International J.Math. Combin. Vol.3 (2010), 45-56

Some Fixed Point Theorems in Fuzzy n-Normed Spaces

Sayed Khalil Elagan

Department of Mathematics and Statistics, Faculty of Science, Taif University,
Taif , El-Haweiah, P.O.Box 888, Zip Code 21974, Kingdom of Saudi Arabia (KSA)

Mohamad Rafi Segi Rahmat

School of Applied Mathematics, The University of Nottingham Malaysia Campus,
Jalan Broga, 43500 Semenyih, Selangor D.E, Malaysia

Email: sayed_khalil2000@yahoo.com

Abstract: The main purpose of this paper is to study the existence of a fixed points in fuzzy n-normed spaces. we proved our main results, a fixed point theorem for a self mapping and a common fixed point theorem for a pair of weakly compatible mappings on fuzzy n-normed spaces. Also we gave some remarks on fuzzy n-normed spaces.

Key Words: Smarandache space, Pseudo-Euclidean space, fuzzy n-normed spaces, n-seminorm.

AMS(2000): 34A10, 34C10

§1. Introduction

A Pseudo-Euclidean space is a particular Smarandache space defined on a Euclidean space $\mathbb{R}^n$such that a straight line passing through a point p may turn an angle $\theta_p \geq 0$. If $\theta_p \geq 0$, then p is called a non-Euclidean point. Otherwise, a Euclidean point. In this paper, normed spaces are considered to be Euclidean, i.e., every point is Euclidean.In [7], S. Gähler introduced n-norms on a linear space. A detailed theory of n-normed linear space can be found in [8,10,12-13]. In [8], H. Gunawan and M. Mashadi gave a simple way to derive an $(n-1)$- norm from the n-norm in such a way that the convergence and completeness in the n-norm is related to those in the derived $(n-1)$-norm. A detailed theory of fuzzy normed linear space can be found in [1,3,4,5,6,9,11]. In [14], A. Narayanan and S. Vijayabalaji have extend n-normed linear space to fuzzy n-normed linear space. In section 2, we quote some basic definitions, and we show that a fuzzy n-norm is closely related to an ascending system of n-seminorms. In section 3, we introduce a locally convex topology in a fuzzy n-normed space. In section 4, we consider finite dimensional fuzzy n-normed linear spaces. In section 5, we give some fixed point theorem in fuzzy $n-$ normed spaces.

[1]Received June 1, 2010. Accepted September 8, 2010.

§2. Fuzzy n-norms and ascending families of n-seminorms

Let n be a positive integer, and let X be a real vector space of dimension at least n. We recall the definitions of an n-seminorm and a fuzzy n-norm [14].

Definition 2.1 *A function $(x_1, x_2, \ldots, x_n) \mapsto \|x_1, \ldots, x_n\|$ from X^n to $[0, \infty)$ is called an n-seminorm on X if it has the following four properties:*

(S1) $\|x_1, x_2, \ldots, x_n\| = 0$ *if* $x_1, x_2, \ldots, x_n$ *are linearly dependent;*

(S2) $\|x_1, x_2, \ldots, x_n\|$ *is invariant under any permutation of* $x_1, x_2, \ldots, x_n$*;*

(S3) $\|x_1, \ldots, x_{n-1}, cx_n\| = |c|\|x_1, , \ldots, x_{n-1}, x_n\|$ *for any real* c*;*

(S4) $\|x_1, \ldots, x_{n-1}, y + z\| \leqslant \|x_1, \ldots, x_{n-1}, y\| + \|x_1, \ldots, x_{n-1}, z\|$.

An n-seminorm is called a n-norm if $\|x_1, x_2, \ldots, x_n\| > 0$ whenever $x_1, x_2, \ldots, x_n$ are linearly independent.

Definition 2.1 *A fuzzy subset N of $X^n \times \mathbb{R}$ is called a fuzzy n-norm on X if and only if :*

(F1) For all $t \leqslant 0$, $N(x_1, x_2, \ldots, x_n, t) = 0$*;*

(F2) For all $t > 0$, $N(x_1, x_2, \ldots, x_n, t) = 1$ *if and only if* $x_1, x_2, \ldots, x_n$ *are linearly dependent;*

(F3) $N(x_1, x_2, \ldots, x_n, t)$ *is invariant under any permutation of* $x_1, x_2, \ldots, x_n$*;*

(F4) For all $t > 0$ *and* $c \in \mathbb{R}$, $c \neq 0$,

$$N(x_1, x_2, \ldots, cx_n, t) = N(x_1, x_2, \ldots, x_n, \frac{t}{|c|});$$

(F5) For all $s, t \in \mathbb{R}$,

$$N(x_1, \ldots, x_{n-1}, y + z, s + t) \geq \min\{N(x_1, \ldots, x_{n-1}, y, s), N(x_1, \ldots, x_{n-1}z, t)\}.$$

(F6) $N(x_1, x_2, \ldots, x_n, t)$ *is a non-decreasing function of* $t \in \mathbb{R}$ *and*

$$\lim_{t\to\infty} N(x_1, x_2, \ldots, x_n, t) = 1.$$

The following two theorems clarify the relationship between Definitions $2,1$ and 2.2.

Theorem 2.1 *Let N be a fuzzy n-norm on X. As in [14] define for $x_1, x_2, \ldots, x_n \in X$ and $\alpha \in (0, 1)$*

$$\|x_1, x_2, \ldots, x_n\|_\alpha := \inf\{t : N(x_1, x_2, \ldots, x_n, t) \geq \alpha\}. \tag{1}$$

Then the following statements hold.

(A1) For every $\alpha \in (0, 1)$, $\|\bullet, \bullet, \ldots, \bullet\|_\alpha$ *is an n-seminorm on X;*

(A2) If $0 < \alpha < \beta < 1$ *and* $x_1, \ldots, x_n \in X$ *then*

$$\|x_1, x_2, \ldots, x_n\|_\alpha \leqslant \|x_1, x_2, \ldots, x_n\|_\beta;$$

(A3) If $x_1, x_2, \ldots, x_n \in X$ *are linearly independent then*

$$\lim_{\alpha \to 1-} \|x_1, x_2, \ldots, x_n\|_\alpha = \infty.$$

Proof (A1) and (A2) are shown in [14, Theorem 3.4]. Let $x_1, x_2, \ldots, x_n \in X$ be linearly independent, and $t > 0$ be given. We set $\beta := N(x_1, x_2, \ldots, x_n, t)$. It follows from (F2) that $\beta \in [0, 1)$. Then (F6) shows that, for $\alpha \in (\beta, 1)$,

$$\|x_1, x_2, \ldots, x_n\|_\alpha \geqslant t.$$

This proves (A3). □

We now prove a converse of Theorem 2.1.

Theorem 2.2 *Suppose we are given a family* $\|\bullet, \bullet, \ldots, \bullet\|_\alpha$, $\alpha \in (0, 1)$, *of n-seminorms on X with properties (A2) and (A3). We define*

$$N(x_1, x_2, \ldots, x_n, t) := \inf\{\alpha \in (0, 1) : \|x_1, x_2, \ldots, x_n\|_\alpha \geqslant t\}. \tag{2}$$

where the infimum of the empty set is understood as 1. *Then* N *is a fuzzy n-norm on* X.

Proof (F1) holds because the values of an n-seminorm are nonnegative.

(F2): Let $t > 0$. If $x_1, \ldots, x_n$ are linearly dependent then $N(x_1, \ldots, x_n, t) = 1$ follows from property (S1) of an n-seminorm. If $x_1, \ldots, x_n$ are linearly independent then $N(x_1, \ldots, x_n, t) < 1$ follows from (A3).

(F3) is a consequence of property (S2) of an n-seminorm.

(F4) is a consequence of property (S3) of an n-seminorm.

(F5): Let $\alpha \in (0, 1)$ satisfy

$$\alpha < \min\{N(x_1, \ldots, x_{n-1}, y, s), N(x_1, \ldots, x_{n-1}, z, s)\}. \tag{3}$$

It follows that $\|x_1, \ldots, x_{n-1}, y\|_\alpha < s$ and $\|x_1, \ldots, x_{n-1}, z\|_\alpha < t$. Then (S4) gives

$$\|x_1, \ldots, x_{n-1}, y + z\|_\alpha < s + t.$$

Using (A2) we find $N(x_1, \ldots, x_{n-1}, y+z, s+t) \geqslant \alpha$ and, since α is arbitrary in (3), (F5) follows.

(F6): Definition 2.2 shows that N is non-decreasing in t. Moreover, $\lim_{t \to \infty} N(x_1, \ldots, x_n, t) = 1$ because seminorms have finite values. □

It is easy to see that Theorems 2.1 and 2.2 establish a one-to-one correspondence between fuzzy n-norms with the additional property that the function $t \mapsto N(x_1, \ldots, x_n, t)$ is left-continuous for all $x_1, x_2, \ldots, x_n$ and families of n-seminorms with properties (A2), (A3) and the additional property that $\alpha \mapsto \|x_1, \ldots, x_n\|_\alpha$ is left-continuous for all $x_1, x_2, \ldots, x_n$.

Example 2.3([14,Example 3.3] Let $\|\bullet,\bullet,\ldots,\bullet\|$ be a n-norm on X. Define $N(x_1,x_2,\ldots,x_n,t)$ $=0$ if $t\leqslant 0$ and, for $t>0$,

$$N(x_1,x_2,\ldots,x_n,t)=\frac{t}{t+\|x_1,x_2,\ldots,x_n\|}.$$

Then the seminorms (2.1) are given by

$$\|x_1,x_2,\ldots,x_n\|_\alpha=\frac{\alpha}{1-\alpha}\|x_1,x_2,\ldots,x_n\|.$$

§3. The locally convex topology generated by a fuzzy n-norm

In this section (X,N) is a fuzzy n-normed space, that is, X is real vector space and N is fuzzy n-norm on X. We form the family of n-seminorms $\|\bullet,\bullet,\ldots,\bullet\|_\alpha$, $\alpha\in(0,1)$, according to Theorem 2.1. This family generates a family $\mathcal{F}$ of seminorms

$$\|x_1,\ldots,x_{n-1},\bullet\|_\alpha,\quad \text{where } x_1,\ldots,x_{n-1}\in X \text{ and } \alpha\in(0,1).$$

The family $\mathcal{F}$ generates a locally convex topology on X; see [15, Def. (37.9)], that is, a basis of neighborhoods at the origin is given by

$$\{x\in X: p_i(x)\leqslant\epsilon_i \text{ for } i=1,2,\ldots,n\},$$

where $p_i\in\mathcal{F}$ and $\epsilon_i>0$ for $i=1,2\ldots,n$. We call this the locally convex topology generated by the fuzzy n-norm N.

Theorem 3.1 *The locally convex topology generated by a fuzzy n-norm is Hausdorff.*

Proof Given $x\in X$, $x\neq 0$, choose $x_1,\ldots,x_{n-1}\in X$ such that $x_1,\ldots,x_{n-1},x$ are linearly independent. By Theorem 2.1(A3) we find $\alpha\in(0,1)$ such that $\|x_1,\ldots,x_{n-1},x\|_\alpha>0$. The desired statement follows; see [15, Theorem 37.21]. □

Some topological notions can be expressed directly in terms of the fuzzy-norm N. For instance, we have the following result on convergence of sequences. We remark that the definition of convergence of sequences in a fuzzy n-normed space as given in [20, Definition 2.2] is meaningless.

Theorem 3.2 *Let $\{x_k\}$ be a sequence in X and $x\in X$. Then $\{x_k\}$ converges to x in the locally convex topology generated by N if and only if*

$$\lim_{k\to\infty}N(a_1,\ldots,a_{n-1},x_k-x,t)=1 \tag{4}$$

for all $a_1,\ldots,a_{n-1}\in X$ and all $t>0$.

Proof Suppose that $\{x_k\}$ converges to x in (X,N). Then, for every $\alpha\in(0,1)$ and all $a_1,a_2,\ldots,a_{n-1}\in X$, there is K such that, for all $k\geqslant K$, $\|a_1,a_2,\ldots,a_{n-1},x_k-x\|_\alpha<\epsilon$. The latter implies

$$N(a_1,a_2,\ldots,a_{n-1},x_k-x,\epsilon)\geqslant\alpha.$$

Since $\alpha \in (0,1)$ and $\epsilon > 0$ are arbitrary we see that (4) holds. The converse is shown in a similar way. □

In a similar way we obtain the following theorem.

Theorem 3.3 *Let $\{x_k\}$ be a sequence in X. Then $\{x_k\}$ is a Cauchy sequence in the locally convex topology generated by N if and only if*

$$\lim_{k,m\to\infty} N(a_1,\ldots,a_{n-1},x_k - x_m,t) = 1 \tag{5}$$

for all $a_1,\ldots,a_{n-1} \in X$ and all $t > 0$.

It should be noted that the locally convex topology generated by a fuzzy n-norm is not metrizable, in general. Therefore, in many cases it will be necessary to consider nets $\{x_i\}$ in place of sequences. Of course, Theorems 3.2 and 3.3 generalize in an obvious way to nets.

§4. Fuzzy n-norms on finite dimensional spaces

In this section (X,N) is a fuzzy n-normed space and X has finite dimension at least n. Since the locally convex topology generated by N is Hausdorff by Theorem 3.1 Tihonov's theorem [15, Theorem 23.1] implies that this locally convex topology is the only one on X. Therefore, all fuzzy n-norms on X are equivalent in the sense that they generate the same locally convex topology.

In the rest of this section we will give a direct proof of this fact (without using Tihonov's theorem). We will set $X = \mathbb{R}^d$ with $d \geqslant n$.

Lemma 4.1 *Every n-seminorm on $X = \mathbb{R}^d$ is continuous as a function on X^n with the euclidian topology.*

Proof For every $j = 1,2,\ldots,n$, let $\{x_{j,k}\}_{k=1}^{\infty}$ be a sequence in X converging to $x_j \in X$. Therefore, $\lim\limits_{k\to\infty} \|x_{j,k} - x_j\| = 0$, where $\|x\|$ denotes the euclidian norm of x. From property (S4) of an n-seminorm we get

$$|\|x_{1,k},x_{2,k},\ldots,x_{n,k}\| - \|x_1,x_{2,k},\ldots,x_{n,k}\|| \le \|x_{1,k} - x_1,x_{2,k},\ldots,x_{n,k}\|.$$

Expressing every vector in the standard basis of $\mathbb{R}^d$ we see that there is a constant M such that

$$\|y_1,y_2,\ldots,\ y_n\| \le M\,\|y_1\|\ldots\|y_n\| \text{ for all } y_j \in X.$$

Therefore,

$$\lim_{k\to\infty} \|x_{1,k} - x_1,\ x_{2,k},\ldots,\ x_{n,k}\| = 0$$

and so

$$\lim_{k\to\infty} |\|x_{1,k},\ x_{2,k},\ldots,\ x_{n,k}\| - \|x_1,\ x_{2,k},\ldots,\ x_{n,k}\|| = 0.$$

We continue this procedure until we reach

$$\lim_{k\to\infty} \|x_{1,k},\ x_{2,k},\ldots,\ x_{n,k}\| = \|x_1,\ x_2,\ldots,\ x_n\|\,. \qquad \square$$

Lemma 4.2 *Let $(\mathbb{R}^d, N)$ be a fuzzy n-normed space. Then $\|x_1, x_2, \ldots, x_n\|_\alpha$ is an n-norm if $\alpha \in (0,1)$ is sufficiently close to 1.*

Proof We consider the compact set

$$S = \left\{(x_1,\ x_2, \ldots,\ x_n) \in \mathbb{R}^{dn} : x_1,\ x_2, \ldots,\ x_n \text{ is an orthonormal system in } \mathbb{R}^d\right\}.$$

For each $\alpha \in (0,1)$ consider the set

$$S_\alpha = \{(x_1, x_2, \ldots, x_n) \in S : \|x_1, x_2, \ldots, x_n\|_\alpha > 0\}.$$

By Lemma 4.1, S_α is an open subset of S. We now show that

$$S = \bigcup_{\alpha \in (0,1)} S_\alpha. \tag{6}$$

If $(x_1,\ x_2, \ldots,\ x_n) \in S$ then $(x_1,\ x_2, \ldots,\ x_n)$ is linearly independent and therefore there is β such that $N(x_1, x_2, \ldots, x_n, 1) < \beta < 1$. This implies that $\|x_1,\ x_2, \ldots,\ x_n\|_\beta \geq 1$ so (6) is proved. By compactness of S, we find $\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_m$ such that

$$S = \bigcup_{i=1}^{m} S_{\alpha_i}.$$

Let $\alpha = \max\{\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_m\}$. Then $\|x_1,\ x_2, \ldots,\ x_n\|_\alpha > 0$ for every $(x_1,\ x_2, \ldots,\ x_n) \in S$.

Let $x_1,\ x_2, \ldots,\ x_n \in X$ be linearly independent. Construct an orthonormal system $e_1,\ e_2, \ldots,\ e_n$ from $x_1,\ x_2, \ldots,\ x_n$ by the Gram-Schmidt method. Then there is $c > 0$ such that

$$\|x_1,\ x_2, \ldots,\ x_n\|_\alpha = c\,\|e_1,\ e_2, \ldots,\ e_n\|_\alpha > 0.$$

This proves the lemma. □

Theorem 4.1 *Let N be a fuzzy n-norm on $\mathbb{R}^d$, and let $\{x_k\}$ be a sequence in $\mathbb{R}^d$ and $x \in \mathbb{R}^d$.*

(a) $\{x_k\}$ converges to x with respect to N if and only if $\{x_k\}$ converges to x in the euclidian topology.

(b) $\{x_k\}$ is a Cauchy sequence with respect to N if and only if $\{x_k\}$ is a Cauchy sequence in the euclidian metric.

Proof (a) Suppose $\{x_k\}$ converges to x with respect to euclidian topology. Let $a_1, a_2, \ldots, a_{n-1} \in X$. By Lemma 4.1, for every $\alpha \in (0,1)$,

$$\lim_{k\to\infty} \|a_1,\ a_2, \ldots,\ a_{n-1},\ x_k - x\|_\alpha = 0.$$

By definition of convergence in $(\mathbb{R}^d, N)$, we get that $\{x_k\}$ converges to x in $(\mathbb{R}^d, N)$. Conversely, suppose that $\{x_k\}$ converges to x in $(\mathbb{R}^d, N)$. By Lemma 4.2, there is $\alpha \in (0,1)$ such that $\|y_1, y_2, \ldots, y_n\|_\alpha$ is an n-norm. By definition, $\{x_k\}$ converges to x in the n-normed space $(\mathbb{R}^d, \|\cdot\|_\alpha)$. It is known from[8, Proposition 3.1] that this implies that $\{x_k\}$ converges to x with respect to euclidian topology.

(b) is proved in a similar way. □

Theorem 4.2 *A finite dimensional fuzzy n-normed space (X, N) is complete.*

Proof This follows directly from Theorem 3.4. □

§5. Some fixed point theorem in fuzzy $n-$ normed spaces

In this section we prove some fixed point theorems.

Definition 5.1 *A sequence a $\{x_k\}$ in a fuzzy n-normed space (X, N) is said to be fuzzy n-convergent to $x^* \in X$ and denoted by $x_k \rightsquigarrow x^*$ as $k \to \infty$ if*

$$\lim_{k\to\infty} N(x_1, \cdots, x_{n-1}, x_k - x^*, t) = 1$$

for every $x_1, \cdots, x_{n-1} \in X$ and x^* is called the fuzzy n-limit of $\{x_k\}$.

Remark 5.1 It is noted that if (X, N) is a fuzzy n-normed space then the fuzzy n-limit of a fuzzy n-convergent sequence is unique. Indeed, if $\{x_k\}$ is a fuzzy n-convergent sequence and suppose it converges to x^* and y^* in X. Then by definition $\lim_{k\to\infty} N(x_1, \cdots, x_{n-1}, x_k - x^*, t) = 1$ and $\lim_{k\to\infty} N(x_1, \cdots, x_{n-1}, x_k - y^*, t) = 1$ for every $x_1, \cdots, x_{n-1} \in X$ and for every $t > 0$. By $(N5)$, we have

$$\begin{aligned} N(x_1, \cdots, x_{n-1}, x - y, t) &= N(x_1, \cdots, x_{n-1}, x^* - x_k + x_k - y^*, t/2 + t/2) \\ &\geqslant \min\{N(x_1, \cdots, x_{n-1}, x^* - x_k, t/2), N(x_1, \cdots, x_{n-1}, x_k - y^*, t/2)\}. \end{aligned}$$

By letting $k \to \infty$, we obtain $N(x_1, \cdots, x_{n-1}, x^* - y^*, t) = 1$, which implies that $x^* = y^*$.

Definition 5.2 *A sequence $\{x_k\}$ in a fuzzy n-normed space (X, N) is said to be fuzzy n-Cauchy sequence if*

$$\lim_{k,m\to\infty} N(x_1, \ldots, x_{n-1}, x_k - x_m, t) = 1$$

for every $x_1, \cdots, x_{n-1} \in X$ and for every $t > 0$.

Proposition 5.1 *In a fuzzy n-normed space (X, N), every fuzzy n-convergent sequence is a fuzzy n-Cauchy sequence.*

Proof Let $\{x_k\}$ be a fuzzy n-convergent sequence in X converging to $x^* \in X$. Then $\lim_{k\to\infty} N(x_1, \cdots, x_{n-1}, x_k - x^*, t) = 1$ for every $x_1, \cdots, x_{n-1} \in X$ and for every $t > 0$. By (N5),

$$\begin{aligned} &N(x_1, \cdots, x_{n-1}, x_k - x_m, t) \\ &\quad = N(x_1, \cdots, x_{n-1}, x_k - x^* + x^* - x_m, t/2 + t/2) \\ &\quad \geqslant \min\{N(x_1, \cdots, x_{n-1}, x_k - x^*, t/2), N(x_1, \cdots, x_{n-1}, x^* - x_m, t/2)\}. \end{aligned}$$

By letting $n, m \to \infty$, we get,

$$\lim_{k,m\to\infty} N(x_1, \cdots, x_{n-1}, x_k - x_m, t) = 1$$

for every $x_1, \cdots, x_{n-1} \in X$ and for every $t > 0$, i.e., $\{x_k\}$ is a fuzzy n-Cauchy sequence. □

If every fuzzy n-Cauchy sequence in X converges to an $x^* \in X$, then (X, N) is called a complete fuzzy n-normed space. A complete fuzzy n-normed space is then called a fuzzy n-Banach space.

Theorem 5.1 *Let (X, N) be a fuzzy n-normed space. Let $f : X \to X$ be a map satisfies the condition:*
There exists a $\lambda \in (0,1)$ such that for all $x, x_1, \cdots, x_{n-1} \in X$ and for all $t > 0$, one has

$$N(x_1, \cdots, x_{n-1}, x, t) > 1 - t \quad \Rightarrow \quad N(x_1, \cdots, x_{n-1}, f(x), \lambda t) > 1 - \lambda t. \tag{7}$$

Then

(i) For any real number $\epsilon > 0$ there exists $k_0(\epsilon) \in \mathbb{N}$ such that $f^k(x) \rightsquigarrow \theta$.

(ii) f has at most a fixed point, that is the null vector of X. Moreover, if f is a linear mapping, f has exactly one fixed point.

Proof (i) Note that if f satisfies the condition (1), then for every $\epsilon \in (0,1)$, there exists a $k_0 = k_0(\epsilon)$ such that, for all $k \geqslant k_0$, and for every $x, x_1, \cdots, x_{n-1} \in X$

$$N(x_1, \cdots, x_{n-1}, f^k(x), \epsilon) > 1 - \epsilon$$

holds. Indeed, one has easily that

$$N(x_1, \cdots, x_{n-1}, x, 1 + \epsilon) > 1 - (1 + \epsilon).$$

Then by condition (1), for all $x, x_1, \cdots, x_{n-1} \in X$ and $k \geqslant 1$,

$$N(x_1, \cdots, x_{n-1}, f^k(x), \lambda^k(1 + \epsilon)) > 1 - \lambda^k(1 + \epsilon)$$

holds. Indeed, for each $\epsilon > 0$ there exists a $k = k_0$ implies that $\lambda^n(1 + \epsilon) \leqslant \epsilon$, from which, because of condition (N6), there exists a $k_0 \in \mathbb{N}$ such that for $k \geqslant k_0$,

$$\begin{aligned} N(x_1, \cdots, x_{n-1}, f^k(x), \epsilon) &\geqslant N(x_1, \cdots, x_{n-1}, f^k(x), \lambda^k(1 + \epsilon)) \\ &> 1 - \lambda^k(1 + \epsilon) \\ &\geqslant 1 - \epsilon. \end{aligned}$$

Since ϵ is an arbitrary, we have $f^k(x) \rightsquigarrow \theta$ as required.

(ii) Assume that $f(x) = x$. By applying part (i), for all $\epsilon \in (0,1)$ one has

$$N(x_1, \cdots, x_{n-1}, x, \epsilon) > 1 - \epsilon$$

for every $x_1, \cdots, x_{n-1} \in X$. This implies that

$$N(x_1, \cdots, x_{n-1}, x, 0+) = 1$$

for every $x_1, \cdots, x_{n-1} \in X$, i.e., $x = \theta$. □

Lemma 5.1 *Let $\{x_k\}$ be a sequence in a fuzzy n-normed space (X, M). If for every $t > 0$, there exists a constant $\lambda \in (0,1)$ such that*

$$N(x_1, \ldots, x_{n-1}, x_k - x_{k+1}, t) \geq N(x_1, \ldots, x_{n-1}, x_{k-1} - x_k, t/\lambda) \tag{8}$$

for all $x_1, \cdots, x_{n-1} \in X$, then $\{x_k\}$ is a fuzzy n-Cauchy sequence in X.

Proof Let $t > 0$ and $\lambda \in (0,1)$. Then for $m \geqslant k$, by using (N5) and the inequality (1), we have

$$\begin{aligned}
&N(x_1, \ldots, x_{n-1}, x_k - x_m, t)\\
&\quad \geqslant \min\{N(x_1, \ldots, x_{n-1}, x_k - x_{k+1}, (1-\lambda)t),\\
&\qquad N(x_1, \ldots, x_{n-1}, x_{k+1} - x_m, \lambda t)\}\\
&\qquad \ldots\\
&\quad \geqslant \min\{N(x_1, \ldots, x_{n-1}, x_0 - x_1, \frac{(1-\lambda)t}{\lambda^k}),\\
&\qquad N(x_1, \ldots, x_{n-1}, x_{k+1} - x_m, \lambda t)\}
\end{aligned}$$

Also,

$$\begin{aligned}
&N(x_1, \ldots, x_{n-1}, x_{k+1} - x_m, \lambda t)\\
&\quad \geqslant \min\{N(x_1, \ldots, x_{n-1}, x_{k+1} - x_{k+2}, (1-\lambda)\lambda t),\\
&\qquad N(x_1, \ldots, x_{n-1}, x_{k+2} - x_m, \lambda^2 t)\}\\
&\qquad \ldots\\
&\quad \geqslant \min\{N(x_1, \ldots, x_{n-1}, x_0 - x_1, \frac{(1-\lambda)t}{\lambda^k}),\\
&\qquad N(x_1, \ldots, x_{n-1}, x_{k+2} - x_m, \lambda^2 t)\}
\end{aligned}$$

By repeating these argument, we get

$$\begin{aligned}
&N(x_1, \ldots, x_{n-1}, x_k - x_m, t)\\
&\quad \geqslant \min\{N(x_1, \ldots, x_{n-1}, x_0 - x_1, \frac{(1-\lambda)t}{\lambda^k}),\\
&\qquad N(x_1, \ldots, x_{n-1}, x_{m-1} - x_m, \lambda^{m-n-1} t)\}\\
&\qquad \ldots\\
&\quad \geqslant \min\{N(x_1, \ldots, x_{n-1}, x_0 - x_1, \frac{(1-\lambda)t}{\lambda^k}),\\
&\qquad N(x_1, \ldots, x_{n-1}, x_0 - x_1, \frac{t}{\lambda^k})\}
\end{aligned}$$

Since $(1-\lambda)\frac{t}{\lambda^k} \leq \frac{t}{\lambda^k}$ and the property (F6), we conclude that

$$N(x_1, \ldots, x_{n-1}, x_k - x_m, t) \geq N(x_1, \ldots, x_{n-1}, x_0 - x_1, \frac{(1-\lambda)t}{\lambda^k}).$$

Therefore, by letting $m \geqslant k \to \infty$, we get

$$\lim_{k,m\to\infty} N(x_1, \cdots, x_{n-1}, x_k - x_m, t) = 1$$

for every $x_1, \cdots, x_{n-1} \in X$ and for every $t > 0$, i.e., $\{x_k\}$ is a fuzzy n-Cauchy sequence. $\square$

Definition 5.3 *A pair of maps (f, g) is called weakly compatible pair if they commute at coincidence point, i.e., $fx = gx$ implies $fgx = gfx$.*

Theorem 5.2 *Let (X, M) be a fuzzy n-normed space and let $f, g\colon X \to X$ satisfy the following conditions:*

(i) $f(X) \subseteq g(X)$;

(ii) *any one $f(X)$ or $g(X)$ is complete;*

(iii) $N(x_1, \ldots, x_{n-1}, f(x)-f(y), t) \geqslant N(x_1, \ldots, x_{n-1}, g(x)-g(y), t/\lambda)$, *for all* $x, y, x_1, \cdots, x_{n-1} \in X$, $t > 0$, $\lambda \in (0, 1)$.

Then f and g have a unique common fixed point provided f and g are weakly compatible on X.

Proof Let $x_0 \in X$. By condition (i), we can find $x_1 \in X$ such that $f(x_0) = g(x_1) = y_1$. By induction, we can define a sequence y_k in X such that

$$y_{k+1} = f(x_k) = g(x_{k+1}),$$

$n = 0, 1, 2, \cdots$. We consider two cases:
Case I: If $y_r = y_{r+1}$ for some $r \in \mathbb{N}$, then

$$y_r = f(x_{r-1}) = f(x_r) = g(x_r) = g(x_{r+1}) = y_{r+1} = z$$

for some $z \in X$. Since $f(x_r) = g(x_r)$ and f, g are weakly compatible, we have $f(z) = fg(x_r) = gf(x_r) = g(z)$. By condition (iii), for all $x_1, \cdots, x_{n-1} \in X$ and for all $t > 0$, we have

$$\begin{aligned} N(x_1, \cdots, x_{n-1}, f(z) - z, t) &= N(x_1, \cdots, x_{n-1}, f(z) - f(x_r), t) \\ &\geqslant N(x_1, \cdots, x_{n-1}, g(z) - g(x_r), t/\lambda) \\ &\geqslant \cdots \geq N(x_1, \cdots, x_{n-1}, g(z) - g(x_r), t/\lambda^k). \end{aligned}$$

Clearly, the righthand side of the inequality approaches 1 as $k \to \infty$ for every $x_1, \ldots, x_{n-1} \in X$ and $t > 0$. Hence, $N(x_1, \cdots, x_{n-1}, f(z) - z, t) = 1$. This implies that $f(z) = z = g(z)$, i.e., z is a common fixed point of f and g.
Case II $y_k \neq y_{k+1}$, for each $k = 0, 1, 2, \cdots$. Then, by condition (ii) again, we have

$$\begin{aligned} N(x_1, \cdots, x_{n-1}, y_k - y_{k+1}, t) &= N(x_1, \cdots, x_{n-1}, g(x_k) - g(x_{k+1}), t) \\ &= N(x_1, \cdots, x_{n-1}, f(x_{k-1}) - f(x_k), t) \\ &\geq N(x_1, \cdots, x_{n-1}, g(x_{k-1}) - g(x_k), t/\lambda) \\ &= N(x_1, \cdots, x_{n-1}, y_{k-1} - y_k, t) \end{aligned}$$

Then, by Lemma 5.1, $\{y_k\}$ is a Cauchy sequence (with respect to fuzzy n-norm) in X. Since $g(X)$ is complete, there exists $w \in g(X)$ such that

$$\lim_{k\to\infty} y_k = \lim_{k\to\infty} g(x_k) = w.$$

Also, since $w \in g(X)$, we can find a $p \in X$ such that $g(p) = w$. Note that

$$w = g(p) = \lim_{k\to\infty} g(x_k) = \lim_{k\to\infty} f(x_k).$$

Thus, by (iii), we have

$$\begin{aligned} N(x_1, \cdots, x_{n-1}, f(p) - g(p), t) &= \lim_{k\to\infty} N(x_1, \cdots, x_{n-1}, f(p) - f(x_k), t) \\ &\geq \lim_{k\to\infty} N(x_1, \cdots, x_{n-1}, g(p) - g(x_k), t/\lambda) \\ &= N(x_1, \cdots, x_{n-1}, g(p) - w, t/\lambda) \\ &= N(x_1, \cdots, x_{n-1}, w - w, t/\lambda), \end{aligned}$$

which implies that $w = f(p) = g(p)$ is a common fixed point of f and g. Furthermore, f and g are weakly compatible maps, we have

$$f(w) = fg(w) = gf(w) = g(w).$$

But than, by (iii),

$$\begin{aligned} N(x_1, \cdots, x_{n-1}, f(w) - w, t) &= N(x_1, \cdots, x_{n-1}, f(w) - f(p), t) \\ &\geq N(x_1, \cdots, x_{n-1}, g(w) - g(p), t/\lambda) \\ &= N(x_1, \cdots, x_{n-1}, f(w) - f(p), t/\lambda) \\ &\geq \cdots \geq N(x_1, \cdots, x_{n-1}, g(w) - g(p), t/\lambda^k). \end{aligned}$$

Clearly, the expression on the righthand side approaches 1 as $k \to \infty$ for every $x_1, \ldots, x_{n-1} \in X$ and $t > 0$, which implies that $f(w) = w$. Therefore, w is a common fixed point of f and g. The uniqueness of fixed point is immediate from condition (iii). $\square$

References

[1] T. Bag and S. K. Samanta, Finite dimensional fuzzy normed linear spaces, *J. Fuzzy Math.* 11 (2003), no. 3, 687-705.

[2] S. Berberian, *Lectures in Functional Analsysis and Operator Theory*, Springer-Verlag, New York, 1974.

[3] S.C. Chang and J. N. Mordesen, Fuzzy linear operators and fuzzy normed linear spaces, *Bull. Calcutta Math. Soc.* 86 (1994), no. 5, 429-436.

[4] C. Felbin, Finite- dimensional fuzzy normed linear space, *Fuzzy Sets and Systems* 48(1992), no. 2, 239-248.

[5] ———, The completion of a fuzzy normed linear space, *J. Math. Anal. Appl.* 74(1993), no. 2, 428-440.

[6] ————, Finite dimensional fuzzy normed linear space. II., *J. Anal.* 7(1999), 117-131.

[7] S. Gähler, Untersuchungen uber verallgemeinerte m-metrische Raume, I, II, III., *Math. Nachr.* 40(1969), 165-189.

[8] H. Gunawan and M. Mashadi, On n-normed spaces, *Int. J. Math. Math. Sci.* 27(2001), No.10, 631-639.

[9] A. K. Katsaras, Fuzzy topological vector spaces. II., *Fuzzy Sets and Systems* 12(1984), no. 2, 143-154.

[10] S. S. Kim and Y. J. Cho, Strict convexity in linear $n-$ normed spaces, *Demonstratio Math.* 29(1996), No. 4, 739-744.

[11] S. V, Krish and K. K. M. Sarma, Separation of fuzzy normed linear spaces, *Fuzzy Sets and Systems* 63(1994), No. 2, 207-217.

[12] R. Malceski, Strong n-convex n-normed spaces, *Math. Bilten No.* 21(1997), 81-102.

[13] A. Misiak, n-inner product spaces, *Math. Nachr.* 140(1989), 299-319.

[14] Al. Narayanan and S. Vijayabalaji, Fuzzy $n-$ normed linear spaces, *Int. J. Math. Math. Sci.* 27(2005), No. 24, 3963-3977.

[15] R. Rado, A theorem on infinite series, *J. Lond. Math. Soc.* 35(1960), 273-276.

[16] M. R. S. Rahmat, Fixed point theorems on fuzzy inner product spaces, Mohu Xitong Yu Shuxue, (*Fuzzy Systems and Mathematics. Nat. Univ. Defense Tech.*, Changsha) 22 (2008), No. 3, 89-96.

[17] G. S. Rhie, B. M. Choi, and D. S. Kim, On the completeness of fuzzy normed linear spaces, *Math. Japon.* 45(1997), No. 1, 33-37.

[18] G. S. Rhie, B. M. Choi, and D. S. Kim, On the completeness of fuzzy normed linear spaces, *Math. Japon.* 45(1997), No. 1, 33-37.

[19] A. Smith, Convergence preserving function: an alternative discussion, *Amer. Math. Monthly* 96(1991), 831-833.

[20] S. Vijayabalaji and N. Thilligovindan, Complete fuzzy n-normed space, *J. Fund. Sciences* 3 (2007), 119-126 (available online at www.ibnusina.utm.my/jfs)

[21] G. Wildenberg, Convergence preserving functions, *Amer. Math. Monthly* 95(1988), 542-544.