given by the following Cayley table:

$*$	$0$	$a$	$b$	$c$	$d$
$0$	$0$	$0$	$0$	$0$	$0$
$a$	$a$	$0$	$0$	$0$	$a$
$b$	$b$	$0$	$0$	$0$	$b$
$c$	$c$	$0$	$0$	$0$	$c$
$d$	$d$	$d$	$d$	$d$	$0$

Note that  $\Omega_1 = \{0, a, d\}$ ,  $\Omega_2 = \{0, b, d\}$ , and  $\Omega_3 = \{0, c, d\}$  are BCK-algebras.

In what follows, let  $G$  and  $\Omega$  denote a qS-gBCK-algebra and a nontrivial proper BCK-algebra of order more than 3, respectively, unless otherwise specified.

**Definition 3.11.** A nonempty subset  $I$  of  $G$  is called a quasi Smarandache gBCK-ideal of  $G$  related to  $\Omega$  (briefly, **qS $\Omega$ -ideal**) of  $G$  if it satisfies the following conditions:

- (b1)  $I * \Omega := \{a * x \mid a \in I, x \in \Omega\} \subseteq I$ ,

$$(b2) (\forall x \in \Omega) (\forall a, b \in I) (x * ((x * a) * b) \in I).$$

**Example 3.12.** (1) Let  $G = \{0, a, b, c, d\}$  be the qS-gBCK-algebra with  $\Omega = \{0, a, b, c\}$  in Example 3.10(2). Then the sets  $I = \{0, a, d\}$  and  $J = \{0, b, d\}$  are  $qS_{\Omega}$ -ideals of  $G$ .

(2) Let  $G = \{0, a, b, c\}$  be a gBCK-algebra described in Example 3.2. Then  $G$  is a qS-gBCK-algebra with  $\Omega = \{0, a, b\}$ . The sets  $I_1 = \{0, a\}$  and  $I_2 = \{0, b\}$  are  $qS_{\Omega}$ -ideals of  $G$ . But  $J = \{0, c\}$  is not a  $qS_{\Omega}$ -ideal of  $G$  since

$$b * ((b * c) * 0) = b * (0 * 0) = b * 0 = b \notin J.$$

(3) Consider the qS-gBCK-algebra  $G_0$  in Example 3.10(3) with  $\Omega_1 = \{0, a, b\}$  and  $\Omega_2 = \{0, a, c\}$ . Then a set  $I = \{0, b\}$  is a  $qS_{\Omega_2}$ -ideal of  $G$ , but not a  $qS_{\Omega_1}$ -ideal of  $G$  since  $a * ((a * b) * b) = a \notin I$ . The sets  $J_1 = \{0, a\}$  and  $J_2 = \{0, c\}$  are  $qS_{\Omega_4}$ -ideals of  $G_0$ , but the set  $J_3 = \{0, e\}$  is not a  $qS_{\Omega_4}$ -ideal of  $G_0$ .

**Theorem 3.13.** For any element  $a \in G$ , the set  $[a] := \{x \in G \mid x * a = 0\}$  is a  $qS_{\Omega}$ -ideal of  $G$ .

*Proof.* Let  $x \in [a]$  and  $y \in \Omega$ . Then  $x * a = 0$ , and so

$$(x * y) * a = (x * a) * (y * a) = 0 * (y * a) = 0.$$

Hence  $x * y \in [a]$ , i.e.,  $[a] * \Omega \subseteq [a]$ . Let  $z \in \Omega$  and  $x, y \in [a]$ . Then  $x * a = 0$  and  $y * a = 0$ . Hence

$$\begin{aligned} (z * ((z * a) * y)) * a &= (z * a) * (((z * x) * y) * a) \\ &= (z * a) * (((z * x) * a) * (y * a)) \\ &= (z * a) * (((z * a) * (x * a)) * (y * a)) \\ &= (z * a) * (((z * a) * 0) * 0) \\ &= (z * a) * (z * a) \\ &= 0, \end{aligned}$$

and so  $z * ((z * x) * y) \in [a]$ . Therefore  $[a]$  is a  $qS_{\Omega}$ -ideal of  $G$ . □

**Proposition 3.14.** Every  $qS_{\Omega}$ -ideal  $I$  of  $G$  satisfies the following implication:

$$(\forall x \in \Omega) (\forall a \in I) (x * a = 0 \Rightarrow x \in I).$$

*Proof.* Let  $x \in \Omega$  and  $a \in I$  satisfy  $x * a = 0$ . Taking  $b = 0$  in (b2) and using (G4), it follows that  $x = x * 0 = x * (x * a) = x * ((x * a) * 0) \in I$ . This completes the proof. □

**Lemma 3.15.** Let  $I$  be a nonempty subset of  $G$  such that

$$(c1) 0 \in I,$$

$$(c2) (\forall x \in \Omega) (\forall y \in I) (x * y \in I \Rightarrow x \in I).$$

Then we have  $(\forall x \in \Omega) (\forall a \in I) (x * (x * a) \in I)$ .

*Proof.* Assume that  $I$  satisfies (c1) and (c2). Let  $x \in \Omega$  and  $a \in I$ . Then

$$(x * (x * a)) * a = (x * a) * (x * a) = 0 \in I$$

by (c1). It follows from (c2) that  $x * (x * a) \in I$ . □

**Definition 3.16.** A nonempty subset  $I$  of  $G$  is called a **Smarandache gBCK-ideal** of  $G$  related to  $\Omega$  (briefly, **S $_{\Omega}$ -ideal**) of  $G$  if it satisfies the conditions (c1) and (c2).

**Theorem 3.17.** Every  $qS_{\Omega}$ -ideal is an  $S_{\Omega}$ -ideal.

*Proof.* Let  $I$  be a  $qS_{\Omega}$ -ideal of  $G$ . Obviously, (c1) is valid by using (G2) and (b1). Suppose that  $x * y \in I$  for all  $x \in \Omega$  and  $y \in I$ . Then  $(x * (x * y)) = x * ((x * y) * 0) \in I$  by (b2), and so

$$x = x * 0 = x * ((x * (x * y)) * (x * (x * y))) = x * ((x * a) * b) \in I$$

by (b2) where  $a = x * y$  and  $b = x * (x * y)$ . Hence (c2) is valid. □

**Proposition 3.18.** Every  $qS_{\Omega}$ -ideal  $I$  of  $G$  satisfies the following inclusion:

$$(1) \quad (\forall a, b \in I) ([a, b] \subseteq I),$$

where  $[a, b] = \{x \in G \mid (x * a) * b = 0\}$ .

*Proof.* Let  $I$  be a  $qS_{\Omega}$ -ideal of  $G$ . Let  $a, b \in I$  and  $z \in [a, b]$ . Then  $(z * a) * b = 0$ , and so  $z \in I$ . Hence  $[a, b] \subseteq I$ . □

**Theorem 3.19.** If a nonempty subset  $I$  of  $G$  satisfies (1), then  $I$  is an  $S_{\Omega}$ -ideal of  $G$ .

*Proof.* Suppose that  $[a, b] \subseteq I$  for every  $a, b \in I$ . Note that  $(0 * a) * a = 0 * a = 0$  so that  $0 \in [a, a] \subseteq I$ . Let  $x \in \Omega$  and  $y \in I$  satisfy  $x * y \in I$ . Using (G2) and (G3), we have  $(x * (x * y)) * y = (x * y) * (x * y) = 0$  and so  $x \in [x * y, y] \subseteq I$ . Hence  $I$  is an  $S_{\Omega}$ -ideal of  $G$ . □

**Theorem 3.20.** Let  $I$  be an  $S_{\Omega}$ -ideal of  $G$  that satisfies the following inclusion:

$$\Omega * I := \{x * a \mid x \in \Omega, a \in I\} \subseteq \Omega.$$

Then  $I$  is a  $qS_{\Omega}$ -ideal of  $G$ .

*Proof.* Let  $x \in \Omega$  and  $a \in I$ . Then  $(a * x) * a = 0 \in I$  by (c1), and so  $a * x \in I$  by (c2). Now that  $x \in \Omega$  and  $a, b \in I$ . Then  $x * a \in \Omega$  by assumption, and thus

$$(x * ((x * a) * b)) * a = (x * a) * ((x * a) * b) \in I$$

by Lemma 3.15. Hence (b2) is valid. Therefore  $I$  is a  $qS_{\Omega}$ -ideal of  $G$ . □

REFERENCES

[1] S. M. Hong, Y. B. Jun and M. A. Ozturk, *Generalizations of BCK-algebras*, Sci. Math. Jpn. **58** (2003), no. 3, 603–611.  
 [2] Y. B. Jun, *Smarandache BCI-algebras*, Sci. Math. Japon. Online e-2005 (2005), 271–276.  
 [3] Y. B. Jun, *Smarandache fresh and clean ideals of Smarandache BCI-algebras*, Kyungpook Math. J. (to appear).  
 [4] J. Meng and Y. B. Jun, *BCK-algebras*, Kyungmoon Sa Co. Korea, 1994.  
 [5] R. Padilla, *Smarandache algebraic structures*, Bull. Pure Appl. Sci., Delhi, **17E** (1998), no. 1, 119–121; <http://www.gallup.unm.edu/~smarandache/alg-s-tx.txt>.  
 [6] W. B. Vasantha Kandasamy, *Smarandache groupoids*, <http://www.gallup.unm.edu/~smarandache/Groupoids.pdf>.

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